3.4 EXISTENCE OF GEOMETRIC 4-CONFIGURATIONS

We start with a quick summary description of the construction methods detailed in Section 3.3.

The (5m) construction is illustrated in Figure 3.3.1. It starts with an arbitrary (m_3) configuration and yields an $((5m)_4)$ configuration.

The (5/2m) construction is illustrated in Figure 3.3.2. It starts with appropriate configurations $((2m)_3)$ and yields a $((5m)_4)$ configuration; the criteria for usable (m_3) configurations are given on page 3.3.3.

The (4m) construction starts with an astral configuration $((2m)_3)$ and yields a 4-orbit dihedral configuration $((4m)_4)$. As explained on page 3.3.7, it works for most (but not all) such configurations with $m \geq 5$.

The (6m) construction starts with a 3-orbit configuration $((3m)_3)$ and yields a 6-orbit configuration $((6m)_4)$. It assumes that $m \geq 3$ is odd. Some details are given on page 3.3.6.

(3m+) denotes the construction described in detail on page 3.3.18. It starts with an (m_4) configuration and yields an $((3m+p+q)_4)$

Deleted Unions constructions (DU-1) and (DU-2). Using (DU-1), from suitable configurations $C_1 = ((n_1)_4)$ and $C_2 = ((n_2)_4)$ we obtain a configuration with $n_1 + n_2 - 1$ points and as many lines. In particular, we can go from any (n_4) to $((2n-1)_4)$. For (DU-2) we delete two disjoint lines and two unconnected points, and obtain $((2n-2)_4)$ from (n_4).

In addition to these, we use the notation (t-A.m) for the multiastral configuration with t orbits and with symmetry group d_m. This implies that each orbit has m points. Details of these configurations and the notation used for them appear in Section 3.5. (2-A.m) denotes astral configurations. If no other indication is given, the references are to the "trivial" choices of parameters such as (1,2,3,1,2,3) or (1,2,3,4,2,1,4,3).

It is relatively simple to show that (n_4) configurations exist for all $n \geq 210$. Indeed, by the (5m) construction there is for each $m \geq 9$ a $((5m)_4)$ configuration with $p = m$ parallel lines. It follows by the (3m+) construction that for all $m \geq 9$ and $1 \leq p \leq m$ there exists a $((15m+p)_4)$ configuration. Since $15m = 5(3m)$, by the (5m) construction we can add $p = 0$ to the range of p. Thus (n_4) configurations exist all values of n such that $15m \leq n \leq 16m$; for $m \geq 14$ these ranges are contiguous or overlapping, and so the claim is established.

For smaller values of n we have to rely on the various constructions described above and in Section 3.3. We found it simplest to arrange the necessary data in a table (Table 3.4.1) in which we list examples of configurations (n_4) for each n. In most cases there are other configurations we could have listed — the present choice is largely accidental.

n	Reference or explanation
18	(6m) for m = 3; Figure 3.3.4
19	Not known
20	(4m) for m = 5; Figure 3.3.9;
21	(3-A.m) , 7#(3,2,1,3,2,1); Figure 3.2.1
22	Not known
23	Not known
24	(2-A.m), 12#(5,4,1,4); Figure XXX; (3-A.m)
25	(5/2m) for m = 10; starting with $(10_3)_{10}$. Figure 3.3.2.
26	Not known
27	(3-A.m)
28	(4m) for m = 7; Figure 3.3.10
29	Bokowski (unpublished)
30	(3-A.m)
31	Bokowski (unpublished)

32	(4m) for m = 8. Figures 3.3.11 and 3.3.12
33	(3-A.m)
34	(DU-2) from two (18_4)
35	(5/2.m) for m = 7; starting from a (14_3) shown Figure 3.3.3
36	(2-A.m) several possibilities with m = 18; (3-A.m), (4-A.m)
37	Not known
38	(DU-2) from two (20_4)
39	(3-A.m)
40	(4-A.m)
41	(DU-1) from (21_4) Figure 3.3.16
42	(3-A.m)
43	Not known
44	(4-A.m)
45	(5-A.m) for m = 9; e.g. 9#(1,2,3,4,2,1,2,3,4,2); (3-A.m)
46	(DU-2) from (24_4) Figure 3.3.18
47	(DU-1) from (24_4)
48	(2-A.m); (3-A.m); (4-A.m); (6-A.m)
49	(7-A.m); Figure 3.4.1.
50	(5-A.m), 10#(1,4,3,2,3,1,4,3,2,3)
51	(3-A.m)
52	(4-A.m)
53	(DU-1) from (27_4)
54	(3-A.m)
55	(5-A.m)
56	(4-A.m)
57	(3-A.m)
58	(DU-2) from (30_4)
59	(DU-1) from (30_4)
60	(2-A.m); (3-A.m); (4-A.m); (5-A.m); (6-A.m) 10#(1,3,2,4,2,3,4,1,3,2,3,2)
61	(DU-1) applied to (21_4) and (41_4)

62	(DU-1) applied to (21_4) and (42_4)
63	(3-A.m)
61 – 63	(3m+) from (20_4)
64 – 66	(3m+) from (21_4)
67	(DU-1) from (33_4) and (35_4), obtained by (5/2m) for m = 14.
68	(4-A.m)
69	(3-A.m)
70	(5-A.m)
71	(DU-1) from (36_4)
72	(2-A.m); (3-A.m); (4.-A.m); (6-A.m)
73 – 76	(3m+) from (24_4), p+q = 4
75	(5/2m), m = 30; (5-A.m)
76 – 80	(3m+) from (25_4) = (5/2m), m = 10, p+q = 5
81	(3-A.m), m = 27; (9-A.m), m = 9, 9#{3,4,2,1,4,1,4,3,2,3,4,2,1,4,1,4,3,2}
82 – 87	(3m+) from (27_4), p + q = 6
88 – 95	(3m+) from (29_4), p + q = 6
91 – 98	(3m+) from (30_4), 30#(4,6,9,4,6,9), p + q = 8
99	(3-A.m), m = 33;
100 – 105	(3m+) from (33_4), p + q = 6
106 – 112	(3m+) from (35_4) = (5/2m), m = 14, p + q = 7
109 – 114	(3m+) from (36_4), 12#(1,2,3,1,2,3), p + q = 6
115	(5/2.m), m = 46; (5-A.m)
116	(4-A.m), m = 29
117	(3-A.m), m = 39
118 – 123	(3m+) from (39_4), 13#(1,5,3,1,5,3), p + q = 6
121 – 128	(3m+) from (40_4) = (5/2m), m = 16, p + q = 8
127 – 132	(3m+) from (42_4), 14#(1,3,5,1,3,5), p+q = 6
133 _ 139	(3m+) from (44_4), 11#(1,2,5,4,2,1,4,5), p+q = 7
136 – 144	(3m+) from (5/2m) = (45_4), m = 18, p+q =9
145 – 152	(3m+) from (48_4), 12#(1,2,5,4,2,1,4,5), p+q = 8

151 – 160	(3m+) from (50_4) = (5/2m), m = 20, p+q = 10
157 – 164	(3m+) from (52_4), 13#(1,2,5,4,2,1,4,5), p+q = 8
165	(5m) (33_3)
166 – 173	(3m+) from (55_4), 11#(1, 2, 3, 4, 5, 1, 2, 3, 4, 5), p+q = 8
172 – 177	(3m+) from (57_4), 19#(1,4,7,1,4,7), p+q = 6
178 – 185	(3m+) from (59_4), p+q = 8, Figure 3.3.17
181 – 192	(3m+) from (60_4) = (5/2m), m = 24, p+q = 12
193 – 200	(3m+) from (64_4), 16#(1, 3, 7, 5, 3, 1, 5, 7), p+q = 8
199 – 207	(3m+) from (66_4), 11#(1, 2, 3, 4, 5, 4, 2, 1, 4, 3, 4, 5), p+q = 9
208 – 213	(3m+) from (69_4), 23#(1,3,5,1,3,5), p+q = 6
211 – 224	(3m+) from (5m), m = 14 = p+q
225	(5m) from (45_3)
226 – 240	(3m+) from (5m), m = 15 = p+q
241 – 256	(3m+) from (5m), m = 16 = p+q
256 – 272	(3m+) from (5m), m = 17 = p+q
271 – 288	(3m+) from (5m), m = 18 = p+q
286 – 304	(3m+) from (5m), m = 19 = p+q.

Table 3.4.1. Desriptions of the construction of (n_4) configurations for $n \leq 304$.

The arguments presented above, together with the data in Table 3.4.1, constitute a proof of Theorem 3.2.4.

The known constructions explained above for the configurations (n_4) with small n (such as 18, 20, 21, 24, 25) all rely on r-fold rotational symmetry with $r \geq 3$. As a consequence, none of these constructions can be carried out in the *rational* projective plane. While there is no proof available showing that some or all these configurations are not realizable in the rational projective plane, it is a challenging problem to decide for which n is such a realization possible. An easy argument shows that if we start with rational configurations then (5m) constructions can be performed so as to yield rational configurations. Similarly for (5/2m) and (3m+) constructions.

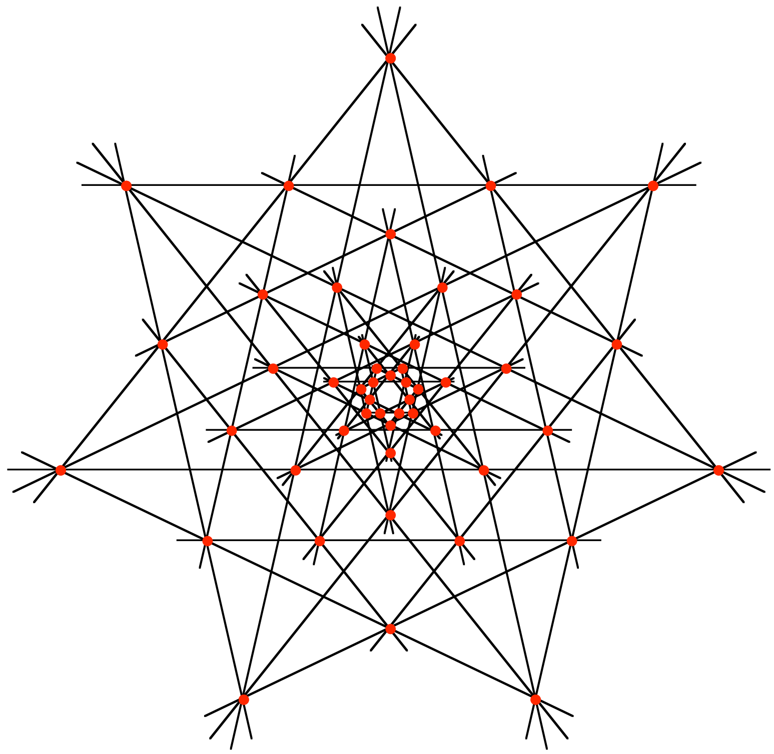

Figure 3.4.1. A (7-A.m) configuration (49_4), with symbol 7#(2,1,2,1,3,2,3,2,1,2,1,3,2,3).

Exercises and problems 3.4

1. Decide whether a suitable affine (or projective) image of the (18_4) configuration shown in Figure 3.3.4 can be put in the rational plane.

2. Determine for which n can one find a configuration (n_4) in the plane over a quadratic extension of the rationals.