3.5 ASTRAL 4-CONFIGURATIONS

In this section we start our investigation of a special class of 4-configurations which we call **k-astral** for some $k \geq 2$. They are of interest for several reasons. To begin with, such configurations were the first 4-configurations for which geometric realizations were found (see, for example, [G50], [G40], [B20], and the other publications that will be mentioned later). Next, they have a clear-cut definition that leads to a natural notation, as well to construction of the configuration given its symbol. Finally, they exhibit a variety of phenomena that add interest to their study, such as the relation of a configuration to its dual (actually, its polar) configuration, and questions of realizability versus representation.

k-astral configurations have appeared under several different names, and with several different definitions – not all of which coincide in all cases. In several publications configurations we call k-astral have been termed *celestial*. The intention in the present account of these configurations is to have an easily implementable decision algorithm for checking the membership of either a given configuration to the class, or of a symbol for correspondence to a geometric configuration.

Definition 3.5.1. A (n_4) configuration C is **k-astral** provided all the following conditions are satisfied:

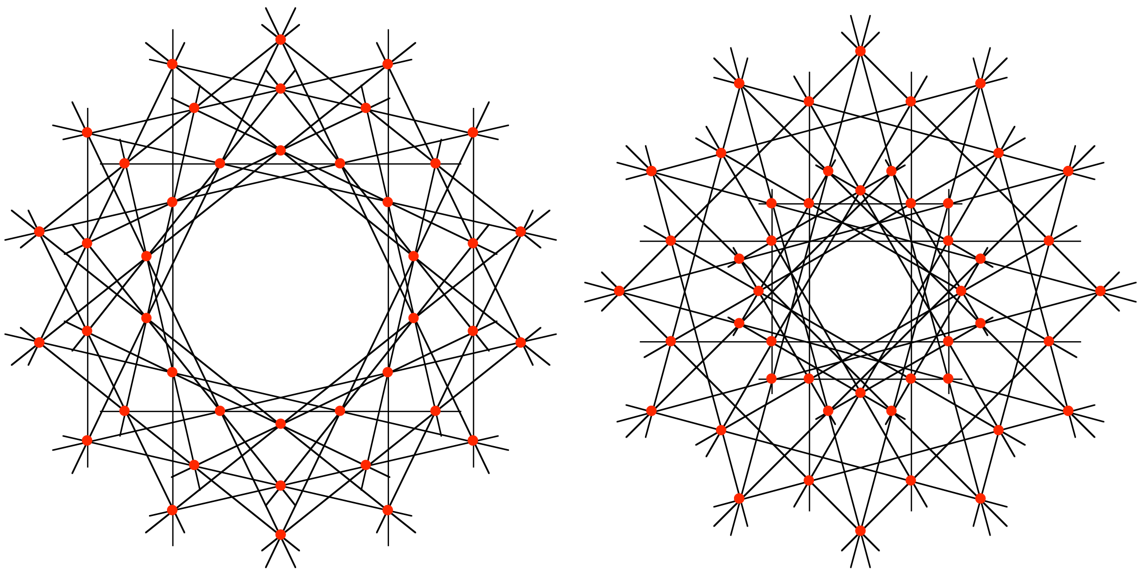
(A1) $k \geq 2$ and $n = k \cdot m$, for some $m \geq 7$.

(A2) The points of C are at the vertices of k regular convex m-gons, with common centers and such that all angles subtended from this center by the various points of C are multiples of π/m.

(A3) C has symmetry group d_m; the vertices of each k-gon form an orbit.

(A4) Each line of C contains two points from each of two m-gons (point orbits); each point is incident with two lines each from two line orbits.

We have already encountered various configurations that are k-astral, for example the ones in Figures 1.1.2 and 1.5.1(a). Two additional examples are shown in Figure 3.5.1.



(a) (b)

Figure 3.5.1. (a) A 3-astral configuration (42_4) with symbol 14#(5,3;2,4;1,3). (b) A 4-astral configuration (45_4) with symbol 12#(5,4;3,2;4,5;2,3).

Some comments deserve to be made regarding k-astral configurations.

(i) In most cases the k regular m-gons have different sizes; however, in some cases with $k \geq 3$ there may be pairs of polygons with the same size. We shall give examples later.

(ii) The conditions in Definition 3.5.1 could be weakened at the expense of complicating the verification.

(iii) It will turn out to be convenient to consider the case of connected k-astral configurations separately from the case of disconnected ones.

Theorem 3.5.1. Each k-astral configuration C can be assigned a symbol $m\#(s_1,t_1;s_2,t_2; \ldots ,s_k,t_k)$ in such a way that C is the only configuration from which that symbol arises. At most 2k distinct symbols correspond to each configuration; such symbols are said to be equivalent. The family of equivalent symbols can be obtained from any one of them by cyclic permutations of an even number of steps of the 2k entries in parentheses, or by reversal of these.

Proof. The proof consists of a description of the steps leading from the configuration to one of the symbols, and observing the stages at which distinct symbols may arise. The main tools in the derivation are a notation for the intersection points of the diagonals determined by each of the regular m-gons, and the "characteristic paths" along lines of the configuration.

For a regular convex m-gon M, the **span** s of a diagonal S is the number of edges of M between the endpoints of S, taken as the smaller of the two possible numbers; hence $s \leq m/2$. Despite talking about "endpoints", by "diagonal" we understand both the elementary-geometric meaning of the term as a segment, as well as the whole line determined by this segment. In Figure 3.5.1, the outer polygon has diagonals of spans 3 and 5 for both configurations (a) and (b).

The intersections of a diagonal S of span s with the other diagonals of span s of the same polygon M are denoted by the symbol (s//t), where t is the position of the intersection points on S, counting from the midpoint of S. (Instead of (s//t), the notation [[s,t]] has also been used.) Thus, for example, each endpoint of S has symbol (s//s). The intersection points are not limited to the diagonal considered as a segment, but continue "outside" and exist for all $t < m/2$. If m is even, one may consider the point-at-infinity on S as having $t = m/2$. An illustration of the notation (s//t) is given in Figure 3.5.2.

The use of polar coordinates is particularly convenient for the intersection points (s//t), since it is easily seen that in the setting of Figure 3.5.2, such a point has coordinates $(\cos s\cdot\pi/m \,/\, \cos t\cdot\pi/m,\ t\cdot\pi/m)$. If the endpoints of the diagonal are not on the unit circle but at distance r, then the first polar coordinate needs to be multiplied by r.

A **characteristic path** P of a (connected) k-astral configuration C consists of k *segments* of lines of the configuration, determined as follows. The procedure we describe here is illustrated by the example in Figure 3.5.3.

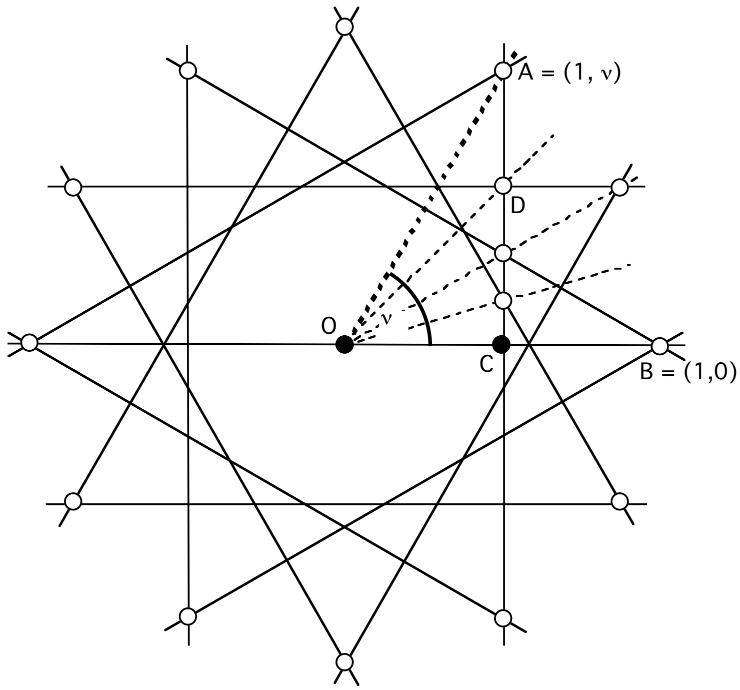


Figure 3.5.2. The determination of the symbol (s//t) of an intersection point of diagonals of a regular m-gon. Here $m = 12$, the diagonals are of span $s = 4$, the vertices of the m-gon with unit radius have polar coordinates $(1, \nu)$, where ν is a multiple of π/m. In the diagram $\nu = 4\pi/m$, angle DOB is $3\pi/m$, so that $s = 4$, $t = 3$. This gives for D the symbol (s//t) = (4//3). The right triangles OCA and OCD imply that $OD = \cos s\cdot\pi/m \,/\, \cos t\cdot\pi/m$, hence D has polar coordinates $(\cos s\cdot\pi/m \,/\, \cos t\cdot\pi/m,\ t\cdot\pi/m)$.

As the first step we orient all lines of C in the same sense, generally taken to be counterclockwise as seen from the center. Next, we choose an arbitrary point P_0 of C and through it an arbitrary line L_1 for which P_0 is the earlier of the two points in the same orbit; this involves the choice of one line from the two orbits of points through P_0. On L_1 we take the first point (in the orientation we adopted) of the *other* orbit of points incident with L_1, and denote it P_1. We choose as line L_2 a line through P_1 that is in the orbit different from L_1, and for which P_1 is the earlier point in its orbit. On L_2 we choose the earlier point in the orbit different from the one of P_1, and denote it P_2. Continuing in the same way, we select the line L_{j+1} through the already selected point P_j that is in the orbit different from L_j, and for which P_j is the earlier point among the points on L_{j+1} belonging

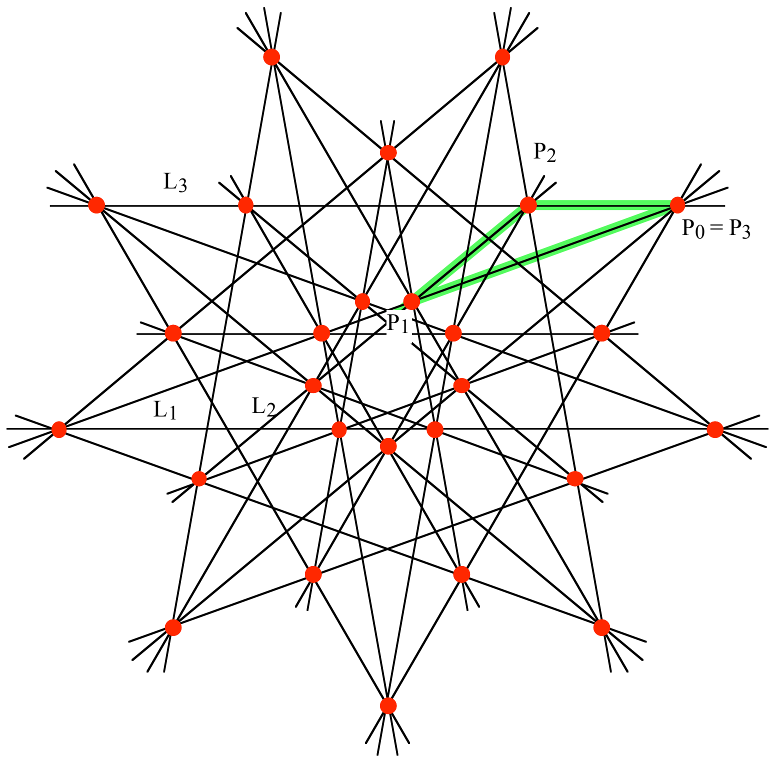


Figure 3.5.3. An illustration of the construction of a characteristic path (shown green) and the corresponding symbol of the configuration, according to the description given in the text. Since $P_1 = (4//2)$, $P_2 = (3//4)$, $P_3 = (2//3)$, and k = 3 while m = 9, the resulting symbol of this (27_4) configuration is 9#(4,2;3,4;2,3).

to the orbit different from the orbit of the earlier point P_j. This continues until we reach the line L_k and the point P_k. (in Figure 3.5.3 we have k = 3.) This point P_k necessarily belongs to the same orbit as the starting point P_0; in the illustration P_k coincides with P_0, but this is not necessarily the case. Figure 3.5.4 illustrates the possibility of P_k being different from P_0. By using the notation $(s_j//t_j)$ for the point P_j, the characteristic path $P_0,L_1,P_1,L_2,P_2, \ldots L_k,P_k$ determines a symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ for the configuration.

What are the possible alternative symbols for a configuration? We arbitrarily chose the orientation of the lines, the starting point of the characteristic path, and the starting line through that point. The choice of orientation does not lead to any new symbols since a k-astral configuration has *dihedral* symmetry group d_k. However, the other

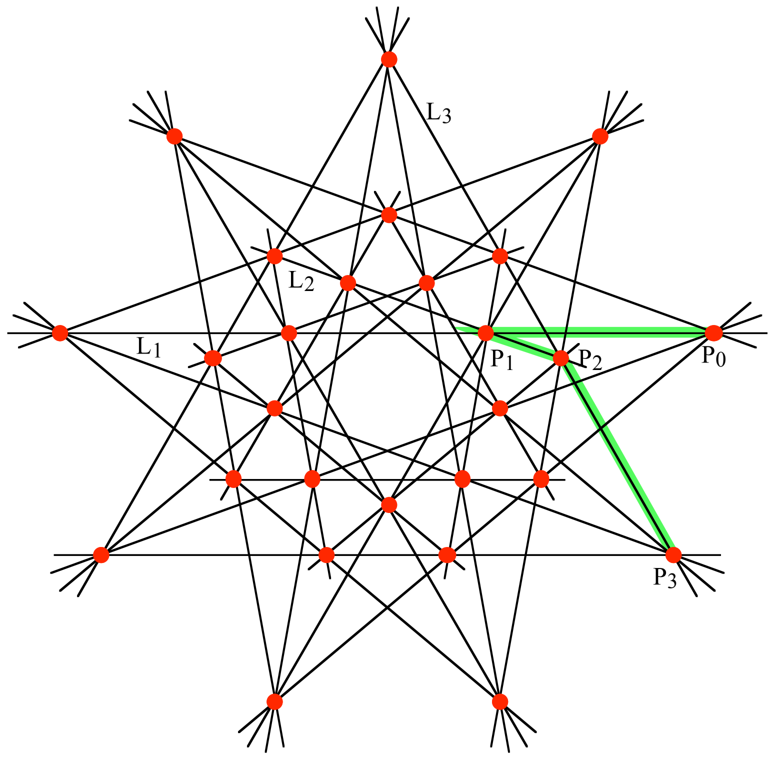


Figure 3.5.4. Another illustration of the construction of a characteristic path and the symbol of the configuration. Since $P_1 = (4//3)$, $P_2 = (2//3)$, $P_3 = (1//3)$, and k = 3 while m = 9, the resulting symbol of this (27_4) configuration is 9#(4,3;2,3;1,3), and $P_3 \neq P_0$.

two choices obviously matter, and in general lead to 2k distinct symbols — k choices of the orbit of the starting point of the path, and two choices for the starting line through that point. As an illustration we show in Figure 3.5.5 the four characteristic paths and the resulting four equivalent symbols for the 2-astral configuration (48_4).

The various equivalent symbols for a given k-astral configuration arise in one of the following two ways. For a given characteristic path, selecting on this path a different point as the starting point clearly permutes the symbols $(s_j//t_j)$ cyclically, that is, by an even number of steps in the symbol $m\#(s_1,t_1;s_2,t_2; \dots ;s_k,t_k)$ of the configuration. This yields up to k distinct symbols. On the other hand, if we consider a diagonal of span s, the symbol (s//t) for the t^{th} intersection point (counting from the midpoint) can be interpreted as saying that on the orbit of all points (s//t) the same diagonal line has span t, and the original endpoints (that gave span s to the diagonal) now have symbol (t//s).

This means the following: Starting with a given characteristic path but traversing it in the opposite direction, will reverse the roles of s_j and t_j in all the diagonals, as well as the order of the points. Hence this leads to the reversal of all the entries in the original symbol, thus accounting for (up to) an additional k symbols.

The construction of the symbols for a k-astral configuration (n_4) leads to several notable properties of the symbols and the configurations. For ease of reference we list them as a continuation of the entries in Theorem 3.5.1.

(A5) Since the symbol (s//s) denotes the endpoints of a diagonal of span s, (hence would not constitute a step in the characteristic path) any two adjacent entries in the symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ must be different; this includes the requirement that s_1 and t_k are distinct.

Next, as obvious from the reasoning concerning the symbol (s//t) and visible in Figure 3.5.2, the polar angle of the point (s/t) differs from 0 by a multiple of π/m. The parity of that multiple is the same as the parity of $s+t$. Since the endpoint of a characteristic path leads to a point in the orbit of the starting point, and the polar angles of any two such points differ by a multiple of $2\pi/m$, it follows that the sum of all entries in the parentheses of a symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ must be even, or equivalently, that

(A6) $$\delta = \tfrac{1}{2} \sum_{1 \le j \le k} (s_j - t_j) \quad \text{is an integer.}$$

If condition (A6) is not satisfied in a symbol that fulfils all other requirements, then the last point of the characteristic path ends midway between points of the orbit of the starting point — and consequently has only two lines incident with it, just as the starting point is incident with only two lines. We shall discuss this in more detail later, but already here we can supply in Figure 3.5.6 an example of such a situation.

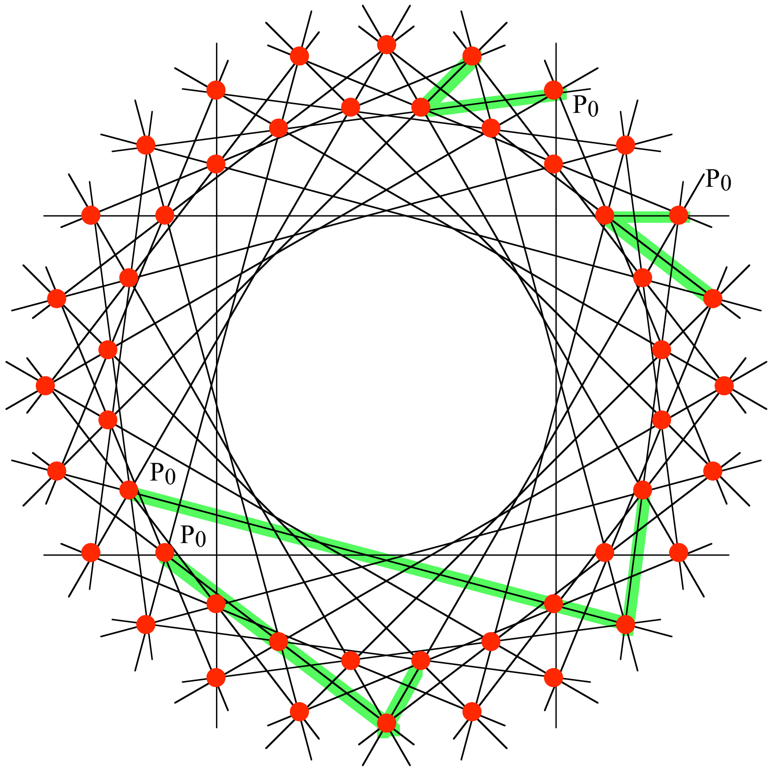


Figure 3.5.5. Four characteristic paths (green) for a 2-astral (48_4) configuration; all proceed counterclockwise. In order to avoid excessive clutter, in each path only the starting point is labeled. The path on top starts in the outer ring of points; it leads to the symbol 24#(5, 2; 7, 8), since the first point of the inner ring that is encountered by the path has symbol (5//2), and the first point met after that in the outer ring has symbol (7//8). The other characteristic paths lead to the symbols 24#(7, 8; 5, 2), 24#(2, 5; 8, 7), and 24#(8, 7; 2, 5), respectively, in counterclockwise order of the starting points.

One additional — and very important and useful — property of all k-astral 4-configurations follows from the comments we made after the introduction of the (s//t) notation. Since the radius of a point (s//t) of a regular convex m-gon with circumradius r is $r \cdot \cos(s \cdot \pi/m) / \cos(t \cdot \pi/m)$, the distance of the point P_j from the center is (assuming the starting point of the characteristic path is at unit distance from the center):

$$\prod_{1 \le i \le j} (\cos(s_i \cdot \pi/m) / \cos(t_i \cdot \pi/m)).$$

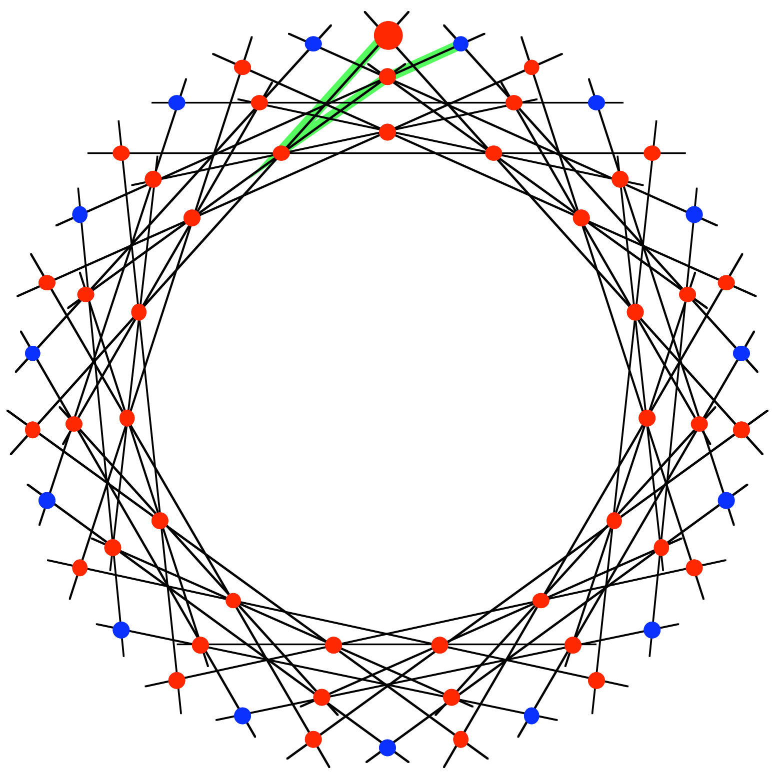


Figure 3.5.6. The symbol 15#(4,2;1,3;2,3) satisfies all conditions for a valid symbol of a 3-astral 4-configuration (45_4), except (A6). The characteristic path (green) that starts at the top point (large red) leads to a point (blue) at an in-between position. Both the starting point and the end point of the path are incident with just two lines each — hence the symbol does not correspond to any 4-configuration.

Since the endpoint of any characteristic path is in the same orbit as the starting point, this yields

(A7) $$\prod_{1\le j\le k} \cos(s_j\cdot\pi/m) = \prod_{1\le j\le k} \cos(t_j\cdot\pi/m)$$

It is easy to verify that in all examples of k-astral configurations presented in this section the condition (A7) is fulfilled.

The appropriateness of the characteristic path approach to the notation for k-astral 4-configurations can be seen in the straightforward translation of the characteristic path into the reduced Levi diagram of the configuration. Without entering in lengthy descriptions of the procedure (which is essentially taken from Boben and Pisanski [B20]), we show in Figure 3.5.7 a typical example. The configuration 20#(9,8;2,6;1,4) and a characteristic path leading to this symbol are shown in part (a), while part (b) presents a reduced Levi diagram of this configuration. In part (c) we show the reduced Levi diagram of the

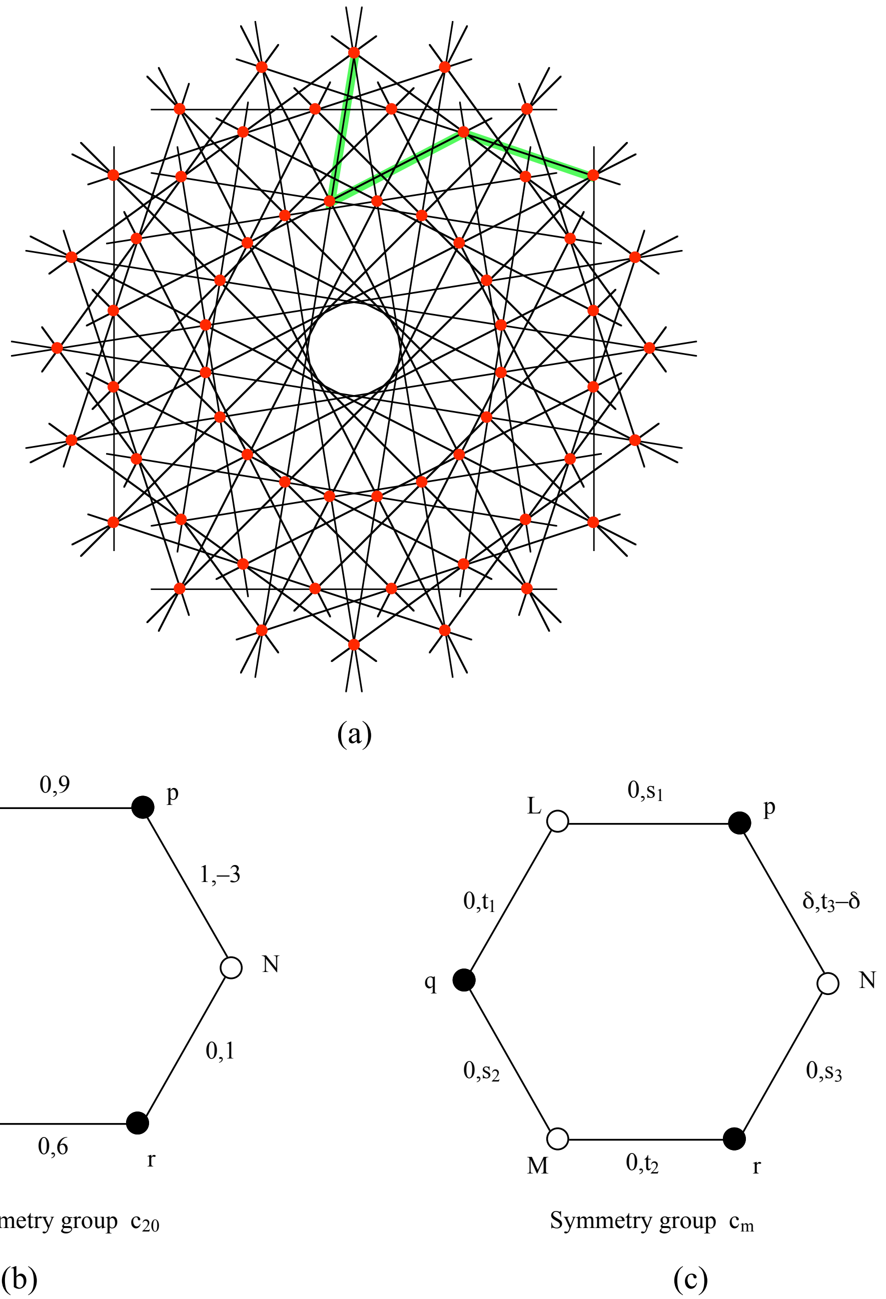


Figure 3.5.7. (a) The 3-astral configuration 20#(9,8;2,6;1,4) and a characteristic path . (b) The corresponding reduced Levi graph of 20#(9,8;2,6;1,4) . (c) The reduced Levi graph of the 3-astral configuration $m\#(s_1,t_1;s_2,t_2;s_3,t_3)$.

general case of a 3-astral configuration $m\#(s_1,t_1;s_2,t_2;s_3,t_3)$. The corresponding situation for a k-astral configuration differs only in the length of the circuit, so that it contains k white and k black points. The value of δ is determined by (A6).

Next, we explore what happens if the 2k entries between parentheses of a symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ of a k-astral configuration C are changed by a cyclic permutation that moves them an *odd* number of steps. What — if anything — can we say about a configuration C* that would correspond to $m\#(t_1,s_2;t_2, \ldots ,s_k;t_k,s_1)$?

Considering the well-known relations between points and lines polar to them with respect to a circle of a given radius and center (illustrated in Figure 3.5.8, see also, for example, [<C12 Chapter 6]), we see that for a configuration corresponding to the symbol $m\#(t_1,s_2;t_2, \ldots ,s_k;t_k,s_1)$, the distance of the line L_j of C* that is polar to the point P_j of C with respect to a circle of unit radius should satisfy

$$\text{distance}(O, L_j) = OP_j{}^* = \prod_{1\le i\le j} \big(\cos(t_i\cdot\pi/m) / \cos(s_i\cdot\pi/m)\big) = 1/OP_j =$$

$$1\Big/\prod_{1\le i\le j} \big(\cos(s_i\cdot\pi/m) / \cos(t_i\cdot\pi/m)\big) = \prod_{1\le i\le j} \big(\cos(t_i\cdot\pi/m) / \cos(s_i\cdot\pi/m)\big).$$

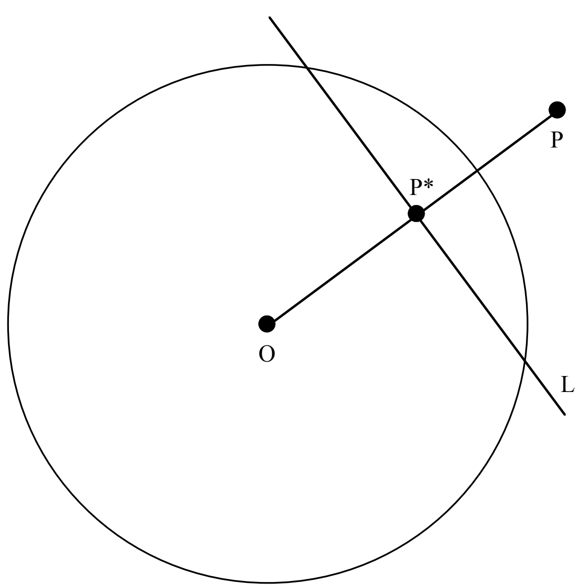


Figure 3.5.8. If the point P and the line L are polars of each other with respect to the circle of radius r and center O, then the distance between O and L is the same as the distance OP*, and the relation between the distances is $OP\cdot OP^* = r^2$.

Hence distances from the center O of all lines of the putative configuration C* are correct for them being the polars of the points of C, and since incidences and symmetry are all preserved under polarity, we can conclude:

Theorem 3.5.2. If the symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ corresponds to a k-astral 4-configuration then the symbol $m\#(t_1,s_2;t_2, \ldots ,s_k;t_k,s_1)$ corresponds to a k-astral 4-configuration that is polar to the former with respect to the unit circle with center at the common center of both configurations.

A notational remark. Unless there is a definite reason to do otherwise, we shall always strive to use the lexicographically highest symbol for each k-astral 4-configuration.

Several concepts simplify the listing and classification of possible k-astral configurations. One is based on the observation that if we switch the positions of two entries separated by an odd number of other entries in the symbol of an astral configuration, the modified symbol automatically satisfies all the conditions listed above, except possibly (A5). By repeated application of this observation while avoiding a violation of (A5), we arrive to the conclusion that it is sensible to introduce the **cohort concept** and **notation**. For a k-astral configuration with symbol $m\#(s_1,t_1;s_2,t_2; \ldots ;s_k,t_k)$ the **cohort symbol** is $m\#\{s, t\} = m\#\{\{s_1,s_2, \ldots ,s_k\},\{t_1,t_2, \ldots ,t_k\}\}$; it stands for all the valid assignments of suitable permutations of the s_i's and permutations of the t_i's into a symbol for a k-astral configuration. For example, for the configuration in Figure 3.5.1(a) the symbol is 14#(5,3;2,4;1,3), and the cohort symbol is 14#{{5,2,1},{4,3,3}}; This cohort symbol indicates, and is shared by, the six distinct 3-astral configurations 14#(5,4;2,3;1,3), 14#(5,3;2,4;1,3). 14#(5,3;2,3;1,4), 14#(5,4;1,3;2,3), 14#(5,3;1,4;2,3), and 14#(5,3;1,3;2,4).

The second comes from the observation that all the conditions, except possibly (A5), are satisfied if in the cohort symbol m#{s,t} the sets s and t are the same. As an example, the configuration we used in Figure 3.5.3 has symbol 9#(4,2;3,4;2,3), hence s = t ={4,3,2}. Since the condition (A7) is satisfied without the need to make any calculations, we shall say that the cohort 9#{{4,3,2},{4,3,2}} is **trivial**. On occasion we shall use "trivial" also for an individual configuration in a trivial cohort. For odd k, a typical representative of a trivial cohort is m#(a,b;c, ... ;u,v;w,a;b,c; ... ,u;v,w), while for even k we can use m#(a,b;c,d; ... ;v,w;b,a;d,c; ... ;w,v). We should mention here that there cannot be any 2-astral configurations of the trivial type.

We shall say that a cohort symbol m#{s,t} of k-astral configurations is **systematic** provided m and the elements of s and t depend on one or more parameters in such a way that the validity on (A7) can be ascertained using only trigonometric identities and without the need to calculate specific values of the parametrized s_i's and t_i's. As we shall illustrate in Section 3.6, the cohort with m = 6k, s = {3k-j,j}, t = {3k-2j,2k} is a systematic 2-astral cohort.

If a k-astral configuration belongs neither to a trivial cohort nor to a systematic one, we shall say that the configuration and its cohort are **sporadic.** For k = 2 all sporadic configurations are known, and we list them in Section 3.6. However, already for k = 3 we have only examples of such configurations (as discussed in Section 3.7), but no complete characterization.

If a cohort symbol m#{s,t} of a k-astral configurations contains the same integer in both s and t, a cohort symbol of a (k-1)-astral configurations may result if this integer is deleted from both s and t. As is easily verified, all the conditions for (k-1)-astral symbol are automatically verified, except possibly (A5). We call **clade** of m#{s,t} all the cohorts resulting from one or several applications of this procedure. This will be illustrated in Section 3.7.

* * * * * *

We end this section with an unsolved problem of methodology in the study of k-astral configurations. We required in the definition that the symmetry group of every astral configuration is d_k. In fact, the other conditions show that this happens automatically if we require that the cyclic group c_k is a symmetry group of the configuration. The characteristic path, the symbol, and the reduced Levi graph of each k-astral configuration are all based on the cyclic group. The reason is that (as far as I know) nobody has come up with a reasonable version of all these based on the dihedral symmetry group. The construction of a reduced Levi graph that is based on the dihedral symmetry is certainly feasible – but does not appear to be useful. How come?

Exercises and problems 3.5

1. Devise symbols for the configurations in Figure 3.5.9.

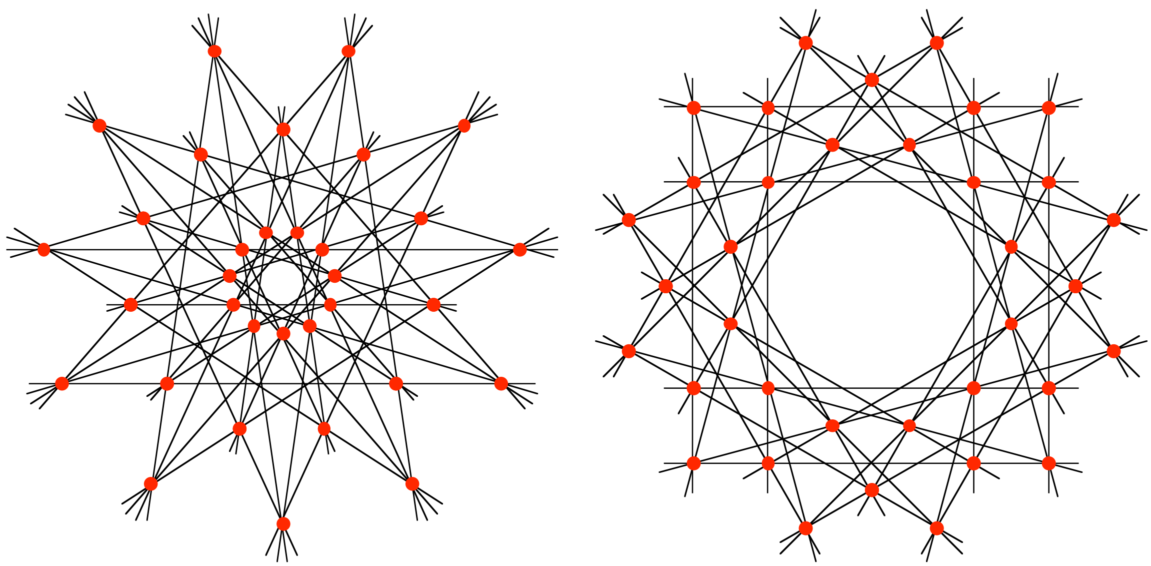


Figure 3.5.9. Two 3-astral 4-configurations.

2. What are the symbols for the objects in Figure 3.5.10. Devise a characteristic path in each and find out.

3. Superimpose each object in Figure 3.5.10 with a copy rotated 12° about the center. What is the result? Can you find a symbol for it?

4. Find the dual configurations of the ones in Figure 3.5.9.

5. Find a symbol for the 4-configuration in Figure 3.5.11.

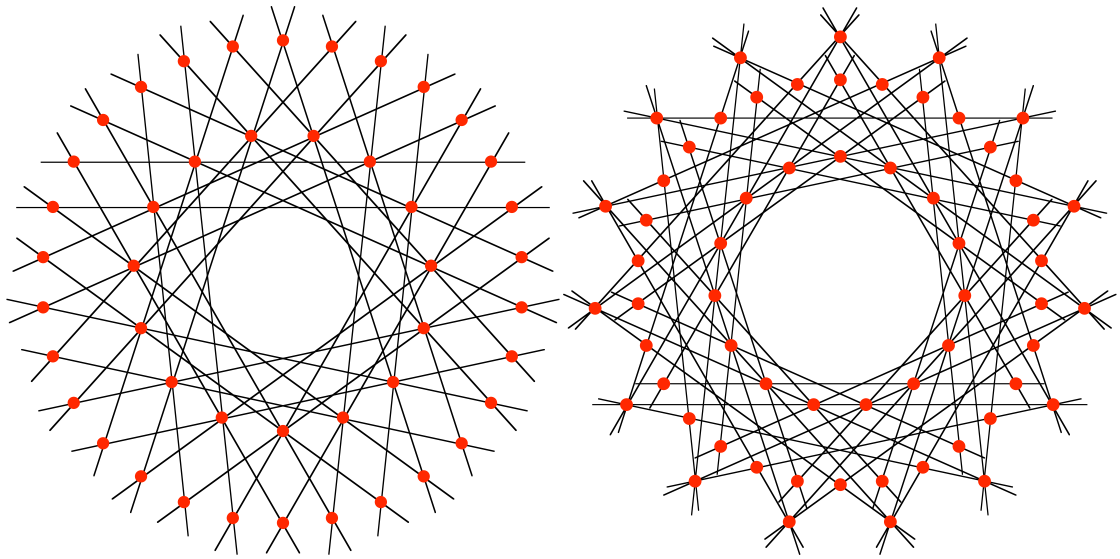


Figure 3.5.10. Not configurations!

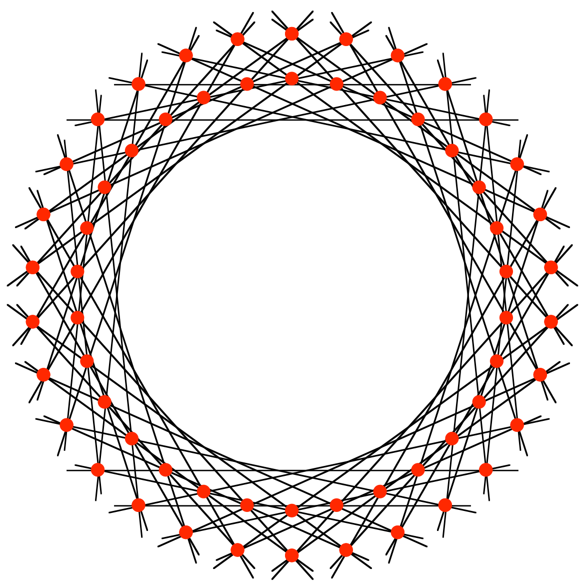


Figure 3.5.11. A configuration (60_4).