4.1 5-CONFIGURATIONS

The history of 5-configurations is even shorter than that of 4-configurations, and the knowledge is also much skimpier. However, there are several interesting aspects that do not appear in 3- and 4-configurations.

From the obvious necessary conditions it follows that any (n_5) configuration must satisfy $n \geq 21$. The build-up of a combinatorial configuration (21_5) using the "greedy" approach (as for (7_3) in Table 2.2.2 and for (13_4) in Table 3.1.1) can probably be carried out without undue effort. However, it seems more interesting to note that (21_5) is the cyclic configuration based on (0,3,4,9,11). As noted by Gropp [G8], while it is obvious that this cyclic basis works for all $n \geq 2 \cdot 11 + 1 = 23$, its validity for $n = 21$ is unexpected but easily verified. The configuration is presented in Table 4.1.1. Gropp [G32] seems also to be the first to discover that (0,1,4,9,11) is a cyclic basis for (n_5) for all $n \geq 23$ as well; but it does not yield (21_5). Gropp establishes a connection of these bases with the "Golomb rulers" — combinatorial objects interesting in their own right; for some details see [G18], [G4], [G5][1].

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3
9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8
11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10

Table 4.1.1. The cyclic combinatorial configuration (21_5) generated by the basis (0,3,4,9,11). This basis work also for each $n \geq 23$ to yield a configuration (n_5).

So far we avoided mentioning the configuration (22_5). It is a particularly interesting one because — in contrast to the situation we encountered for 3- and 4-configurations — this configuration does not exist even combinatorially. The proof of this requires tools that are outside the scope of this text.

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[1] Much additional information can be found on the Internet. See, for example, [C4], [S13], and, in particular, "Golomb ruler" in the Wikipedia.

Except for the existence of two non-isomorphic cyclic combinatorial configurations (23_5) there seems to be no information available regarding the numbers of distinct (n_5). It is easy to construct, for all $n \geq 25$, additional cyclic bases such as (0,3,4,10,12), (0,1,4,10,12), or (0,1,6,10,12); but neither their number, nor possible isomorphisms, nor the existence of non-cyclic configurations seem to have been investigated.

For 4-configurations we have seen in Section 3.4 that one needs to increase the number of points only slightly from the minimal value n = 13 to reach values for which topological or geometric configurations exist — n = 17 for the former, and n = 18 for the latter. Moreover, all these are best possible values. In case of 5-configurations the information available is far less satisfactory.

It is obvious that the configuration LC(5) (see definition in Section 1.1) is geometrically realizable; however, with $5^5 = 3125$ points and lines there is no intelligible realization. The first description of a graphically presentable 5-configuration appeared in [G50]; it is a (60_5) that is 3-astral in the extended Euclidean plane, and is also shown in [G46] and as Figure 4.1.1 below. (By the convention adopted in Section 1.5, we may call such 3-astral 5-configurations **astral**.) The construction is based on the idea that many 4-configurations have quadruplets of points aligned on diameters and are such that these diameters are parallel to quadruplets of lines. Then the addition of the diameters gives [5,4]-configurations, for which the addition of points at infinity results in 5-configurations. This construction is also illustrated in Figure 4.1.2 in the case of a (50_5) configuration, which is a smallest such configuration known. Another (50_5) configuration is shown in [G50].

All these configurations are symmetric only in the extended Euclidean plane, since they include points at infinity. Switching to their polars is no remedy due to the lines through the center. Allowing a slight larger size enables one to construct 5-configurations with dihedral symmetry by a slightly different process, starting with 5-astral 4-configurations. An example appears in Figure 4.1.3. It is a (54_5) configuration with d_9 symmetry, that is 6-astral in the extended Euclidean plane.

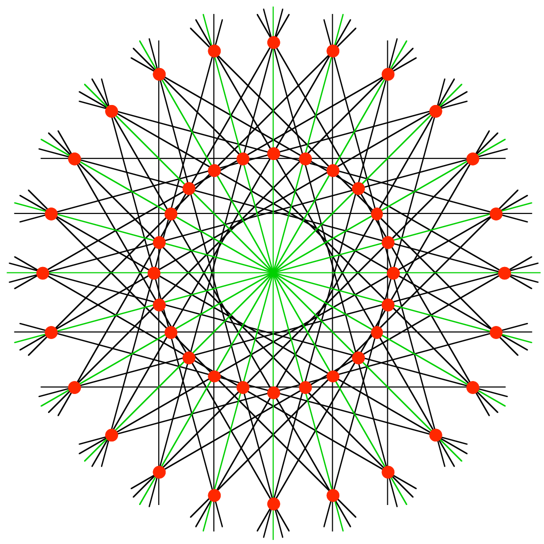


Figure 4.1.1. Deleting the 12 lines (green) through the center yields the astral (48_4) configuration (2) 12#(5,4;1,4). With these lines it is a (48_5, 60_4) configuration. Adding 12 points at infinity, in the directions of the ten green lines, results in a (60_5) configuration that is astral in the extended Euclidean plane.

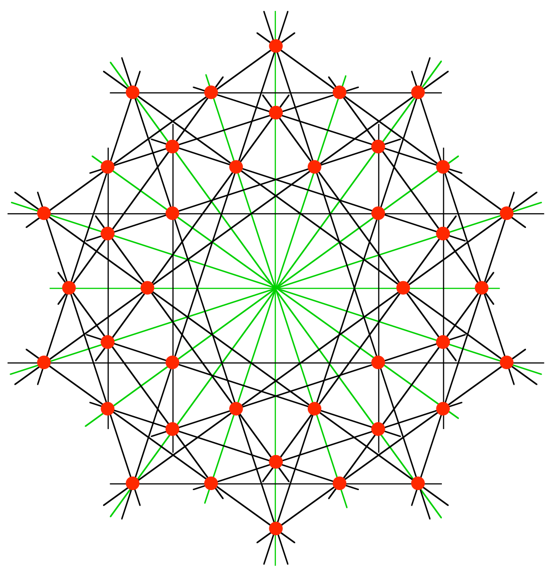


Figure 4.1.2. Deleting the ten lines (green) through the center yields the 4-astral configuration 10(#4,3,2,3,1,2,1,2). With these lines it is a (40_5, 50_4) configuration. Adding ten points at infinity, in the directions of the ten green lines, results in a (50_5) configuration that is 5-astral in the extended Euclidean plane.

The smallest 5-configuration discovered so far is the (48_5), found by L. Berman and shown in Figure 4.1.4. It has cyclic symmetry c_{12}; moreover, it is 4-astral in the Euclidean plane.

As mentioned above, the configuration (60_5) illustrated in Figure 4.1.1 has the advantage of being *astral* — but only in the *extended Euclidean* plane E^{2+}. One of the long-standing conjectures (see [G46], [B6]) is:

Conjecture 4.1.1. There are no 5-configurations 3-astral in the *Euclidean* plane E^2.

The existence of certain types of astral 5-configurations in the Euclidean plane has been ruled out in the recent paper [B11], but the more general question is still open.

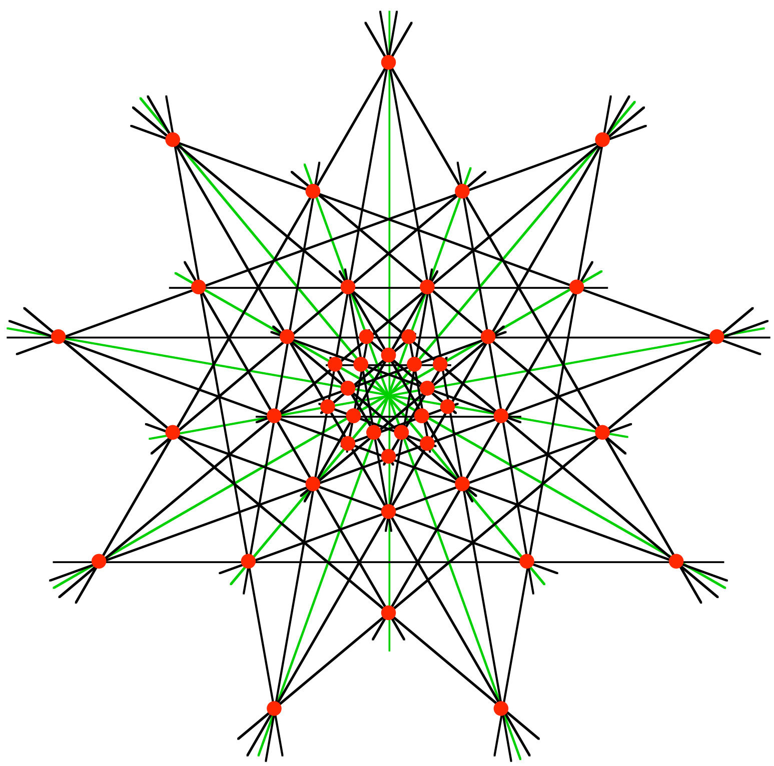


Figure 4.1.3. The addition of 9 diameters (green) to the 5-astral configuration 9#(3,4;1,3;2,3;4,1;3,2) together with the inclusion of 9 points at infinity in the direction of quintuplets of parallel lines, yields a 6-astral (54_5) configuration.

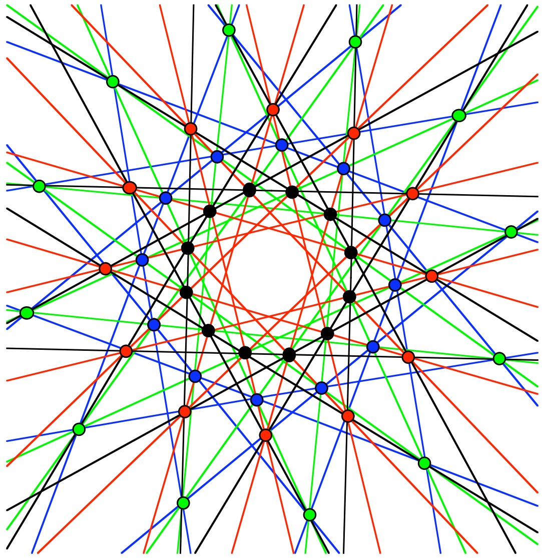


Figure 4.1.4. The smallest 5-configuration known is this 4-astral (48_5). (L. Berman, private communication)

One of the basic differences in the knowledge about 5-configurations compared to 3- and 4-configurations is our ignorance whether geometric configurations (n_5) exist for **all** n that are greater than some fixed bound. On the other hand, a similarity appears to exist: Among the known 5-configurations, there are topological ones that are smaller than the smallest known geometric configuration. One of several topological (42_5) configurations is shown in Figure 4.1.5. This is to be compared with the result mentioned in the proof of Theorem 3.2.1 to the effect that any topological (n_5) must satisfy $n \geq 25$. Although the gap from 25 to 42 is still large, it is not unexpected: There has been no investigation of 5-configurations — topological or geometric — till very recently, and no systematic approaches have been developed so far.

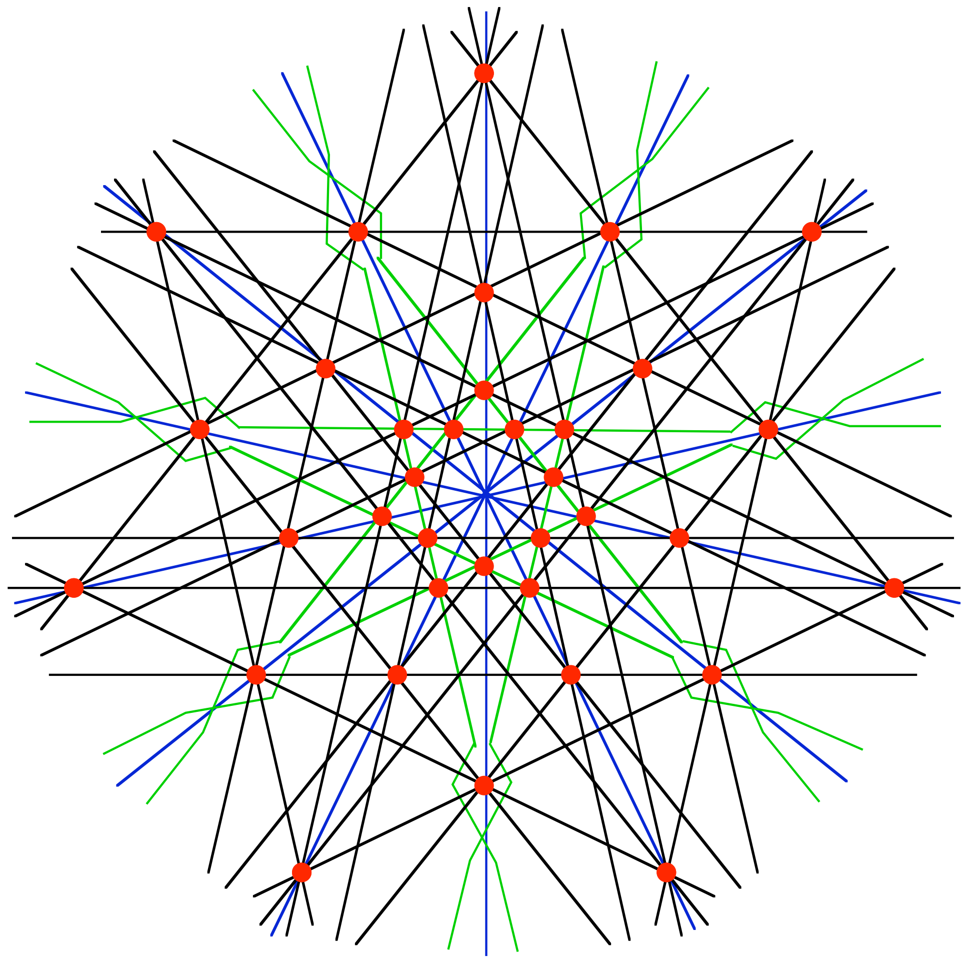
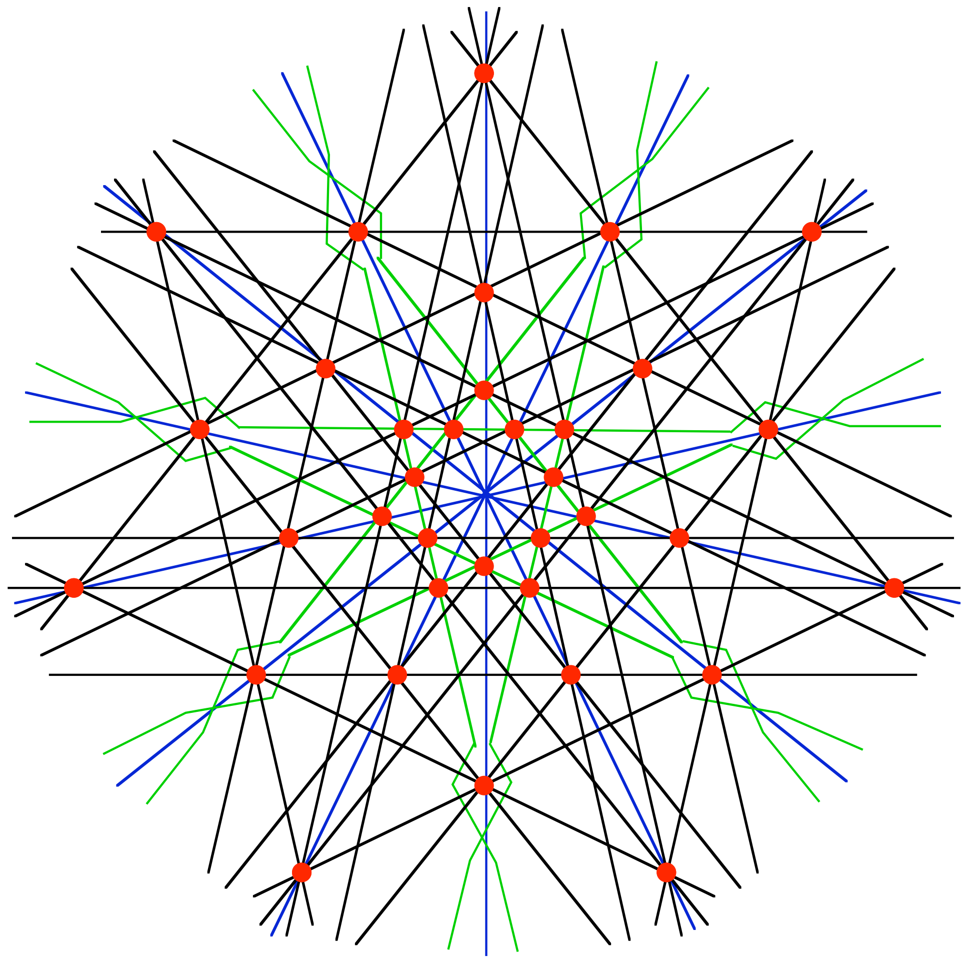


Figure 4.1.5. The geometric configuration 7#(2,1;2,1;3,2;1,2;1,3) has unintended incidences, and is just a prefiguration. If these incidences are avoided by using pseudolines we obtain a topological (35_4) configuration formed by the black lines and green pseudolines. Adding the seven blue lines yields a $(35_5, 42_4)$ configuration, and adding also the seven points at infinity (in the directions of the quintuplets of lines/pseudolines) results in a topological (42_5) configuration.



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Exercises and problems 4.1.

1. Determine the 4-configurations that can be turned into 5-configurations by adding lines, and points at infinity. It seems that all 4-astral 4-configurations can be used, admitting duplication if necessary. Are there any others?

2. The configuration in Figure 4.1.1 was constructed in an obvious way from two copies of the astral (24_4) configuration. Can this method be applied to all astral 4-configurations?

3. Are there any 4-astral 5-configurations in the *Euclidean* plane that have dihedral symmetry?

4. Decide whether any of the configurations in Figures 4.1.1 to 4.1.3 is selfpolar.

5. Decide whether there are geometric (n_5) configurations for any $n < 48$.

6. Decide whether there are topological (n_5) configurations for any $n < 42$.

7. Find a useful and convenient way of encoding symmetric 5-configurations.

8. Show that the 4-astral configuration 10#(4,3;1,2;3,4;2,1) can be used to construct a configuration (50_5). Determine all 3-astral configurations (40_4) that can be used for that purpose.

9. The constructions we have seen can be generalized. Determine criteria on 4-astral configurations $((4m)_4)$ that make it possible to obtain configurations $((5m)_5)$. Similarly, for $((5m)_4)$ configurations to yield $((6m)_5)$ configurations.