3.7 3-ASTRAL 4-CONFIGURATIONS

The 3-astral 4-configurations have a lot in common with the 2-astral configurations we studied in Section 3.6, but they also have many properties and peculiarities that are not present in the earlier case. This is the main reason for treating them in a separate section.

It seems to me that the 3-astral configurations are arguably the most interesting type of configurations. The reason for this assessment is that they are more general in the opportunities for investigation than the 2-astral 4-configurations considered in the preceding section, but are still experimentally quite accessible. As we shall show, there are many open problems that seem very attractive, as well as tractable with an appropriate investment of effort. Naturally, k-astral configurations with $k \geq 4$ have their own attraction and appeal, but with increasing k they are harder to investigate and, in any case, much less is known about them.

The first graphic presentation in a published paper of any 4-configuration[1] was that of a 3-astral (21_4) in [G50]; here we show it in Figure 3.7.1. As with 2-astral configurations, 3-astral ones will be illustrated in various section; several are shown in Section 5.9.

The notation we use for the k-astral configurations is the one introduced in Section 3.5, based on characteristic paths. In the case of 3-astral configurations there are, in general, 6 different characteristic paths, leading to distinct symbols. We preferentially choose the symbol that is lexicographically the highest.

For 3-astral configurations our approach is completely analogous to the treatment of 2-astral configurations in Section 3.6. It would be nice if at this stage we could formulate a theorem analogous to Theorem 3.6.1, and give necessary and sufficient conditions for symbols corresponding to geometric 3-astral 4-configurations. However, we have

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1 Kárteszi [K5] in 1986 and Zeitler [Z8] in 1987 came very close to finding such 4-configurations. In the diagrams they show one only has to delete some lines, and make all copies of one of the shown lines, to get a 3-astral 4-configuration, with m = 10 in the former and m = 12 in the other.

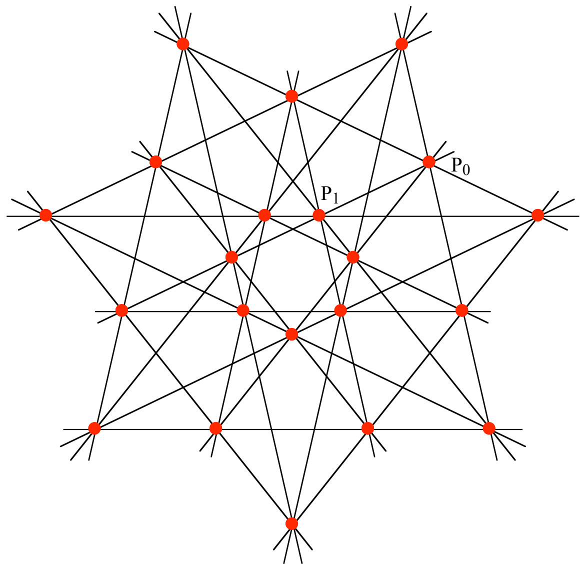

Figure 3.7.1. The 3-astral 4-configuration (21_4). This configuration is 3-astral with symbol 7#(3,2;1,3;2,1) obtained from the characteristic path that starts at P_0 and has as its next point P_1. This configuration belongs to the "trivial" type.

only partial information; the most important criterion is the condition (A7) from Section 3.5, the analog of the trigonometric condition in Theorem 3.6.1:

Theorem 3.7.1. If $m\#(s_1,t_1;s_2,t_2;s_3,t_3)$ is the symbol of a geometric 3-astral 4-configuration then

(*) $$\cos s_1\pi/m \cdot \cos s_2\pi/m \cdot \cos s_3\pi/m = \cos t_1\pi/m \cdot \cos t_2\pi/m \cdot \cos t_3\pi/m.$$

This is an expression of the fact that each characteristic path has its endpoint in the same orbit as its starting point. Conversely, if a symbol satisfies (*) and the natural conditions listed in Section 3.5 then *in general* there exists a geometric 3-astral 4-configuration with the symbol in question.

The natural conditions just mentioned are:

(**) No entry is equal to either of the two adjacent entries, the first and last considered as adjacent;

(***) Each entry is smaller than m/2.

(****) The sum of all s_j and t_j entries in the symbol is even.

There are two deep deficiencies in this theorem. One unsatisfactory aspect of Theorem 3.7.1 (and of the analogous statements one can make for $k \geq 4$) is that we do not have any analogue of Myerson's Theorem 3.6.2, hence we cannot devise a list of all the configurations in question. As is stated in [M21], Myerson's methods could probably lead to a complete, explicit list of solutions of (*), but this appears to be a momentous task — a task that has not been carried out. This is the first big problem concerning 3-astral 4-configurations.

The other problem is euphemistically covered by the italicized words "in general". We shall discuss later in the section the known and the unknown results in this direction.

For the presentation of *known* solutions of (*) satisfying all the necessary conditions it is convenient to distinguish three kinds of symbols $m\#(s_1,t_1;s_2,t_2;s_3,t_3)$ or the corresponding cohorts $m\#\{\{s_1,s_2,s_3\},\{t_1,t_2,t_3\}\}$, which we shall call "trivial", "systematic", and "sporadic". The terminology was introduced in Section 3.5, and here we only briefly remind the reader of the meaning of these terms.

- In accordance with this terminology, **trivial** symbols (and 3-astral configurations) have the form m#(b,c;d,b;c,d), where b, c, d are different positive integers, each less than m/2. Since the terms on the two sides cancel each other without any calculations and the other conditions are automatically satisfied, the label "trivial" seems appropriate — not in any derogatory sense but as describing the mathematically simplest case. In other words, trivial are those astral configurations for which the cohort symbol m#{s,t} is of the special form m#{s,s}. Figure 3.7.1 shown an example of a trivial 3-astral configuration. From the general properties of equivalent symbols and symbols of dual configuration discussed in Section 3.5 it follows that all trivial 3-astral configurations are selfdual; the same applies to all trivial k-astral configurations with odd k. Indeed, m#(b,c,d,b,c,d) has as dual m#(c,d,b,c,d,b), which is the same as the original; the argument is analogous for other odd k. Obviously, the polar of a trivial configuration is itself trivial.

- **Systematic** symbols are those that contain infinite families for which the validity can be verified by *formal* trigonometric calculations, without the need to determine *val-*

ues of the trigonometric functions that depend on specific parameters. At present, four such families m#{s,t} of 3-astral 4-configurations are known, mostly through unpublished work of L. Berman.

(1) m = 2q, s = {q-p,q-2r,p}, t = {q-2p,q-r,r}

(2) m = 3q, s = {q+p,q-p,p}, t = {q,q,3p}

(3) m = 6q, s = {3q-p,r,p}, t = {3q-2p,2q,r}

(4) m = 10q, s = {5q-p,2p,p}, t = {|5q-4p|,4q,2q}.

Here p, q and r are any positive integers, and the possibilities have to be understood in the sense of cohorts, that is, all permutations within s and within t, as well as interchanging s and t, are allowed provided conditions (**) and (***) are satisfied; condition (****) is fulfilled automatically. If all the entries are distinct the cohort contains 12 distinct configurations; equality of some entries reduces this number.

For example, in family (4) with m = 20, q = 2. For p = 1 we have s = {9,2,1} and t = {8,6,4}; this leads to a total of 12 distinct symbols: 20#(9,8;2,6;1,4), 20#(9,8;2,4;1,6), 20#(9,6;2,8;1,4), 20#(9,6;2,4;1,8), 20#(9,4;2,8;1,6), 20#(9,4;2,6;1,8), and six more in which the positions of 1 and 2 are interchanged. Switching the two sets of parameters yields additional 12 symbols, but no new configurations, since these symbols are equivalent to the earlier dozen. For p = 3 we have s = {7,6,3} and t= {8,4,2} leading again to 12 different symbols. For p = 2 and p = 4 we get trivial symbols only, while $p \geq 5$ exceeds the bound in (***). On the other hand, the cohort of 9#{{4,2,1},{3,3,3}} consists of just two configurations, shown in Figure 3.7.2. As mentioned above, in some cases the resulting symbols become trivial.

The verification that the above symbols of the four families satisfy condition (*) is quite straightforward. We illustrate this only for family (2), for which condition (*) reduces to the verification of

$$\cos(q+p)\pi/m \cdot \cos(q-p)\pi/m \cdot \cos p\pi/m = (\cos q\pi/m)^2 \cdot \cos 3p\pi/m.$$

Since q = m/3, the value of $\cos q\pi/m = \cos \pi/3 = \frac{1}{2}$, and standard trigonometric identities yield

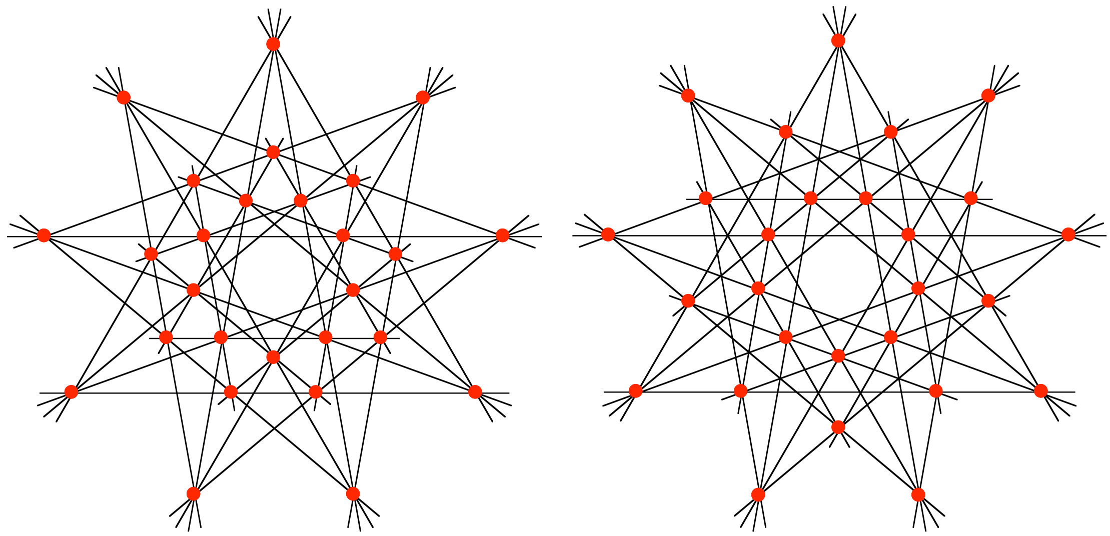

Figure 3.7.2. The only two distinct members in the cohort 9#{{4,2,1},{3,3,3}} of family (2) for q = 3, p = 1. (a) 9#(4,3,2,3,1,3); (b) 9#(4,3,1,3,2,3).

$2 \cos (q+p)\pi/m \cdot (\cos q\pi/m + \cos (q-2p)\pi/m) = \cos 3p\pi/m$, so

$\cos (q+p)\pi/m + 2 \cos (q+p)\pi/m \cdot \cos (q-2p)\pi/m = \cos 3p\pi/m$,

$\cos (p\pi/m + \pi/3) + \cos (2q-p)\pi/m + \cos 3p\pi/m = \cos 3p\pi/m$, and finally

$\cos (p\pi/m + \pi/3) + \cos (-p\pi/m +2\pi/3) = 0$,

which is obviously true.

Similar calculations validate the other families of symbols.

- **Sporadic** symbols and configurations are those that do not belong to any of these two families. For example, 18#(5,4;1,3;1,2) shown in Figure 3.7.3 is sporadic — *at least for the time being*. The reason for this qualification is that although the symbol is neither trivial nor belongs to one of the four families in (ii), it may well be part of a still not discovered infinite (systematic) family.

In Table 3.7.1 we give a list of cohorts of the sporadic 3-astral configurations (n_4) with $n \leq 108$. It was obtained by numerically solving equation (*), and eliminating duplicates and symbols that correspond to trivial or systematic 3-astral configuration. An

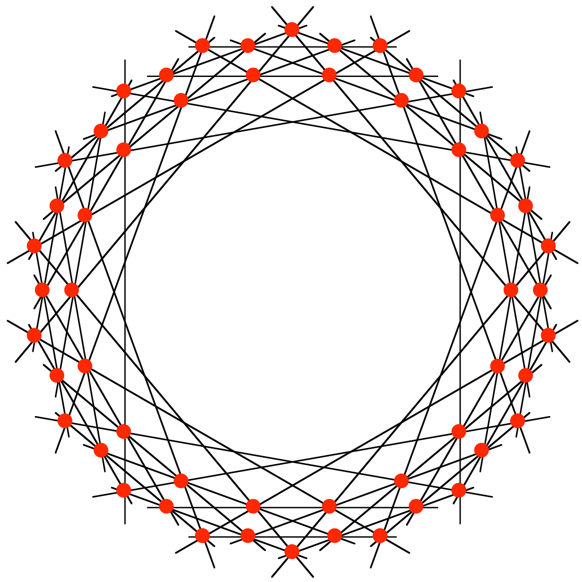

Figure 3.7.3. The sporadic configuration 18#(5,4;1,3;1,2).

unexpected result of these computations is that such sporadic configurations exist only for n that are multiples of 12.

One additional comment concerning Table 3.7.1. Some of the cohort symbols contain the same symbol in both parts. This implies that the crucial relation (*) will be satisfied even if the symbol is deleted from both parts. In such a case the **reduced** cohort symbol belongs to a 2-astral 4-configuration.This brings us to other open problems:

- Do there exist any other systematic families besides the ones in (ii) above?
- Is the list of connected sporadic 3-astral configurations finite?

It is worth noting that there is no known visible cue in a given 3-astral configuration whether it is trivial, systematic, or sporadic. It takes working out its symbol and looking at the criteria in order to decide where it belongs.

We turn now to the second deficiency in Theorem 3.7.1. It parallels the problems with the Steinitz theorem on 3-configurations encountered in Chapter 2, and did not arise for 2-astral 4-configurations.

m = 18

{5, 1, 1}, {4, 3, 2}
{6, 3, 2}, {5, 5, 1}
{7, 4, 2}, {6, 6, 1}
{8, 2, 1}, {6, 6, 5}a
{8, 4, 3}, {7, 7, 1}
{8, 6, 1}, {8, 5, 4}*bA
{8, 7, 2}, {8, 6, 5}*a

{6, 2, 1}, {5, 4, 2}*
{7, 1, 1}, {6, 4, 3}
{7, 6, 1}, {7, 5, 4}*A
{8, 3, 2}, {7, 5, 5}
{8, 5, 4}, {7, 6, 6}b
{8, 6, 3}, {7, 7, 5}

m = 24

{6, 2, 1}, {5, 3, 3}
{8, 3, 2}, {7, 5, 3}*A
{8, 6, 2}, {7, 6, 5}*A
{9, 8, 2}, {9, 7, 5}*A
{10, 5, 3}, {9, 8, 1}
{10, 7, 5}, {8, 8, 8}
{10, 9, 3}, {10, 8, 6}*B
{11, 3, 2}, {9, 8, 7}
{11, 5, 3}, {10, 9, 2}
{11, 6, 5}, {9, 9, 8}
{11, 8, 3}, {10, 9, 7}
{11, 10, 2}, {11, 8, 8}*

{8, 2, 1}, {7, 5, 1}*A
{8, 3, 3}, {7, 6, 1}
{9, 2, 1}, {8, 5, 3}
{10, 3, 2}, {9, 7, 1}
{10, 6, 5}, {9, 9, 1}
{10, 8, 2}, {10, 7, 5}*A
{11, 2, 1}, {8, 8, 8}
{11, 3, 3}, {10, 7, 6}
{11, 6, 2}, {9, 9, 7}
{11, 8, 2}, {11, 7, 5}*A
{11, 9, 3}, {11, 8, 6}*B

m = 30

{7, 2, 1}, {6, 4, 2}*A
{7, 5, 1}, {6, 5, 4}*A
{8, 3, 2}, {6, 6, 3}*B
{8, 4, 2}, {7, 6, 1}a
{8, 7, 1}, {8, 6, 4}*A
{9, 1, 1}, {8, 4, 3}
{9, 4, 2}, {7, 7, 3}
{9, 8, 2}, {9, 6, 6}*cB
{10, 2, 1}, {7, 6, 6}b
{10, 3, 1}, {8, 7, 3}*dC
{10, 3, 2}, {9, 6, 2}*D
{10, 4, 2}, {7, 7, 6}
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{10, 6, 3}, {9, 8, 2}c
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{10, 8, 2}, {10, 6, 6}*B
{10, 9, 1}, {9, 8, 7}*C
{11, 2, 1}, {10, 6, 2}*E
{11, 3, 1}, {9, 8, 2}c
{11, 4, 1}, {10, 6, 4}*E

{7, 3, 1}, {6, 4, 3}*A
{8, 2, 1}, {6, 6, 1}*B
{8, 4, 2}, {6, 6, 4}*aB
{8, 5, 2}, {6, 6, 5}*
{8, 7, 2}, {7, 6, 6}*bB
{9, 2, 1}, {7, 6, 3}
{9, 7, 1}, {9, 6, 4}*eA
{10, 1, 1}, {8, 6, 4}
{10, 2, 1}, {8, 7, 2}*bC
{10, 3, 1}, {9, 6, 1}*dD
{10, 4, 1}, {8, 7, 4}*C
{10, 4, 3}, {9, 6, 4}*eD
{10, 6, 1}, {8, 7, 6}*C
{10, 6, 4}, {8, 7, 7}
{10, 7, 3}, {9, 7, 6}*D
{10, 8, 3}, {9, 8, 6}*D
{11, 1, 1}, {8, 7, 6}
{11, 3, 1}, {9, 6, 6}c
{11, 3, 1}, {10, 6, 3}*cE
{11, 5, 1}, {10, 6, 5}* E

{11, 6, 1}, {10, 8, 2}
{11, 7, 1}, {11, 6, 4}*fA
{11, 7, 6}, {10, 10, 2}j
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{11, 8, 7}, {10, 10, 6}
{11, 10, 1}, {11, 8, 7}*gC
{12, 1, 1}, {10, 8, 4}
{12, 2, 1}, {11, 6, 4}f
{12, 3, 2}, {11, 7, 3}*F
{12, 4, 2}, {11, 7, 4}*hF
{12, 6, 1}, {10, 8, 7}i
{12, 6, 2}, {10, 10, 2}*jG
{12, 6, 3}, {10, 10, 3}*kG
{12, 6, 4}, {10, 10, 4}*lG
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{12, 7, 1}, {12, 6, 4}*lA
{12, 7, 6}, {11, 10, 4}m
{12, 8, 2}, {11, 8, 7}*gF
{12, 8, 6}, {10, 10, 8}*G
{12, 9, 6}, {10, 10, 9}*nG
{12, 10, 2}, {11, 10, 7}oF
{12, 10, 3}, {12, 9, 6}*nD
{12, 11, 1}, {12, 10, 6}*pE
{12, 11, 6}, {11, 10, 10}*G
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{13, 3, 1}, {10, 9, 8}s
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{13, 7, 2}, {12, 10, 2}*K
{13, 7, 3}, {12, 10, 3}*nK
{13, 7, 5}, {12, 10, 5}*K
{13, 7, 6}, {12, 10, 6}*pK
{13, 7, 7}, {12, 11, 4}
{13, 8, 7}, {12, 10, 8}*vK
{13, 9, 6}, {13, 10, 3}*D
{13, 9, 7}, {12, 12, 3}w
{13, 10, 1}, {13, 8, 7}*vC
{13, 10, 3}, {13, 9, 6}*D
{13, 10, 6}, {12, 11, 8}y
{13, 11, 1}, {12, 11, 8}*yJ

{11, 7, 1}, {10, 7, 6}*fE
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{11, 8, 1}, {10, 8, 6}*E
{11, 8, 4}, {10, 10, 1}
{11, 9, 1}, {10, 9, 6}*E
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{12, 2, 1}, {11, 7, 1}*fF
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{12, 6, 2}, {11, 7, 6}*jF
{12, 6, 3}, {11, 9, 1}k
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{13, 11, 4}, {12, 10, 10}x
{13, 11, 4}, {13, 10, 7}*xH
{13, 12, 2}, {13, 11, 7}*zF
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{14, 8, 2}, {13, 11, 6}
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{14, 8, 6}, {13, 11, 8}*N
{14, 9, 4}, {12, 12, 9}*M
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{14, 11, 4},{14, 10, 7}*ffH
{14, 12, 2},{14, 11, 7}*F
{14, 12, 6},{13, 12, 11}*ggN
{14, 12, 8},{13, 13, 10}hh
{14, 13, 2},{14, 11, 10}*
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{14, 4, 3}, {12, 10, 9}w
{14, 4, 3}, {13, 9, 7}w
{14, 6, 1}, {12, 11, 8}y
{14, 6, 2}, {11, 11, 10}bb
{14, 6, 3}, {13, 11, 3}*N
{14, 6, 4}, {13, 10, 7}cc
{14, 6, 5}, {13, 11, 5}*N
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{14, 8, 4}, {13, 12, 1}
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{14, 10, 3}, {13, 11, 9}
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{14, 10, 6}, {13, 11, 10}*eeN
{14, 10, 7}, {12, 12, 11}ff
{14, 10, 10},{13, 12, 11}gg
{14, 11, 4}, {12, 12, 11}*ffM
{14, 12, 1}, {13, 13, 7}
{14, 12, 3}, {14, 10, 9}*L
{14, 12, 6}, {14, 10, 10}*ggG
{14, 13, 1}, {14, 12, 8}*hhJ
{14, 13, 4}, {13, 12, 12}*M

m = 36
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{12, 4, 2}, {11, 7, 2}*B
{12, 5, 2}, {10, 8, 5}*A
{12, 5, 4}, {11, 7, 5}*B
{12, 8, 4}, {11, 8, 7}*B
{12, 9, 4}, {11, 9, 7}*B
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{12, 5, 3}, {11, 8, 1}
{12, 7, 2}, {10, 8, 7}*A
{12, 9, 2}, {10, 9, 8}*A
{12, 10, 4}, {11, 10, 7}*B

{12, 11, 2}, {11, 10, 8}*A	{13, 4, 1}, {12, 7, 3}
{13, 10, 5}, {12, 12, 2}	{13, 12, 2}, {13, 10, 8}*A
{13, 12, 4}, {13, 11, 7}*B	{14, 2, 2}, {13, 6, 5}
{14, 7, 3}, {13, 10, 1}	{14, 11, 7}, {12, 12, 10}
{14, 12, 4}, {14, 11, 7}*B	{14, 13, 5}, {14, 12, 8}*
{15, 2, 1}, {14, 7, 5}	{15, 3, 2}, {13, 10, 5}
{15, 3, 2}, {14, 8, 4}	{15, 3, 3}, {13, 11, 1}
{15, 6, 3}, {14, 10, 2}	{15, 10, 3}, {14, 11, 7}
{15, 12, 2}, {15, 10, 8}*A	{15, 12, 4}, {15, 11, 7}*B
{15, 13, 5}, {15, 12, 8}*	{15, 14, 4}, {15, 12, 10}*
{16, 4, 2}, {15, 10, 3}	{16, 5, 4}, {15, 11, 1}
{16, 7, 5}, {15, 12, 1}	{16, 8, 4}, {15, 12, 3}
{16, 8, 7}, {15, 13, 1}	{16, 10, 8}, {15, 14, 3}
{16, 12, 4}, {16, 11, 7}*B	{16, 13, 5}, {16, 12, 8}*
{16, 15, 3}, {16, 12, 12}*C	{17, 2, 1}, {14, 12, 12}
{17, 2, 1}, {15, 14, 3}	{17, 3, 2}, {14, 13, 11}
{17, 5, 3}, {16, 11, 8}	{17, 5, 4}, {15, 12, 11}
{17, 6, 1}, {14, 14, 10}	{17, 7, 2}, {15, 13, 10}
{17, 7, 3}, {16, 13, 4}	{17, 7, 5}, {15, 15, 3}
{17, 8, 4}, {15, 13, 11}	{17, 8, 7}, {15, 13, 12}
{17, 10, 5}, {15, 14, 11}	{17, 12, 2}, {17, 10, 8}*A
{17, 12, 3}, {16, 13, 11}	{17, 12, 4}, {17, 11, 7}*B
{17, 13, 5}, {17, 12, 8}*	{17, 14, 4}, {17, 12, 10}*
{17, 15, 3}, {17, 12, 12}*C	{17, 16, 2}, {17, 14, 12}*

Table 3.7.1. A list of all sporadic 3-astral 4-configurations (n_4) with $n \leq 72$. In the cohort notation used, each entry corresponds to a cohort of configurations. Reducible cohorts are indicated by an asterisk following the symbol. Equal upper-case letters indicate that the configurations reduce to the same 2-astral configuration. In case of the symbols for m = 30, some have a common factor 2; however, they are not disconnected, since in each case the symbol corresponding to entries one-half of the ones given would violate condition (****). Equal lower-case letters attached to the symbols indicate that the symbols share one of the parentheses; while it is not clear what this commonality implies, it is signaled to ease possible investigations.

The problem is that in the discussions in Sections 3.5, 3.6, and (so far) in 3.7 we did not worry about possible *unintended incidences* of points and lines. However, such incidences may well happen, as is shown by the example in Figure 3.7.4. The reason is easily discerned from the symbol of the configuration, as first pointed out in [B20]. In the language of the characteristic paths this happens when a segment of the path (or the line it determines) passes through a point that is in the same orbit as the endpoint of another segment of the path, but is not itself a vertex of the path. This is illustrated in Figure 3.7.4(b), where the characteristic path starts at the red point of the middle orbit and goes towards the innermost orbit of points, but the second segment contains the blue point of the orbit of the starting point. That causes each point of this orbit to be on six lines. This description is easily translated in the language of the configuration symbols: As we step from one entry of the general symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ to the next, a consecutive string of entries needs to be changed in only its first or else its last terms in order to obtain a valid symbol for an h-astral configuration with h < k and the same m. In the 3-astral example in Figure 3.7.4, its symbol 12#(5,4;1,5;4,1) contains the string 5,4;1,5. If the last entry is changed to 4, the resulting symbol 12#(5,4;1,4) corresponds to the 2-astral configuration we have seen in Figure 3.6.2. Since the configuration in Figure 3.7.4 is selfdual, it is clear that there necessarily are lines that meet six of its points. We formulate this in the general case of k-astral configurations by the following requirement:

(A8) The symbols of a k-astral configuration $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ should not contain a string such that changing one of the ends of the string results in a valid symbol for an h-astral configuration with the same m and with h < k.

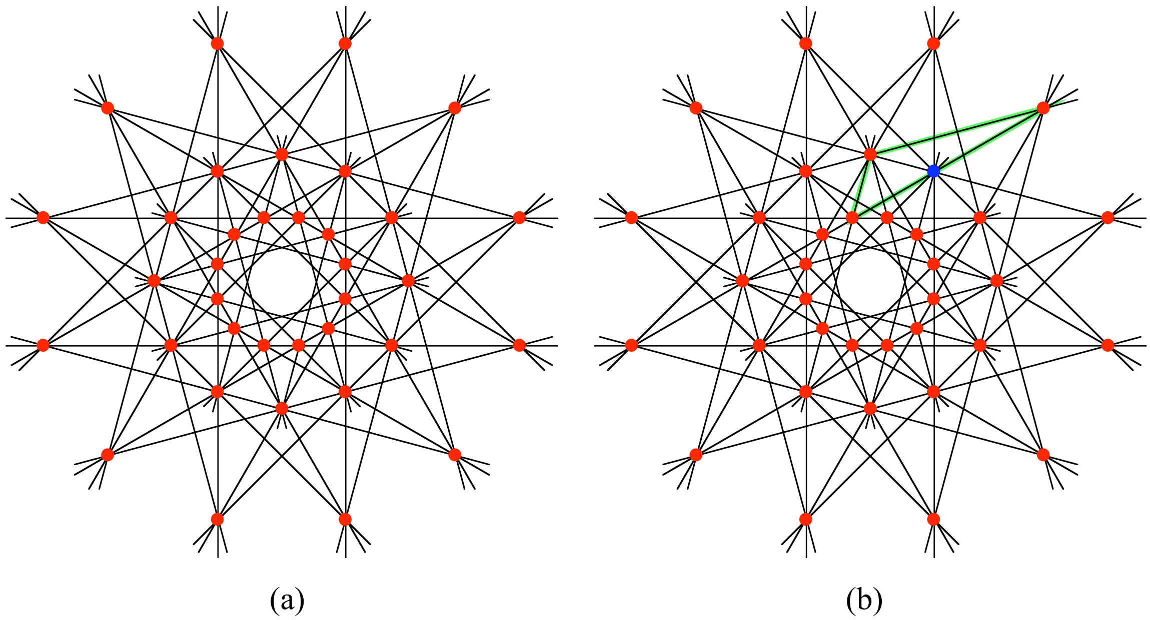


Figure 3.7.4. (a) The trivial 3-astral configuration 12#(5,4,1,5,4,1) is not a configuration at all — it is a prefiguration due to the presence of an orbit of points each incident with six lines, and an orbit of lines each incident with six points. (b) The explanation for this situation, as detailed in the text.

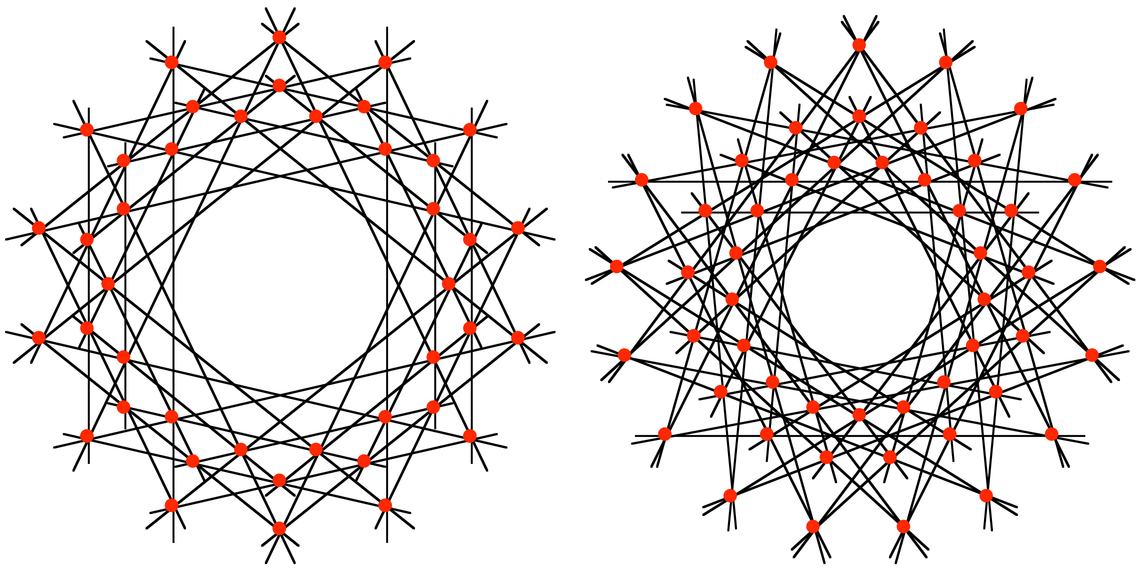
Exercises and problems 3.7

1. Find the symbols for the two configurations in Figure 3.7.5, and decide whether it is trivial, systematic or sporadic. Find all the other configurations that are in the same cohorts.

2. Find other examples of unintended incidences like the ones in Figure 3.7.4.

3. Find all systematic configurations with m = 20, and draw three of them.

4. The configuration in Figure 3.7.5a has its lines parallel in sets of six, three on each side of the center. Find a 3-astral 4-configuration in which the points appear in collinear sets of six.



(a) (b)

Figure 3.7.5. Identify these configurations.

5. Is there a trivial 3-astral configuration in which the points are in collinear sets of six?

6. Find the symbol of the configuration in Figure 3.7.6, and explain your findings.

7. Prove that the configuration (21_3) in Figure 3.7.1 is the only k-astral configuration (21_3) for any k.

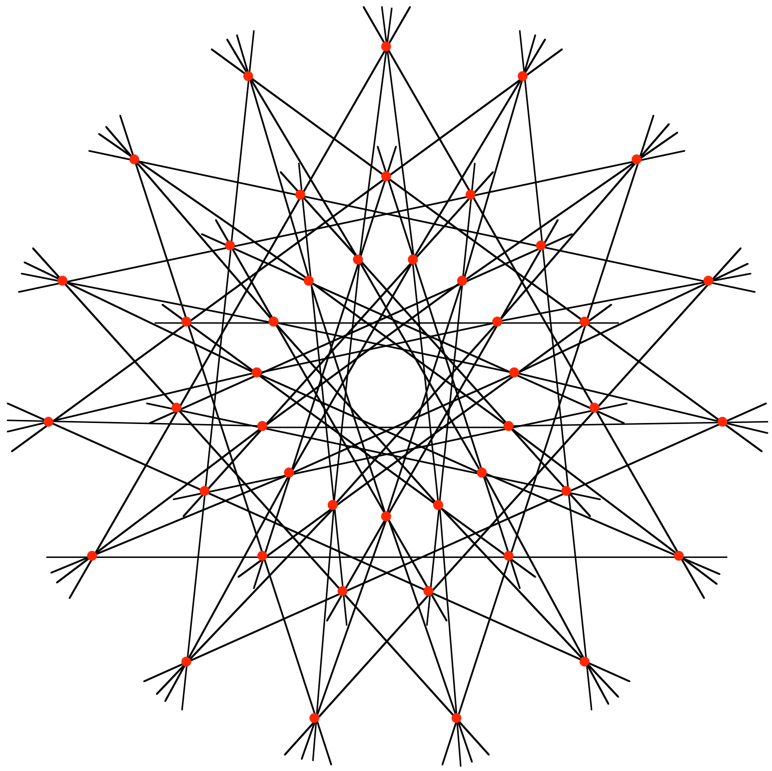


Figure 3.7.6. An interesting configuration.