3.8 k-ASTRAL CONFIGURATIONS FOR k ≥ 4

As in the preceding three sections, configurations with 4 or more orbits of points that are known in greatest detail are those k-astral ones in which a dihedral symmetry group acts transitively on the points (and lines) of k different orbits, and each orbit has the same number of points. We shall discuss these first. Although the definitions are completely analogous to the ones in previous cases, striking differences in properties led us to separate the present case from the cases of 2- and 3-astral configurations. The main change is in the possibility of various unintended incidences not encountered earlier; hence it is clear that in a number of cases we can speak only of representations and not of realizations. But before we get to that, let us review the definitions.

A **k-astral** 4-configuration (n_k), with n = k·m, is a configuration with a dihedral symmetry group d_m that operates transitively on each of k orbits of points situated at vertices of a regular m-gon, and k orbits of lines, provided all orbits have the same number of elements and each element is incident with two elements of the other kind from each of two orbits. As detailed in Section 3.5, we can attach to each k-astral configuration a well-determined set of mutually equivalent symbols, derived from the consideration of the characteristic paths possible in the configuration.

This is illustrated in Figure 3.8.1, where a characteristic path starts at P_0, and goes on to P_1, and other points that are not labeled to avoid clutter. Since P_1 has symbol (2 // 1), and the following points of the characteristic path shown have symbols (5 // 3), (4 // 5), (1 // 2) and (3 // 4), while the orbits have size 11, — the symbol of this configuration is 11#(2,1;5,3;4,5;1,2;3,4). Hence it is a 5-astral configuration.

A similar procedure leads in the case of the symbol 9#(2,1;4,2;1,3;2,3;1,3) to the diagram shown in Figure 3.8.2(a). Here we came face to face with a serious problem: Our graphics, which frequently show only (slightly elongated) segments that are necessary to connect all points that are incident according to the symbol, are misleading. Configurations consist of *lines*, not segments, — and if the segments we used are extended to the rim of the diagram, additional incidences become evident; thus, we do not have a

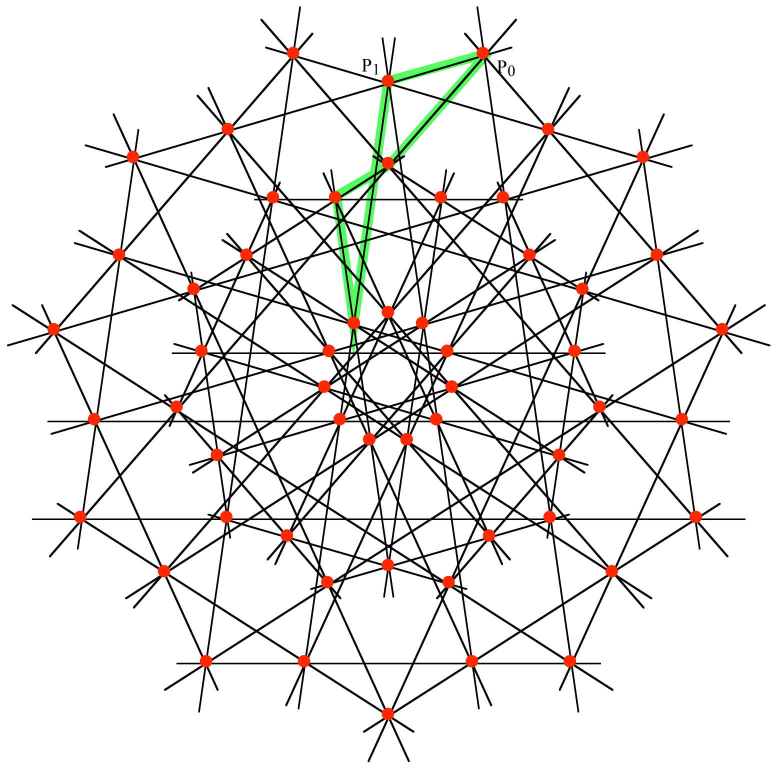
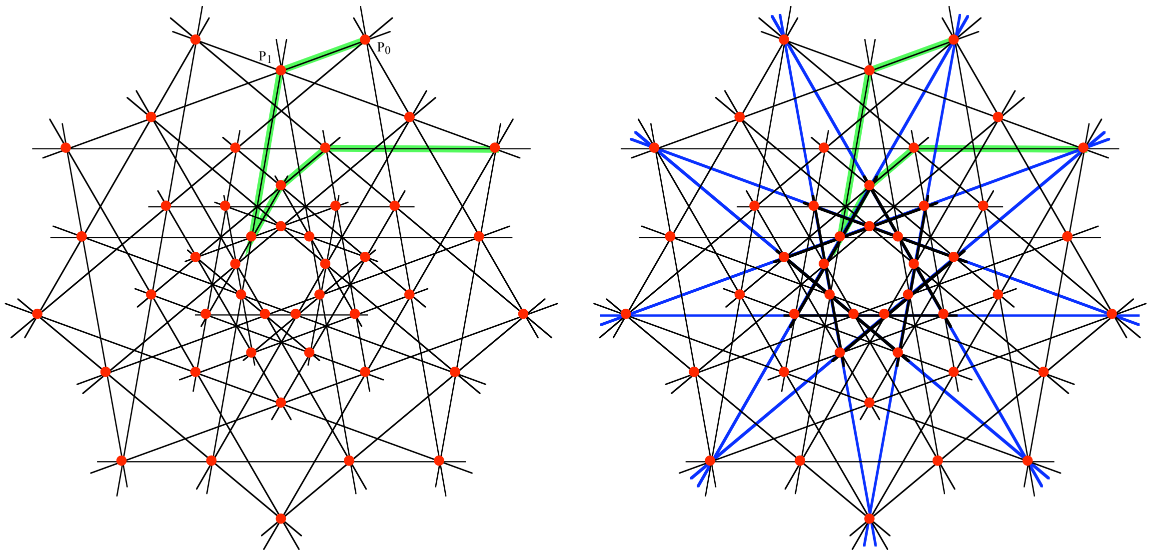


Figure 3.8.1. A 5-astral configuration (55_4) with symbol 11#(2,1;5,3;4,5;1,2;3,4) and symmetry group d_{11}.

configuration at all. Instead, we have a prefiguration, that can be interpreted as a *representation* of the abstract configuration 9#(2,1;4,2;1,3;2,3;1,3).

As mentioned already in Section 3.7, the explanation of the problem is that the characteristic path essentially crosses itself at a configuration point. This is detectable from the symbol of the configuration, and leads to the following condition we repeat here from Section 3.7 in a slight reformulation; the condition was signaled by Boben and Pisanski in [B20]:

(A8) The symbols of a k-astral configuration m#(s_1,t_1;s_2,t_2; ... ;s_k,t_k) should not contain a string of even length such that changing at most one of the ends of the string results in a valid symbol for an h-astral configuration with the same m and with h < k.



(a) (b)

Figure 3.8.2. Problems with realization of 9#(2,1;4,2;1,3;2,3;1,3).

If condition (A8) is not fulfilled, the symbol m#(s_1,t_1;s_2,t_2; ... ;s_k,t_k) encodes for a *representation* by a prefiguration, and not for a *realization* by a configuration. In the example of Figure 3.8.2, the symbol can be written in the equivalent form 9#(1,3;2,1;4,2;1,3;2,3). Then the string 3,2,1,4,2,1 can be replaced by 4,2,1,4,2,1, which leads to the trivial 3-astral configuration 9#(4,2;1,4;2,1).

The change in (A8) consists in the words "at most", which really mean that the string itself should be usable in a configuration symbol. This could not have happened with 3-astral configurations, but can happen in the situation considered here.

As an example we show in Figure 3.8.3 the result of drawing the configuration that corresponds to the symbol 12#(5,4;1,4;1,4;5,4). The expected (48_4) configuration did not come about. Instead, we obtained a prefiguration that looks as having three orbits of points and lines but points in one orbit are incident with six lines while lines of one orbit are incident with six points.

The explanation for the (mis)behavior of the symbols in these cases is the rather obvious failure of (A8): The characteristic path returns to one orbit three times; in other

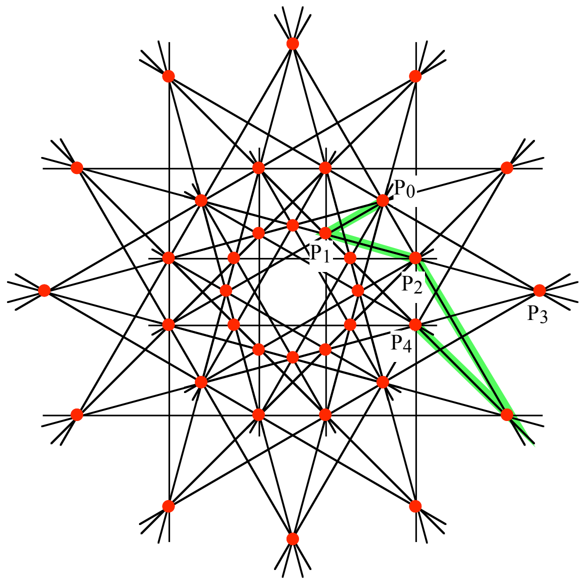


Figure 3.8.3. Given the symbol 12#(5,4;1,4;1,4;5,4) for a 4-configuration (48_4), the software produced this result which has three orbits of points and three of lines; it would be a (36_4) configuration, if it were not for the middle orbit of points: Each appears to be on *six* lines! In fact, these are two coinciding points of the configuration, and the lines with span 5 also represent two lines of the configuration.

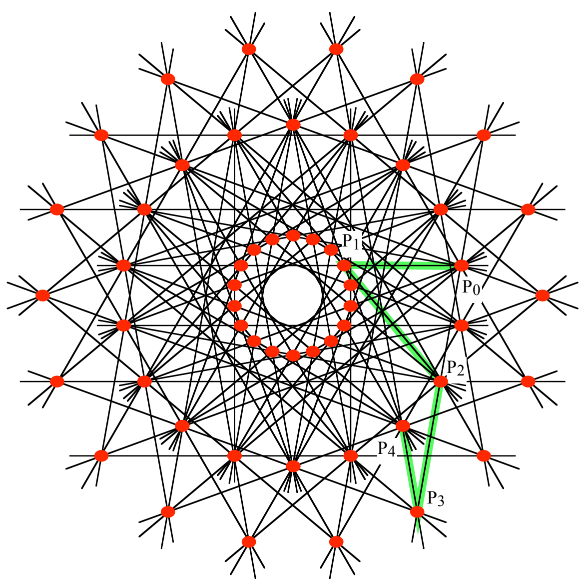


Figure 3.8.4. For the symbol 18#(8,6;1,7;2,5;7,6) we obtain the prefigurations shown. It does have four orbits of lines, each incident with two points from each of two point orbits; but there appear to be only three point orbits, one of which consists of points incident with eight lines. These actually represent pairs of coinciding points of th configuration.

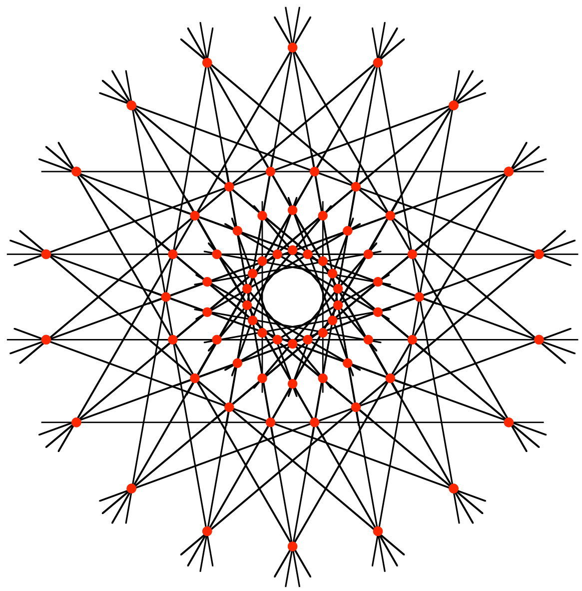


Figure 3.8.5. The symbol 18#(8,6;7,5;2,7;1,6) leads to a prefiguration (polar to the one in Figure 3.8.4) in which each line of one orbit is incident with eight points (in four different orbits).

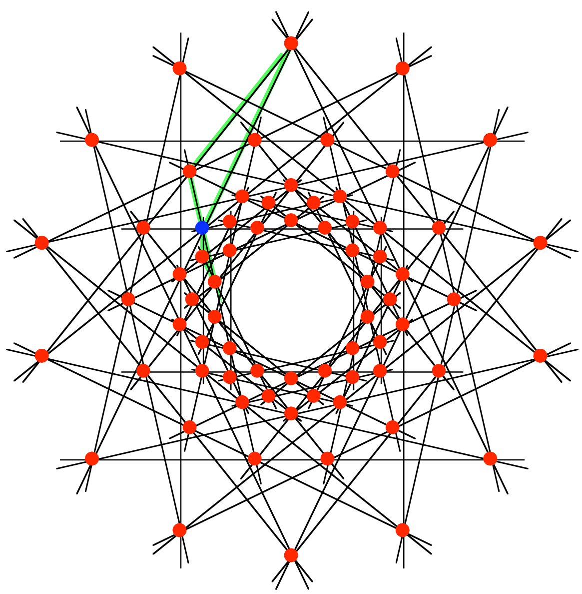


Figure 3.8.6. The characteristic path of the prefiguration corresponding to the 5-astral symbol 14#(5,1;3,2;4,3;2,5;1,4) contains the string 3,2,4,3,2,5; replacing its last entry by 4 we get the symbol of the 3-astral trivial configuration 14#(3,2;4,3;2,4). The blue point is a vertex of the characteristic path, but is also in the relative interior of a different segment of this path; hence an orbit of points that are on six lines each. The line of that segment is clearly incident with six points, as are all lines in its orbit.

words, a proper string of the symbol already codes for a 4-configuration, and so does the remaining part, hence there are six or more lines through each point of the appropriate orbit. In fact, the points of the middle orbit are doubled-up, and so are the span-5 lines. Other examples are given in Figures 3.8.4 and 3.8.5.

The example in Figure 3.8.6 shows that if the relative interior of a segment of the characteristic path contains the endpoint of another segment, unexpected incidences occur as well.

There is one more set of circumstances in which unexpected incidences of a different kind occur; it was also signaled by Boben and Pisanski in [B20]. It is illustrated by Figure 3.8.7, in which points of one orbit are on five lines while lines of one orbit contain five points. To avoid such incidences, the following condition is imposed by Boben and Pisanski beyond the ones we already require:

(A9) The symbol of a k-astral configuration $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ should not contain a string of odd length, such as $s_i,t_i;s_{i+1},t_{i+2}; \ldots ;s_j$, such that

$s_i + t_i + s_{i+1} + t_{i+2} + \ldots + s_j$ is an even integer and

$$\prod_{i\leq g\leq j} \cos(s_g\cdot\pi/m) = \prod_{i\leq g\leq j-1} \cos(t_g\cdot\pi/m) .$$

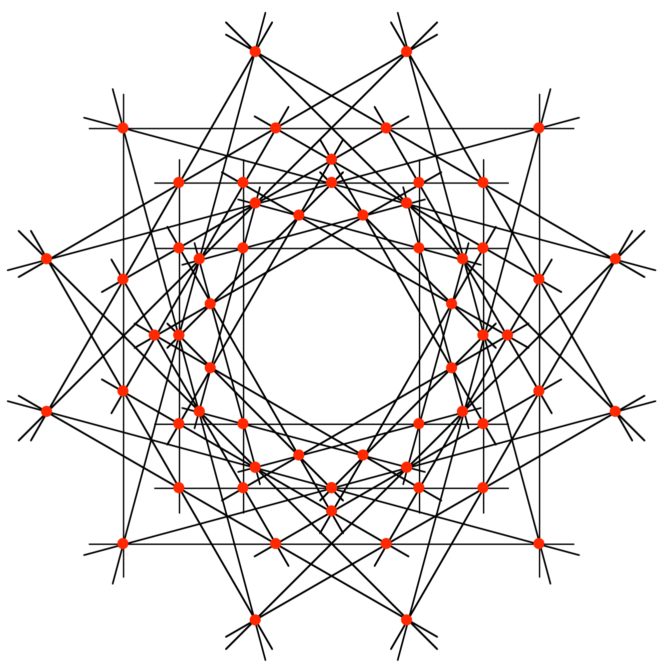


Figure 3.8.7. 12#(3,2,3,4,2,3,1,3,4,1) is not a 4-configuration.

Clearly, the 5-astral configuration 12#(3,2,3,4,2,3,1,3,4,1) in Figure 3.8.7 violates this condition: The sum of the string of the first five entries is 14, and

$$\cos \pi/4 \cos \pi/4 \cos \pi/6 = \cos \pi/6 \cos \pi/3.$$

Hence the unintended incidences.

Exercises and problems 3.8

1. The symbol of the prefiguration in Figure 3.8.7 contains many pairs of equal entries. Explain why canceling any such pair would not yield an example violating condition (A9).

2. The string s_1,t_1,s_2 with $m = 12$ is in a sense the only *known* source of examples violating condition (A9). By this is meant that one can obviously add the same numbers to the even and odd position (2 was added in the example of Figure 3.8.7), and one can use multiples of the string with the appropriate multiples of m. Decide whether there are any essentially different strings.

3. Prove that no 4-astral configuration can violate condition (A9).

4. The symbol that yielded the example in Figure 3.8.2 belongs to the cohort m#{{4,2,2,1,1},{3,3,3,2,1}} with $m = 9$, while the one in Figure 3.8.1 corresponds to $m = 11$. Why are the results different in the two cases?

5. Does the cohort 9#{{4,2,2,1,1},{3,3,3,2,1}} contain any geometrically realizable configurations?

6. List all the configuration symbols for 4-astral configurations $\{28_4\}$.

7. Draw the (potential) configuration 7#(3,2;1,3;1,3;2,1;3,1) and describe what happens.

8. Find some systematic families for 4-astral configurations, other than the ones that arise from an h-astral configuration with $h < k$ by insertion of matched pairs.