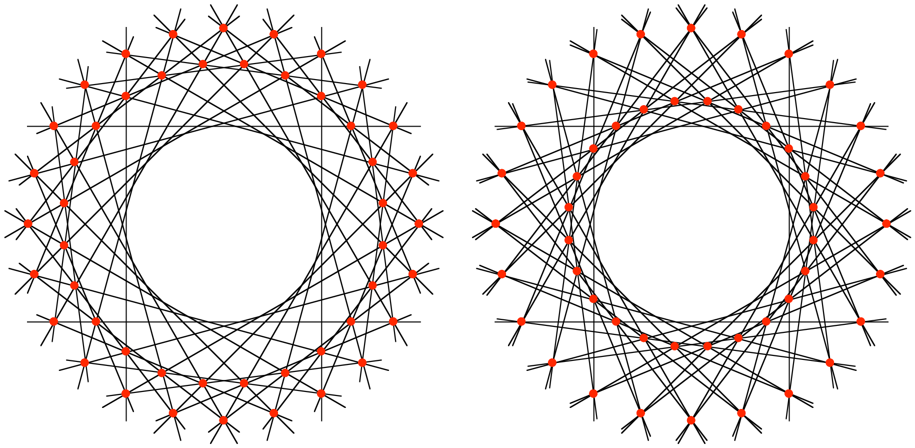
### 3.6 2-ASTRAL 4-CONFIGURATIONS

Following the terminology introduced in Section 3.5, a geometric 4-configuration (that is, an $(n_4)$ configuration for some integer $n$) is called **2-astral** provided there are precisely two orbits of points and two orbits of lines under the symmetry group of the configurations, and the other conditions spelled out in Section 3.5 are satisfied. Since $k = 2$ is the smallest value of $k$ possible in a 4-configuration, following the convention proposed in Section 1.5 we shall call such configurations **astral** for short. An astral 4-configuration cannot have points at infinity, since any line through such a point would have to have three points of a single orbit in the finite part of the plane. Hence we need to consider only what happen in the Euclidean plane.

The astral 4-configurations have been completely characterized. To present this characterization we need an appropriate notation; this was set up in Section 3.5. Here we shall present the list of these astral configurations (Theorem 3.6.1). Before giving the proof that our list is complete we have to digress into explanations of some of the detailed results about intersection of diagonals in regular polygons — a topic that has its own interesting and convoluted history. Finally, a proof of completeness of the list will be given; the first such proof is that of L. Berman [B3], [B4].

The notation for astral 4-configurations has evolved in several stages since the first publication on the topic in [G39]. The notation used here, introduced in Section 3.5, is the one that was found most suitable for the present purpose as well as for the generalization to k-astral 4-configurations that we shall consider in Section 3.7. The notation is explained by the example of a $(48_4)$ astral configuration shown in Figure 3.5.5. One of its symbols is 24#(8,7;2,5); the configuration belongs to the cohort 24#{{8,2},{7,5}}; this cohort contains only one other configuration, with symbol 24#(8,5;2,7). Both configurations are shown in Figure 3.6.1.

In the next two figures we show the smallest astral 4-configurations. The unique $(24_4)$ is shown in Figure 3.6.2, while the six configurations $(36_4)$ appear in Figure 3.6.3. Additional illustrations appear in several other sections, but in particular in Section 5.9.



(a) (b)

Figure 3.6.1. The only two 2-astral configurations ($48_4$) in the cohort 24#{{8,2},{7,5}}. (a) The configuration 24#(8,7;2,5). (b) The configuration 24#(8,5;2,7).

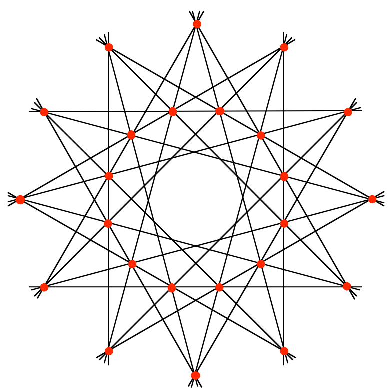
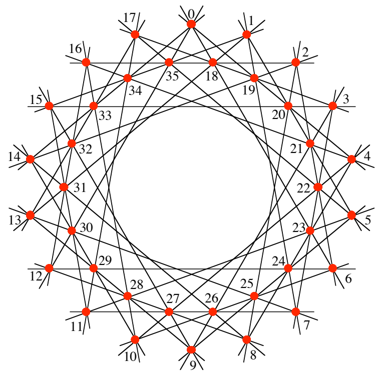
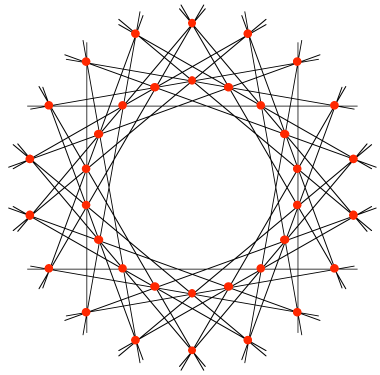


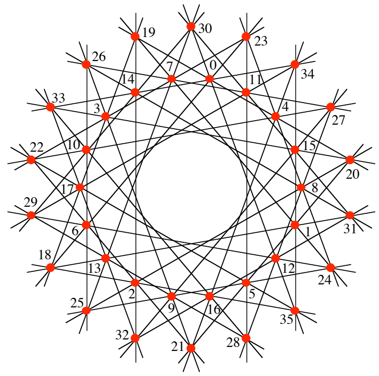
Figure 3.6.2. The smallest astral 4-configuration. It is a sporadic and selfdual ($24_4$), with symbol 12#(5, 4; 1, 4).



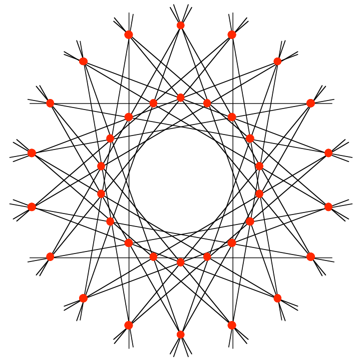

18#(6, 5; 1, 4)



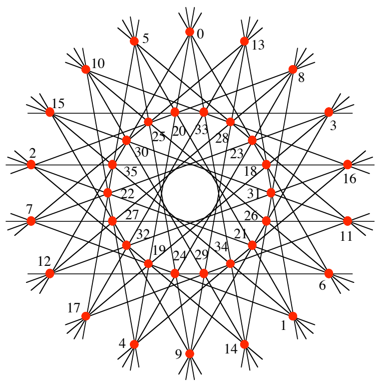
18#(6, 4; 1, 5)



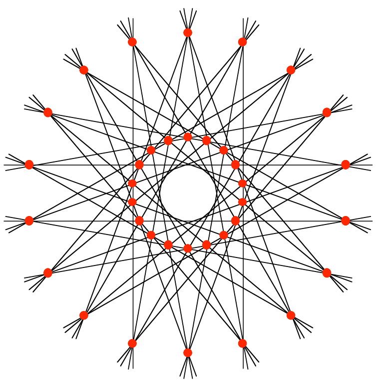

18#(7,6;2,5)



18#(7,6;2,5)




18#(8,7;1,6)



18#(8,6;1,7)

Figure 3.6.3. The six configurations $(36_4)$ belong to three cohorts: 18#{{6,1},{5,4}}, 18#{{7,2},{6, 5}, 18#{{8,1},{7,6}}. Near each configuration we show the lexico-

graphically highest among its symbols. Although it is not obvious from the symbols or the diagrams, the three configurations at left are isomorphic. This isomorphism is established by the labels near their vertices. Since these configurations are isomorphic, their polars (shown at right) are also isomorphic to each other; they are not isomorphic to the other three configurations.

After these preliminaries, here is the detailed result.

**Theorem 3.6.1.**[1] Astral 4-configurations $m\#(s_1, t_1; s_2, t_2)$ must satisfy all the conditions from Section 3.5, and in particular the equation (A7):

$$\cos(s_1\pi/m)\cdot\cos(s_2\pi/m) = \cos(t_1\pi/m)\cdot\cos(t_2\pi/m).$$

The symbols of these configurations are:

(i) The *systematic configurations* with symbols (6k)#(3k-j, 2k; j, 3k-2j) for $k \geq 2$, $1 \leq j < 3k/2$, with $j \neq k$.

(ii) The *systematic configurations* with symbols (6k)#(2k, j; 3k-2j, 3k-j) for $k \geq 2$, $1 \leq j < 3k/2$, with $j \neq k$. By the general results of Section 3.5, these configurations are polar to the ones in (i) with the same values of k and j.

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**Comment:**

For even k and j = k/2, the configurations in (i) and (ii) are selfpolar, hence coincide. If $k = f\cdot g$ and $j = f\cdot h$, with $f \geq 2$, $g \geq 2$, then both (6k)#(3k-j, 2k; j, 3k-2j) and (6k)#(2k, j; 3k-2j, 3k-j) are disconnected. Each consists of f equidistributed copies of (6g)#(3g-h, 2g; h, 3g-2h) or (6g)#(2g, h; 3g-2h, 3g-h) and is denoted by (f) (6g)#(3g-h, 2g; h, 3g-2h) or (f) (6g)#(2g, h; 3g-2h, 3g-h), respectively.

For simpler formulation, we can say that the configurations in (i) and (ii) are in the cohorts of (6k)#{{3k-j},{3k-2,2k}} for $k \geq 2$, $1 \leq j < 3k/2$, with $j \neq k$.

(iii) The 27 symbols of the *sporadic configurations* listed in Table 3.6.1, and their multiples.

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[1] I am indebted to L. Berman and T. Pisanski for a number of comments and corrections. These led to the present formulation, which I hope is more informative and useful than the statements in previous publications.

| | |
|---|---|
| 30#(7,6;1,4) | 30#(7,4;1,6) |
| 30#(8,6;2,6) | |
| 30#(11,10;1,6) | 30#(11,6;1,10) |
| 30#(12,10;6,10) | |
| 30#(12,11;2,7) | 30#(12,7;2,11) |
| 30#(13,12;1,8) | 30#(13,8;1,12) |
| 30#(13,12;7,10) | 30#(13,10;7,12) |
| 30#(14,12;4,12) | |
| 30#(14,13;6,11) | 30#(14,11;6,13) |
| 42#(13,12;1,6) | 42#(13,6;1,12) |
| 42#(18,17;6,11) | 42#(18,11;6,17) |
| 42#(19,18,5,12) | 42#(19,12;5,18) |
| 60#(22,21;2,9) | 60#(22,9;2,21) |
| 60#(25,24;5,12) | 60#(25,12;5,24) |
| 60#(27,26;3,14) | 60#(27,14;3,26) |

Table 3.6.1. The complete list of connected sporadic astral 4-configurations. The three stand-alone symbols denote selfpolar configurations, the paired symbols correspond to configurations polar to each other.

Here too, the cohorts notation allows a more condensed listing:
30#{{7,1}{6,4}}, 30#{{8,2},{6,6}}, 30#{{11,1},{10,6}}, 30#{{12,6},{10,10}}, 30#{{12,2},{11,7}}, 30#{{13,1}12,8}}, 30#{{13,7},{12,10}}, 30#{{14,4},{12,12}}, 30#{{14,6},{13,11}},
42#{{13,1},{12,6}}, 42#{{18,6},{17,11}}, 42#{{19,5},{18,12}},
60#{{22,2},{21,9}}, 60#{{25,5},{24,12}}, 60#{{27,3},{26,14}}.

The proof of the theorem will be interwoven with an account of the history of its development. In view of all the interest in configurations during the last quarter of the 19th century (as well as the sporadic interest later), it is hard to understand that *no graphical representation* of **any** 4-configuration appeared in print prior to [G50] in 1990. The configuration shown above as Figure 3.6.2 was one of the configurations shown in that paper. Another was the ($21_4$) configuration that gave the paper its title; we shall encounter it again in Section 3.7.

In the early 1990 I found several 4-configurations in addition to the ones in [G50], with two or three orbits of points (and of lines); these were found by drawing with such software as was available to me (mainly MacDraw), until I was initiated to Mathematica® through friendly persuasion by Stan Wagon. (A few other k-astral 4-configurations with various k's were communicated to my by J. F. Rigby.) With programs in Mathematica it was possible to "experimentally" find all possible astral configurations with reasonably small numbers of vertices. This led to the understanding that there are systematic infinite families of such configurations, as well as an apparently finite number of sporadic configurations. I became convinced that I have a complete description, and presented this in seminars and courses during the 1990s; the results were published in 2000 [G40], together with formal demonstrations of the geometric realizability of these configurations. This covers the existence aspect of Theorem 3.6.1.

The main tool for the proof of completeness was the observation that an astral configuration m#($s_1$, $t_1$; $s_2$, $t_2$) has a realization by straight lines if and only if the same points are reached starting from one of the regular polygons regardless of which of two diagonals we are following. In other words, the points described by ($s_1$//$t_1$) must coincide with the points ($t_2$//$s_2$). (Note that the designation ($s_2$//$t_2$) used in determining the symbol of the configuration refers to the diagonals as looked from the other polygon.) This leads to the following necessary condition for the existence of an astral configuration m#($s_1$,$t_1$;$s_2$,$t_2$)

(1) $$\cos(s_1\pi/m)\cdot\cos(s_2\pi/m) = \cos(t_1\pi/m)\cdot\cos(t_2\pi/m).$$

Due to the dihedral symmetry of such configurations, this is also a sufficient condition for the existence. Moreover, criterion (1) is easily implemented for computational searches; the results of these calculations led to the classes listed in Theorem 3.6.1.

For the convenience of use of Theorem 3.6.1 we list in Table 3.6.2 the cohort symbols of the systematic astral configurations ($n_4$) with $n \leq 100$.

12#{{5,1},{4,4}.

18#{{8,1},{7,6}}, 18#{{7,2},{6,5}}, 18#{{6,1},{5,4}}.

24#{{11,1},{10,8}}, *24#{{10,2},{8,8}},* 24#{{9,3},{8,6}}, 24#{{8,2},{7,5}}.

30#{{14,1},{13,10}}, 30#{{13,2},{11,10}}, 30#{{12,3},{10,9}},
30#{{11,4},{10,7}}, 30#{{10,3},{9,6}}, 30#{{10,1},{8,7}}.

36#{{17,1},{16,12}}, *36#{{16,2},{14,12}}*, *36#{{15,3},{12,12}}*, *36#{{14,4},{12,10}},*
36#{{13,5},{12,6}}, 36#{(12,4},{11,7}}, *36#{{12,2},{10,8}}.*

42#{{20,1},{19,14}}, 42#{{19,2},{20,16}}, 42#{{18,3},{15,14}},
42#{{17,4},{14,13}}, 42#{{16,5},{14,11}}, 42#{{15,6},{14,9}},
42#{{14,5},{13,8}}, 42#{{14,3},{12,9}}, 42#{{14,1},{11,10}}.

48#{{23,1},{22,16}}, *48#{{22,2},{20,16}}*, 48#{{21,3},{18,16}},
*48#{{20,4},{16,16}},* 48#{{19,5},{16,14}}, *48#{{18,6},{16,12}}*,
48#{{17,7},{16,10}}, 48#{{16,6},{15,9}}, *48#{{16,4},{14,10}},*
48#{{16,2},{13,11}}.

Table 3.6.2. The cohort symbols of all systematic astral $(n_4)$ configurations with $n \le 100$. Disconnected configurations are in italics.

Once the characterization of the astral configurations has been guessed, it is easy to see that the symbols listed above correspond to actual geometric configurations, and are not results of an approximation error in the computations.

Indeed, for the symbols in part (i) we have to show that

(2) $$\cos((3k-j)\pi/(6k))\cdot\cos(\pi(6k)) = \cos(2k\pi/(6k))\cdot\cos((3k-2j)\pi/(6k)).$$

In view of the trigonometric identity

(3) $$(\cos a)\cdot(\cos b) = \tfrac{1}{2}\,(\cos(a+b) + \cos(a-b))$$

validity of (2) is equivalent to

$$\tfrac{1}{2}\,(\cos 3k\pi/(6k) + \cos((3k-2j)\pi/(6k))) = (\cos \pi/3)\cdot\cos((3k-2j)\pi/(6k)).$$

Since $\cos \pi/2 = 0$ and $\cos \pi/3 = \tfrac{1}{2}$, this is valid for all k and j; hence (2) is correct. The same calculation shows that the symbols in (ii) correspond to astral geometric configurations as well. The fact that the above arguments did not rely on particular values of the cosines involved shows that (i) and (ii) are symbols of systematic configurations.

The existence of the sporadic configurations proceeds somewhat analogously, but needs to rely on information specific to the angles involved. For example, concerning the configuration 30#(8,6;2,6) we note that (1) becomes

$$\cos 8\pi/30 \cdot \cos 2\pi/30 = (\cos 6\pi/30)^2,$$

which by (3) is equivalent to

$$\tfrac{1}{2}(\cos 10\pi/30 + \cos 6\pi/30) = (\cos 6\pi/30)^2$$

Since $\cos \pi/3 = \tfrac{1}{2}$ and $\cos \pi/5 = \tfrac{1}{4}(1 + \sqrt{5})$, this reduces to

$$(12 + 4\sqrt{5})/16 = (6 + 2\sqrt{5})/8,$$

which is obviously true.

Using other explicit algebraic values for cosines, similar arguments can be made for the other sporadic configurations with symbols that start with 30 or 60. Among the values that can be used are

$$\cos 2\pi/30 = (-1 + \sqrt{5} + \sqrt{6(5 + \sqrt{5})})/8,$$

$$\cos 4\pi/30 = (1 + \sqrt{5} + \sqrt{6(5 - \sqrt{5})})/8,$$

$$\cos 8\pi/30 = (1 - \sqrt{5} + \sqrt{6(5 + \sqrt{5})})/8,$$

and so on.

For the symbols that involve 42 it is convenient to follow a slightly different path. The validity of the first of these symbols, 42#(13,12;1,6), is by (1) and (3) equivalent to

$$\cos \pi/3 + \cos 2\pi/7 = \cos 3\pi/7 + \cos \pi/7$$

that is

$$1 + 2\cos 2\pi/7 + 2\cos 4\pi/7 + 2\cos 6\pi/7 = 0.$$

But this is simply an expression of the fact that the centroid of a regular heptagon, centered at the origin and with one vertex at (1,0), is itself at the origin. An completely analogous reasoning shows the validity of the other symbols involving 42.

What is still missing is a proof that there are no other astral 4-configurations. Since these configurations are determined by intersections of diagonals of regular polygons, and since these have been extensively studied and completely determined, in the late 1990's it seemed to me that it should be very easy to supply the proof of completeness.

In reality this task proved far from simple, and it was first successfully carried out in 2001 in the PhD work of L. Berman [B4], [B3]. Berman's rather complicated argumentation relied on the complete description of intersections of diagonals of regular polygons, given by Poonen and Rubinstein [P8] in 1998. Theirs was a new proof (and a much more convenient presentation) of material that has been contained, to a large extent, in earlier publications of G. Bol [B26] in 1933 (with some misprints) and J. F. Rigby [R3] in 1980[2]. For regular n-gons with prime n, or with any odd n, it has been proved by many authors that there are no intersections of more than two diagonals; references to these papers and other related material can be found in [R3], and especially in [P8].

However, independently of these developments, an approach that is easier to apply for our purposes was published by G. Myerson [M21] in 1993; it came to my attention only recently. Myerson's result (his Theorem 4) that is relevant to our proof can be formulated as follows.

**Theorem 3.6.2. (Myerson [M21])** . The equation

$$\sin \pi/6 \cdot \sin t = \sin(t/2)\,\sin(\pi/2 - t/2)$$

is valid for all t. The only other solutions of the equation

$$\sin x_1\pi \cdot \sin x_2\pi = \sin x_3\pi \cdot \sin x_4\pi \tag{4}$$

in rational $x_1$, $x_2$, $x_3$, $x_4$ with $0 < x_1 < x_3 \le x_4 < x_2 \le ½$ are given in Table 3.6.3.

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[2] In contrast to other writers on the topic, Rigby considers the multiple intersections of diagonals outside the n-gon as well. However, his intended [R3, p. 222] investigation of outside intersections of four or more diagonals seems not to have been published, and remains an open problem.

| Label | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| 1 | 1/21 | 8/21 | 1/14 | 3/14 |
| 2 | 1/14 | 5/14 | 2/21 | 5/21 |
| 3 | 4/21 | 10/21 | 3/14 | 5/14 |
| 4 | 1/20 | 9/20 | 1/15 | 4/15 |
| 5 | 2/15 | 7/15 | 3/20 | 7/20 |
| 6 | 1/30 | 3/10 | 1/15 | 2/15 |
| 7 | 1/15 | 7/15 | 1/10 | 7/30 |
| 8 | 1/10 | 13/30 | 2/15 | 4/15 |
| 9 | 4/15 | 7/15 | 3/10 | 11/30 |
| 10 | 1/30 | 11/30 | 1/10 | 1/10 |
| 11 | 7/30 | 13/30 | 3/10 | 3/10 |
| 12 | 1/15 | 4/15 | 1/10 | 1/6 |
| 13 | 2/15 | 8/15 | 1/6 | 3/10 |
| 14 | 1/12 | 5/12 | 1/10 | 3/10 |
| 15 | 1/10 | 3/10 | 1/6 | 1/6 |

Table 3.6.3. The complete list of sporadic solutions of equation (4) as given by Myerson in [M21].

The result of Theorem 3.6.2 gives an immediate solution to the completeness question of Table 3.6.1. Indeed, we only have to recall that $\sin \alpha = \cos(\pi/2 - \alpha)$ in order to see that the rows of Table 3.6.3 correspond (in an appropriate permutation) to the rows of Table 3.6.1. For example, rows with labels 1, 2, 3 correspond to the entries involving 42 of the earlier table, while those labeled 4, 5, and 14 correspond to the last three rows of table 3.6.1.

This completes the proof of Theorem 3.6.1.

**Exercises and problems 3.6.**

1. Verify the complete correspondence between Myerson's list in Table 3.7.3 and the list of sporadic symbols in Table 3.6.1.
2. Verify the validity of the existence claims made above for all sporadic configurations.
3. Draw the configuration 36#(15,12; 3,12) = (3) 12#(5,4;1,4). Is it selfpolar?

4. Prove that the configurations 18#(6,5;1,4) and 18#(6,4;1,5) shown in Figure 3.6.3 are not isomorphic.

5. Determine whether the pairs of polar configurations in Figure 3.6.3 are in appropriate orientation to exhibit the polarity, or does one member of the pair have to be rotated.