3.3 CONSTRUCTIONS OF GEOMETRIC 4-CONFIGURATIONS

The fact that the first graphic realization of *any* (n_4) configuration (see Figure 3.2.1) is less than twenty years old attests to the difficulties that have to be overcome in realizations of such configurations in any intelligible manner. One reason for this situation is that an (n_4) geometric configuration implies the (non-trivial) satisfaction of 2n collinearity conditions, while on the other hand, any finite set of n points (not all collinear) has an affine image that depends on 2n – 6 parameters. Hence there must be some dependences – obvious or hidden – between the collinearity conditions in every geometric configuration (n_4). For a relevant discussion of this topic see Michalucci and Schreck [M18].

In contrast to the situation concerning (n_3) configurations we have presented in Section 2.4, there is no reasonable method or algorithm to go from a combinatorial configuration (n_4) to a topological or geometric one — even if any of these does exist. Nor are any criteria known to distinguish topological configurations which admit geometric realizations from those that do not. Hence, if we wish to find geometric 4-configurations we are, by necessity, forced to resort to more or less *ad hoc* arguments. This does not preclude constructing by the same method large (even infinite) families of examples; however, *finding* such methods or isolated examples is more of an art than a deductive science.

In this section we shall describe several kinds of such constructions. The various families or constructions will be designated in the form (sm), where s is a suitable integer (or another short symbol); the reason for such a name is that for appropriate values of m, the construction leads to a configuration (n_4) with n = s m (or some other value that depends on m).

Following this preamble, let's turn to some concrete cases. In most instances, the construction starts from some given configuration and yields a 4-configuration.

The first construction, which we call **(5m),** starts with an arbitrary (m_3) configuration C; in the example in Figure 3.3.1 this is the (9_3) configuration shown with blue points and lines. We select in the plane a line L (heavy black line in Figure 3.3.1) which misses all the points of C and is neither parallel nor perpendicular to any line determined by any two points of C. We construct three additional copies of C by stretching C through three different ratios in the direction perpendicular to L; only one such copy is shown (red points and lines) in Figure 3.3.1 in order to avoid crowding. The resulting configuration C* consists of the four replicas of C, together with the m intersection points of C with L (shown as hollow dots, which are also intersection points with L of the copies of C), and of the m lines perpendicular to L (shown dashed green) which pass through the points of C (and the other copies). Hence this construction yields a configuration C* of type (n_4), with $n = 5m$. Since — by Theorem 2.1.3 — (m_3) configurations are well-known to exist if and only if $m \geq 9$, this establishes the existence of configurations (n_4) for all $n \geq 45$ which are divisible by 5. Very important for the sequel is the observation that, as follows from the construction, each such configuration C* contains a set of m parallel lines. Moreover, this construction yields "movable" configurations in the sense explained in Section 5.7.

It should be noted that this construction —as well as the ones discussed below — leads in some cases to unwanted incidences, that is, to *prefigurations*. However, this can in all cases be avoided by selecting appropriate parameters for the construction.

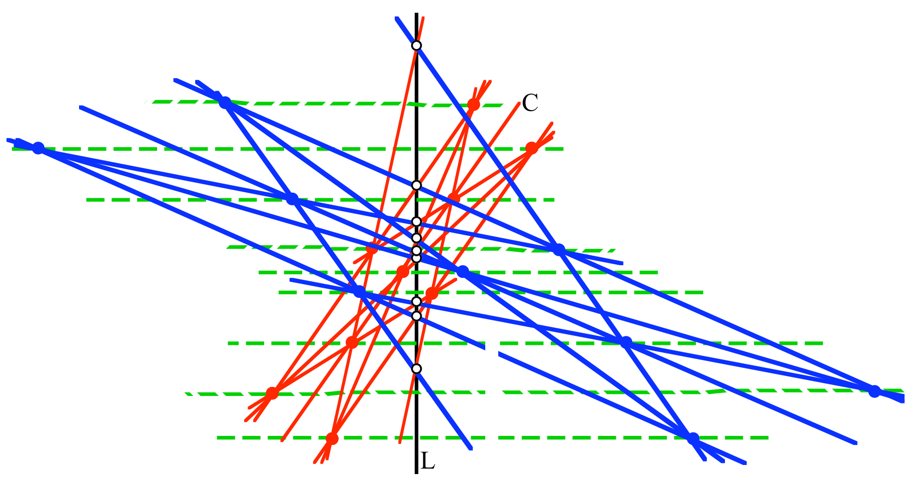


Figure 3.3.1. An illustration of the (5m) construction.

Our second construction is called **(5/2m).** It starts with a $(2m_3)$ configuration C that has a line L of mirror symmetry with the following properties: No point of C lies on the mirror L, no point on L belongs to more than two lines of C, and no line of C is perpendicular to L. It follows from the mirror property of L that there are m points of L at which pairs of lines of C meet. From C another copy is obtained by shrinking C towards L by a certain factor f (say f = ½), and then adding the m intersection points of the lines of the two copies with L, and the m lines perpendicular to L that pass through the points of the two configurations. This is illustrated for (10_4) and (12_4) in Figure 3.3.2, yielding configurations (25_4) and (30_4), respectively. We note that this construction also yields configurations $(5m_4)$ with m parallel lines. Moreover, this construction is **movable**, that is, nontrivial parts of it can be changed in a continuous manner without changing other nontrivial parts. (As already mentioned, we shall discuss movable

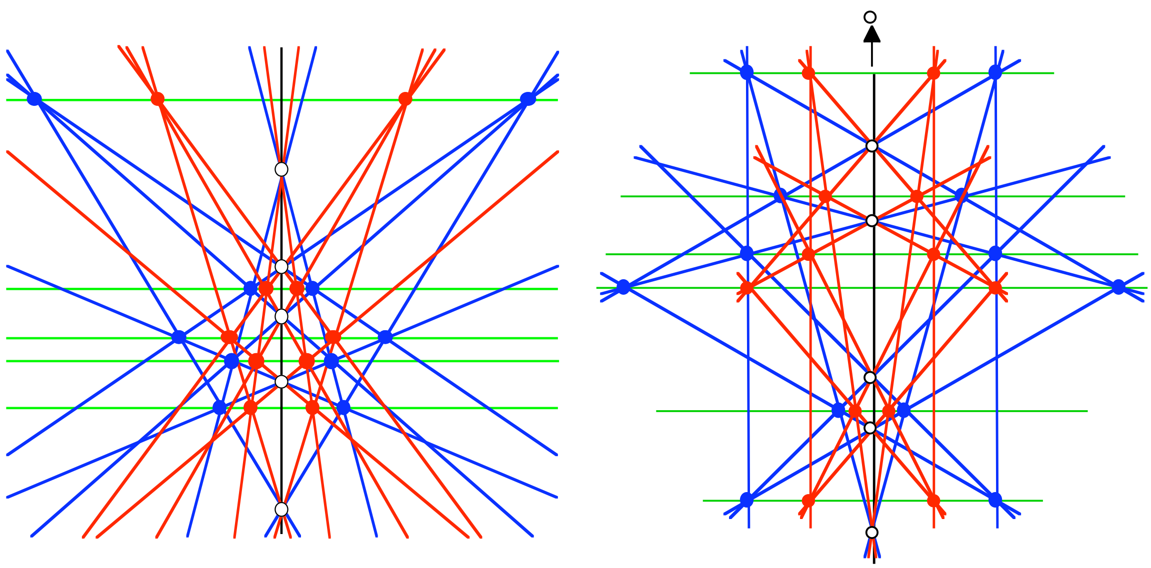


Figure 3.3.2. A (25_4) configuration with five parallel lines, and a (30_4) configuration with six parallel lines. The one at left starts with a (10_3) configuration, the other one with a dihedral astral (12_3) configuration (blue points and lines); copies of these are obtained by shrinking in ratio f = ½ towards the vertical line of symmetry (black line). Adding the five or six intersection points on the line of symmetry (hollow points, at right one at infinity) and five or six horizontal lines (green), completes these typical (5/2m) constructions.

configurations in Section 5.7.) This implies that the cross-ratio of the four points on each of the new (horizontal) lines (which is the same for all m of these lines) can be made equal to any predetermined value by an appropriate choice of f. In Figure 3.3.3. is shown an example of a (14_3) configuration to which the (5/2m) construction is applicable.

A construction of the only known (18_4) configuration was discovered very recently by J. Bokowski and L. Schewe; it is illustrated in Figure 3.3.4, and two different realizations of the same configuration are shown in Figure 3.3.5. This configuration can be considered the smallest member of an infinite family; we shall call this the **(6m) construction** or **family**. The idea to look for such a family came from noticing that the original rendering of the configuration (in Figure 3.3.4) contains a well-known subconfiguration (9_3), which we encountered in Figure 1.1.6, see Figure 3.3.6. This observation led to the construction of a whole family of analogous configurations. The (6m) construction is explained on hand of the typical case illustrated in Figure 3.3.7. The precise membership in the (6m) family has not been determined so far, but the family includes members (n_4) for every $n = 6m$ with odd $m \geq 3$. An additional example is shown in Figure 3.3.8.

The next case to consider is (20_4), first described in [G47], shown in Figure 3.3.9. It too was discovered as a single configuration, and the family to which it belongs was

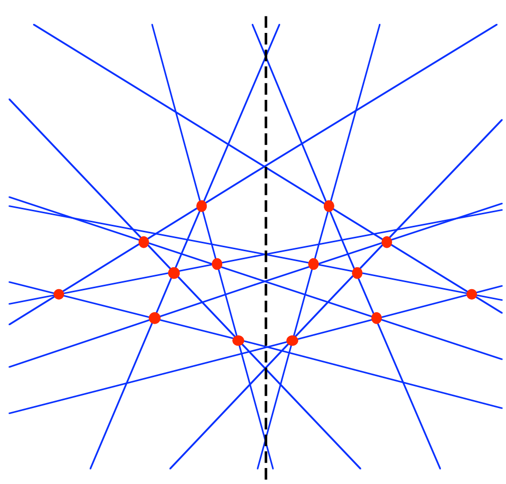


Figure 3.3.3. A (14_3) configuration that can be used to construct a configuration (35_4) by the method in Figure 3.3.2; this (35_4) configuration will have seven parallel lines.

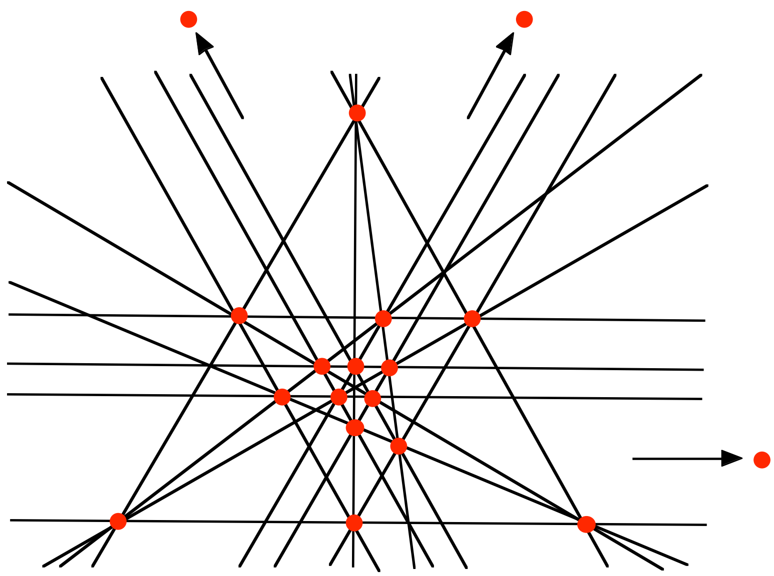


Figure 3.3.4. The only known geometric configuration (18_4) (after [B24]).

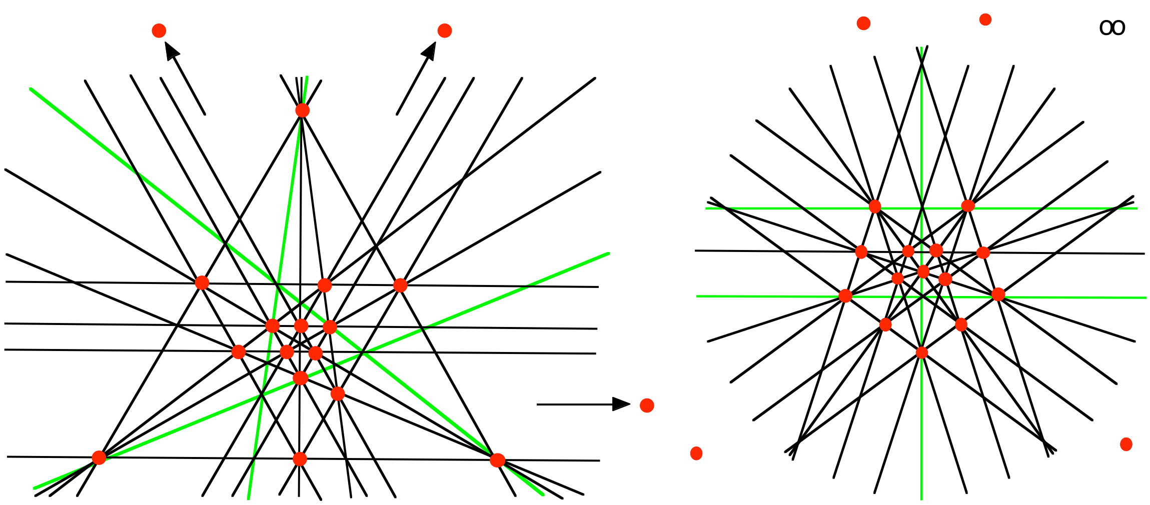


Figure 3.3.5. Two versions of the configuration (18_4) (red points and black lines) from Figure 3.3.4. In each version, adding the three green lines yields a simplicial arrangements of 21 lines (denoted A(21,2) in the catalog [G48]).

found only later; for obvious reasons we call this the **(4m) family** or **construction**. At the time of its discovery the construction seemed quite strange; particularly surprising is the use of two chiral configurations of the same handedness in order to obtain a mirror symmetric configuration. By now we have a much better understanding of the process, although a general proof of the validity of the construction is still not available.

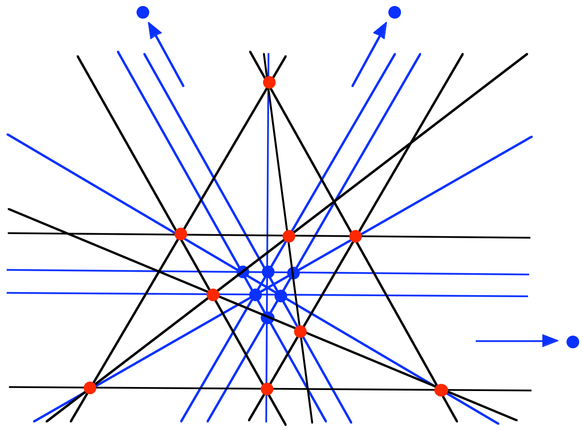


Figure 3.3.6. The configuration (18_4) from Figure 3.3.4 arises from a copy of the configuration $(9_3)_2$ taken from Figure 1.1.6, shown here in red points and black lines, by the addition of nine additional points and lines (shown in blue).

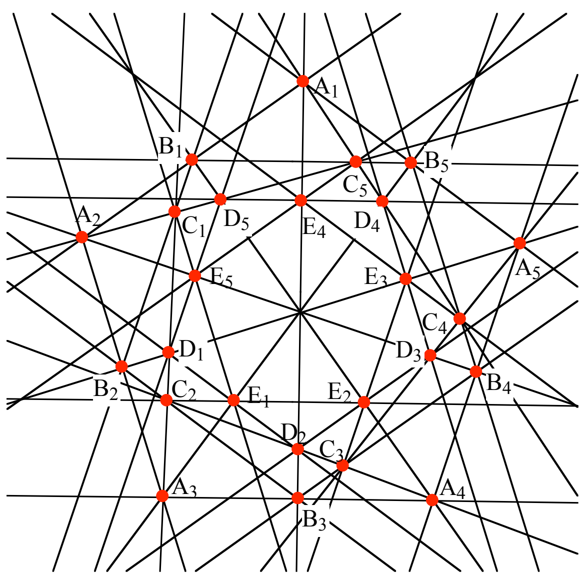


Figure 3.3.7. A (30_4) configuration in the (6m) family, the family that includes the configuration (18_4) in Figure 3.3.4. More generally, the construction of a $((6m)_4)$ configuration starts with a regular m-gon $A_1, \ldots A_m$, where $m \geq 3$ is odd. The point B_i is the midpoint of A_i and A_{i+1}, and C_i is selected on B_i,B_{i+1} so that the line C_i,C_{i+1} passes through A_{i+2}. Then D_i is determined on C_i,C_{i+1} so that $D_iC_i/C_{i+1}C_i = C_iB_{i+1}/B_iB_{i+1}$, and E_i is the midpoint of D_i and D_{i+1}. Finally, m points at infinity (not shown) are added, in the directions A_iA_{i+1}. Lines are: A_iA_{i+1}, B_iB_{i+1}, C_iC_{i+1}, D_iD_{i+1}, E_iE_{i+1} and A_iB_{i+2}. All subscripts are understood mod m.

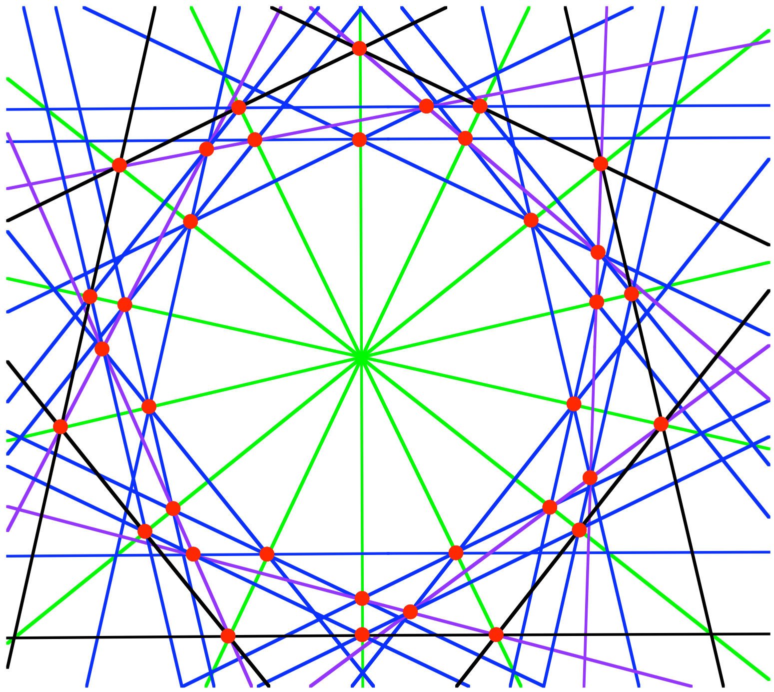


Figure 3.3.8. Shown here is the (6m) construction in the case of the regular 7-gon (black lines), leading to a (42_4) configuration. The seven points at infinity are again not shown; they are in the directions of the quadruplets of parallel black and blue lines.

Extensive experimental evidence led to the general understanding explained below. It leads to the conclusion that geometric configurations (n_4) exist for all $n = 4m$, with $m \geq 5$.

The construction can be described as follows; the explanation is illustrated in Figures 3.3.10 and 3.3.11. We start (see parts (a) in these illustrations) with an astral configuration m#(b,c;d), which we denote C, where $b \geq c > d > 0$ in the notation detailed in Section 2.6. We call this the "outer part" of the construction, and we note that the outermost points of the configuration C determine diagonals of span c. The other m points of C determine diagonals of span b; through each of the outermost points of C passes one of these diagonals. The lines of symmetry of the two diagonals of span c at each outermost point of C (one of these is shown by the green line in (a)) can be used as

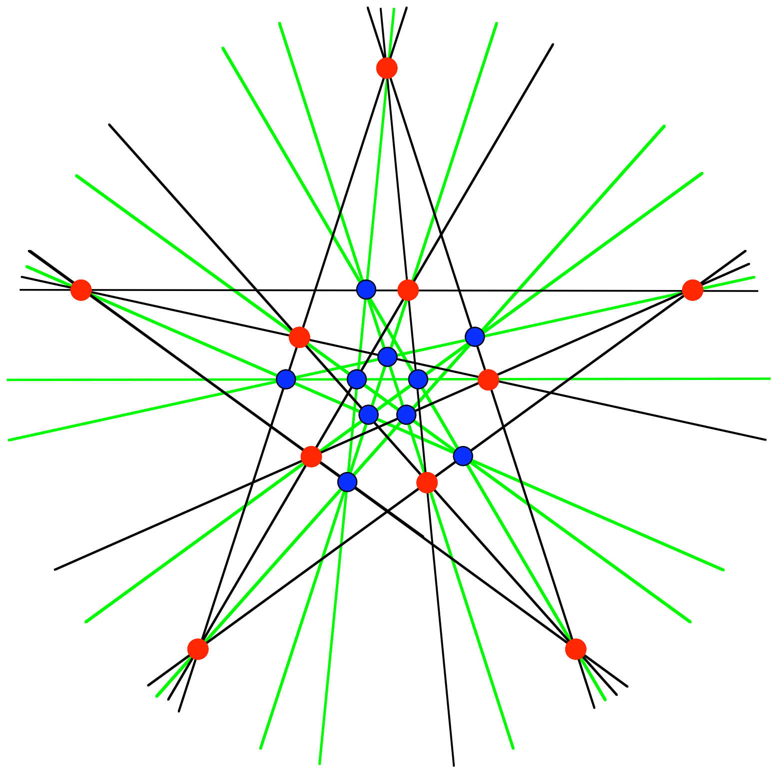


Figure 3.3.9. A (20_4) configuration belonging to the (4m) family. Here m = 5. The construction uses two astral (10_3) configurations; one is shown with red points and black lines, the other with blue points and green lines.

mirrors to reflect the m inner points of C as well as the diagonals of span b (see parts (b) and (c)). The m new points become the outermost points of the "inner part" of the configuration we are constructing. To find the last m (inner) lines, we connect each of the new "outermost" points with one of the original inner points – specifically, we connect it to the $(b+1)^{st}$ of these points, counting in the same orientation as used in calculating the symbol m#(b,c;d). This is indicated by the purple segments in parts (c). The new lines (see parts (d)) pass through previous intersections of two lines, creating the last m points of the $((4m)_4)$ configuration.

It is worth stressing that if the starting outer configuration is the selfpolar m#(b,b;d) as in Figures 3.3.9 and 3.3.10, then the inner configuration is another copy

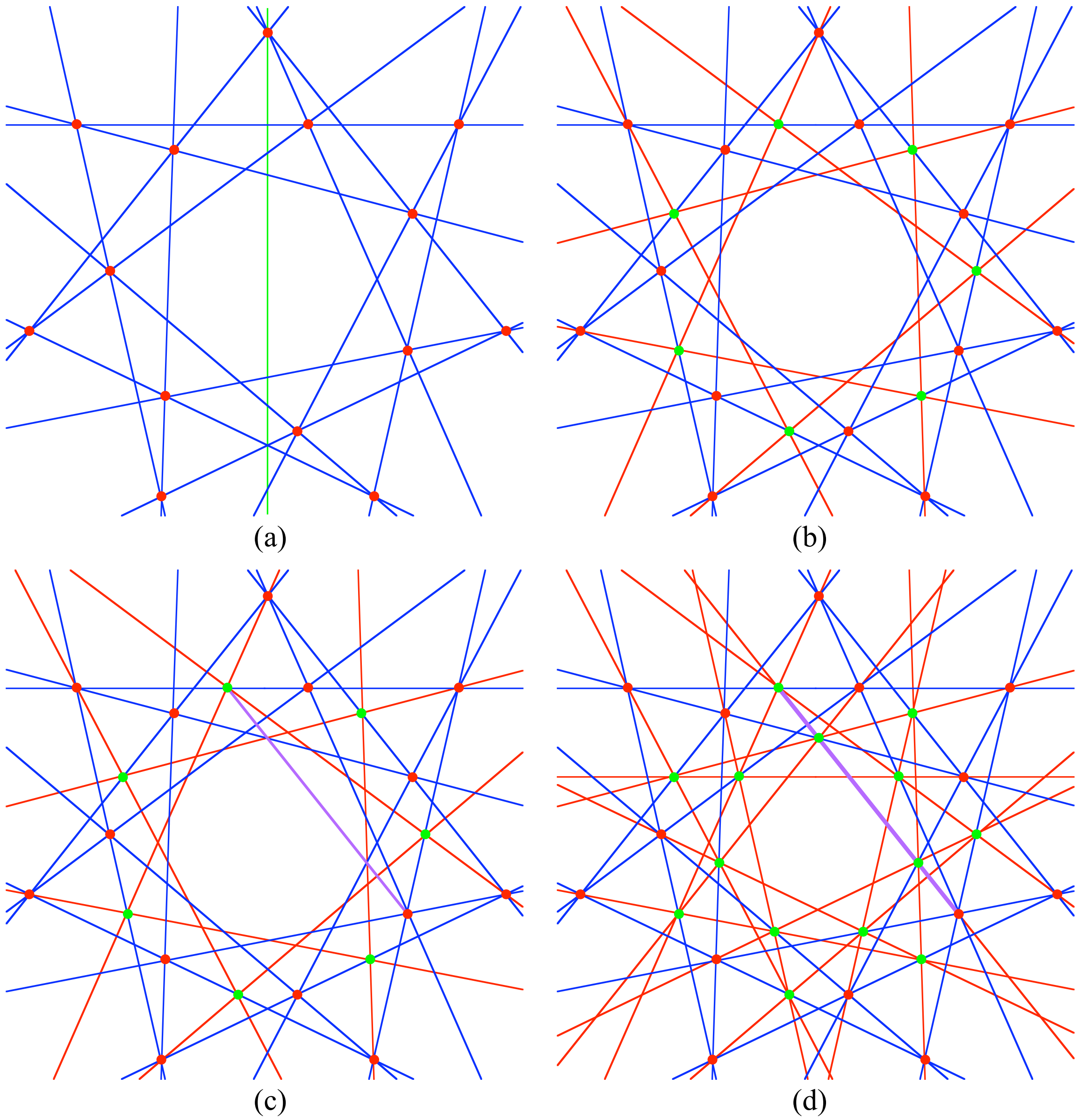


Figure 3.3.10. The steps in the (4m) construction of a (28_4) configuration from the (14_3) configuration 7#(2,2;1), as explained in the text.

(similar to the outer one) of m#(b,b;d). On the other hand, if $b > c$ as in the illustration in Figure 3.3.11, then the outer and inner parts are the two isomorphic and mutually polar configurations with symbol m#(b,c;d).

It is also worth mentioning that if $d > c$ then this construction (or any analogous one I could think of) does not seem to work. This includes the case of selfpolar configurations m#(b,c;d) with $d = (b + c)/2$.

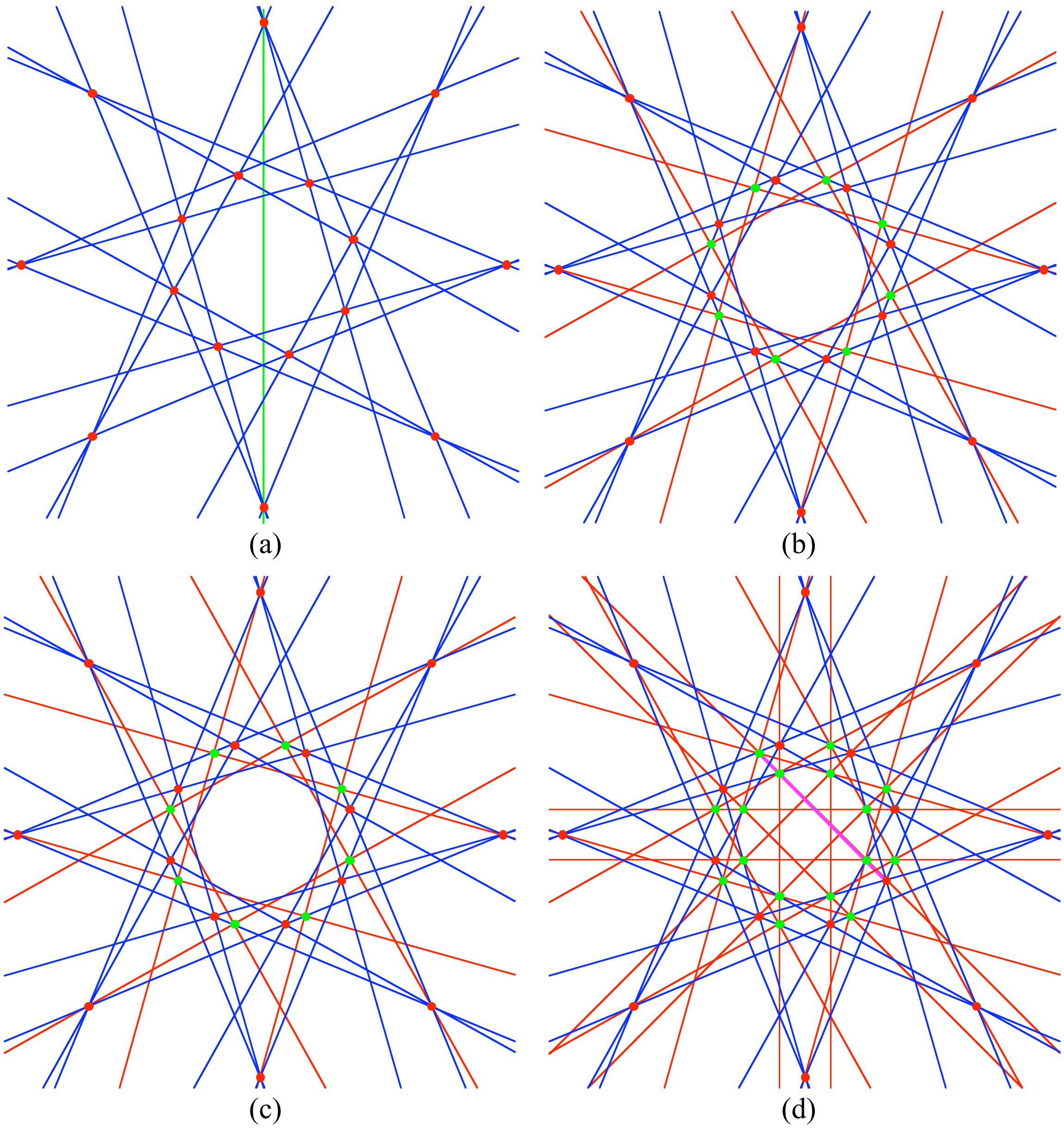


Figure 3.3.11. Another illustration of the construction. We start with a (16_3) configuration $8\#(3,2;1)$ and obtain a (32_4) configuration. Note that the outer and inner parts are not similar, but are polar to each other.

Another infinite family, which we designate as the **(5/6m) family**, is constructed as follows, starting from a 3-astral configuration (n_4) with $n = 6m$, where $m \geq 5$. Let us assume this configuration satisfies the following conditions:

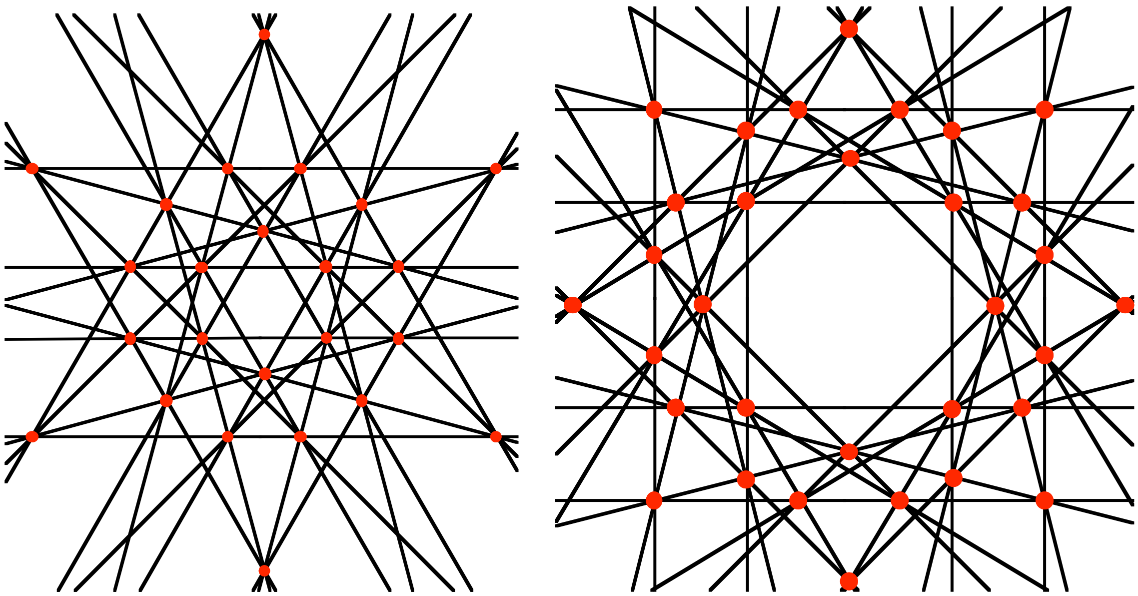


Figure 3.3.12. Configurations (24_4) and (32_4) from the (4m) family; the latter is different from the one in Figure 3.3.11.

(i) It has 2m-gonal dihedral symmetry.

(ii) The configuration is encoded by the symbol $(2m)\#(s_1,t_1;s_2,t_2;s_3,t_3)$, where s_i is the span of the i-th family of diagonals of the i^{th} level polygon P_i, and t_i is the order of the intersection point, counting from the midpoint of the diagonal s_i, and considering only diagonals of span s_i of the polygon P_i. For more details see Section 3.6.

(iii) s_1 and t_3 are distinct, and both are even; this implies $m \geq 5$.

(iv) t_1 and s_3 are odd.

(v) s_2 and t_2 have same parity.

Condition (iii) implies that both kinds of diagonals ending at points of P_1 have even lengths. Therefore, omitting every other point of P_1 and all the lines incident with these points leads to a loss of m points and 2m lines. (Note that, as shown in Figures 3.3.13 and 3.3.14, "level 1" does not mean that P_1 is the "outermost level".) The claim is that the above conditions imply that one can add to the remaining lines and points m suitable lines through the center to obtain a $((5m)_4)$ configuration. The examples in Figures 3.3.13 and 3.3.14 illustrate the construction.

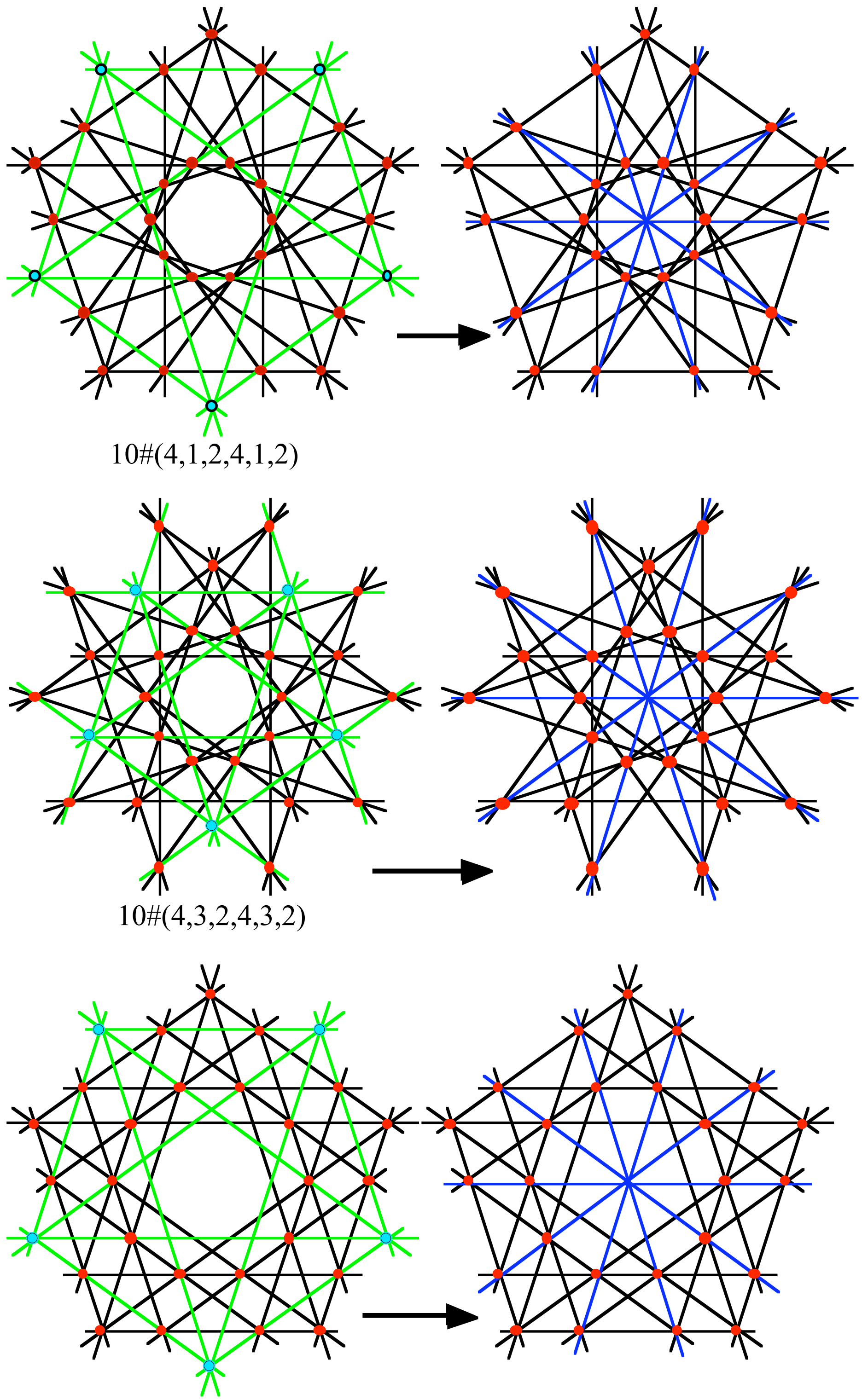
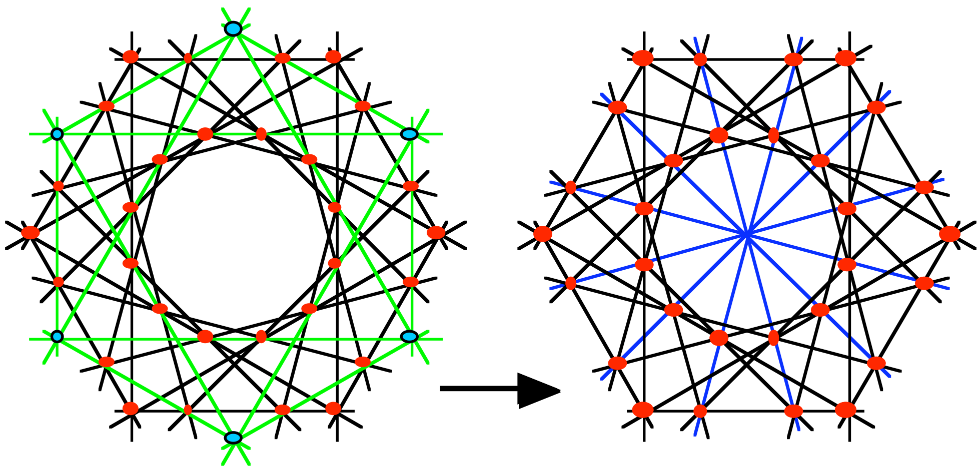
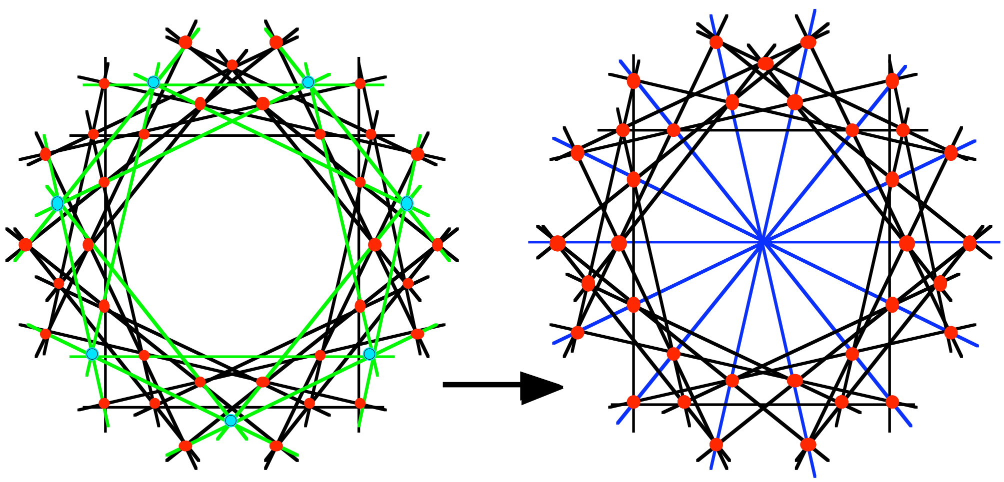


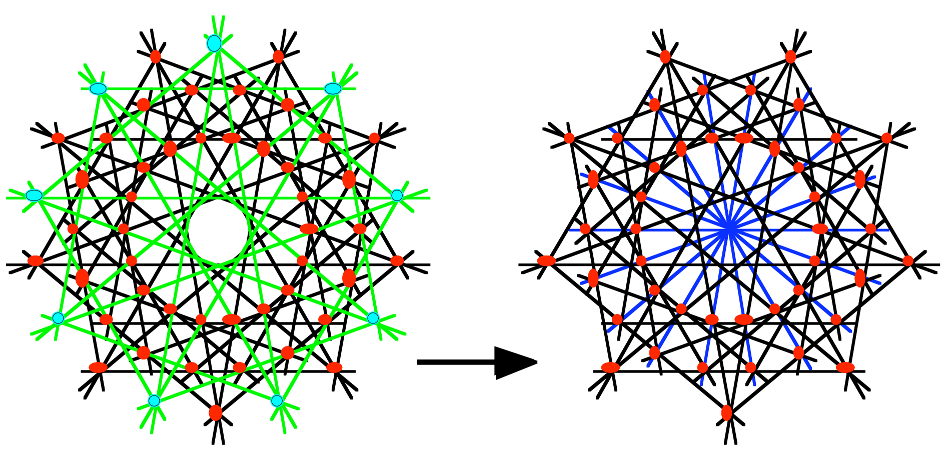
Figure 3.3.13. The three (25_4) configurations obtainable by the (5/6m) construction.



12#(2,1,4,2,1,4)



14#(4,3,2,4,3,2)



18#(4,1,5,1,7,8)

Figure 3.3.14. Configurations (30_4), (35_4) and (45_4) belonging to the (5/6m) family.

The reason the construction works is the following. All points of a 3-astral configuration with (2m)-gonal dihedral symmetry (that is, based on a regular (2m)-gon G) are on lines through the center that are mirrors for the symmetries of the configuration. The points of G are on mirrors that enclose angles that are multiples of p/m. More specifically, the two types of points on an s_i diagonal of P_1 are spaced an even multiple of p/m if s_i and t_i have the same parity, and an odd multiple of p/m if these parities are different.

In view of the above, conditions (iii), (iv) and (v) imply that viewed from the center, the points of level 1 are not aligned with the points of the other two levels. Hence these latter points are aligned, and provide the m lines required for the formation of a $((5m)_4)$ configuration.

Since configurations (2m)#(2,1,4,2,1,4) and (2m)#(2,3,4,2,3,4) exist for all $m \geq 5$, our existence claim is justified. In fact, for every $m \geq 5$ there exist additional possibilities. This is illustrated in Figure 3.3.13, for m = 5. This case was the starting points of this construction. Some of the configurations in Figure 3.3.13 were first constructed, independently and by *ad hoc* methods, by T. Pisanski and J. Bokowski.

A few other configurations in the (5/6m) family are illustrated in Figure 3.3.14.

The constructions we have seen so far started from given configurations that had to satisfy certain conditions. The resulting (n_4) configurations always had as n a composite number — more specifically, a multiple of 4, or 5, or 6. Now we shall describe constructions that are applicable quite generally, but are apt to give (n_4) configurations with other values of n.

The general construction, which we call the **(3m+) construction**, has the interesting feature that it is more easily visualized and explained in 3-space; the resulting configuration is then readily projected into the plane. We start with an (m_4) configuration C in the plane. We assume that this is the (x,y)-plane in a Cartesian (x,y,z)-system of coordinates, and that C has $p \geq 1$ lines parallel to the x axis, and $q \geq 1$ lines parallel to the y axis, such that no two of them have a point of the configuration in common. (Note that by an affine transformation — which does not change incidences — any two sets of parallel lines can be made orthogonal. The orthogonality is assumed only in order to simplify the description.) We select a real number $h > 1$ and keep it constant throughout the discussion; it is convenient (but not necessary) to think of $h = 10$. We construct two copies of C. One is C', obtained from C by stretching C in ratio $(h–1)/h$ (that is, in fact, shrinking it) towards the y-axis, stretching it in ratio $(h+1)h$ towards the x-axis, and then translating it to the level $z = 1$. A schematic representation of a section parallel to the x-axis is shown in Figure 3.3.15.The other is C", obtained similarly but by using the ratio $(h+1)/h$ for stretching towards the y-axis, $(h–1)h$ for the ratio towards the x-axis, and translation to the plane $z = –1$. Thus, C' is obtained from C by the map $f(x, y, 0) = (x(h–1)/h, y(h+1)/h, 1)$, and C" by $g(x, y, 0) = (x(h+1)/h, y(h–1)/h, –1)$. It is easy to check that for each point $A = (x, y, 0)$ the points A, f(A) and g(A) are collinear, and that the points $h(A) = (0, 2y, h)$ and $h^*(A) = (2x, 0, –h)$ are collinear with them. Now, for any four points A_j $(j = 1, 2, 3, 4)$ of C that are on a line L parallel to the x-axis — that is, have the same y-coordinate — the point $h(A_j)$ will be the same since it does not depend on the x-coordinate. Therefore we can conclude that by deleting the line L from the configuration C and its parallels in C' and C", while adding the

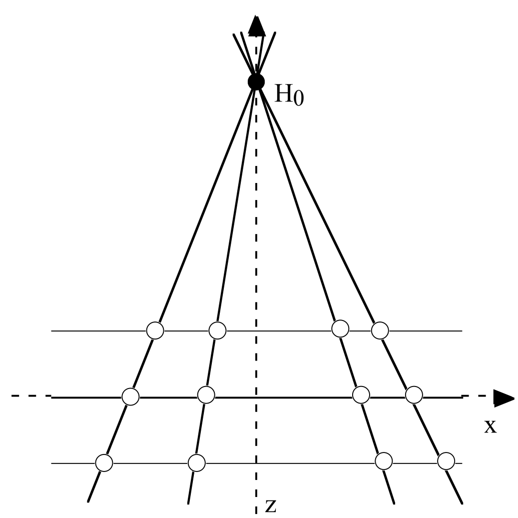


Figure 3.3.15. A schematic illustration of the (3m+) construction.

lines from A_j to $h(A_j)$, the points A_j and the corresponding points in C' and C" will remain incident with four lines, and the new point $h(A_j)$ will also be incident with four lines. We deleted three lines and added four, and also added one point. Thus, from the starting (m_4) configuration we obtained a configuration (n_4) where $n = 3m+1$. Analogously, any four points of C collinear on a line parallel with the y-axis may lead to an additional increase in the number of points and lines; the assumed disjointedness of the two families of parallels is needed here to assure that no 5-point lines arises. Proceeding similarly with some or all lines parallel to either the x-axis or the y-axis, we see that from (m_4) we can obtain configurations (n_4) for each n such that $3m + 1 \leq n \leq 3m + p + q$.

Next, we have the **deleted unions constructions** (DU-1) and (DU-2). Consider any configurations $C_1 = ((n')_4)$ and $C_2 = ((n'')_4)$, such that the cross-ratio of points of C_1 on a certain line coincides with the cross-ratio of lines through a certain point of C_2. Then omitting the line and the point in question, and adjusting the positions and sizes of the deleted configurations appropriately, we obtain a configuration with $n' + n'' - 1$ points. In every case one can use for C_2 a polar of C_1, to go from (n_4) to $((2n-1)_4)$. This is construction (DU-1); illustrations are provided in Figures 3.3.16 and 3.3.17. For (DU-2) we need to delete two disjoint lines and two unconnected points, respectively. An illustration is given in Figure 3.3.18. Again, the only requirement is that the cross-ratios of the appropriate quadruplets of points and of lines be equal.

With this we have completed the description of the various constructions that will enable us to find geometric configurations (n_4) for almost all values of $n \geq 18$. The proof of this assertion, which we have already formulated as Theorem 3.2.4, will be given in the next section. In it we shall utilize various configurations with very high symmetry — astral, multiastral, and other. Since their construction and properties are both interesting and complicated, we are not describing them here; instead, we shall devote to them several later sections.

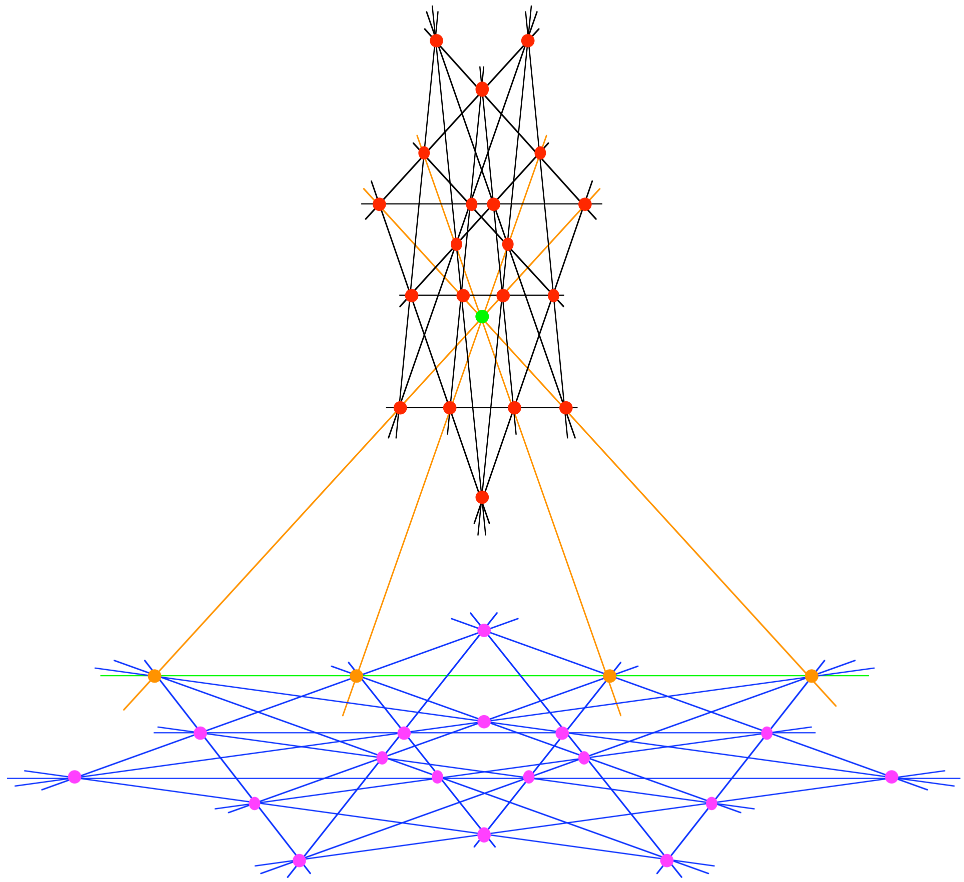


Figure 3.3.16. (41_4) from two copies of (21_4) using (DU-1).

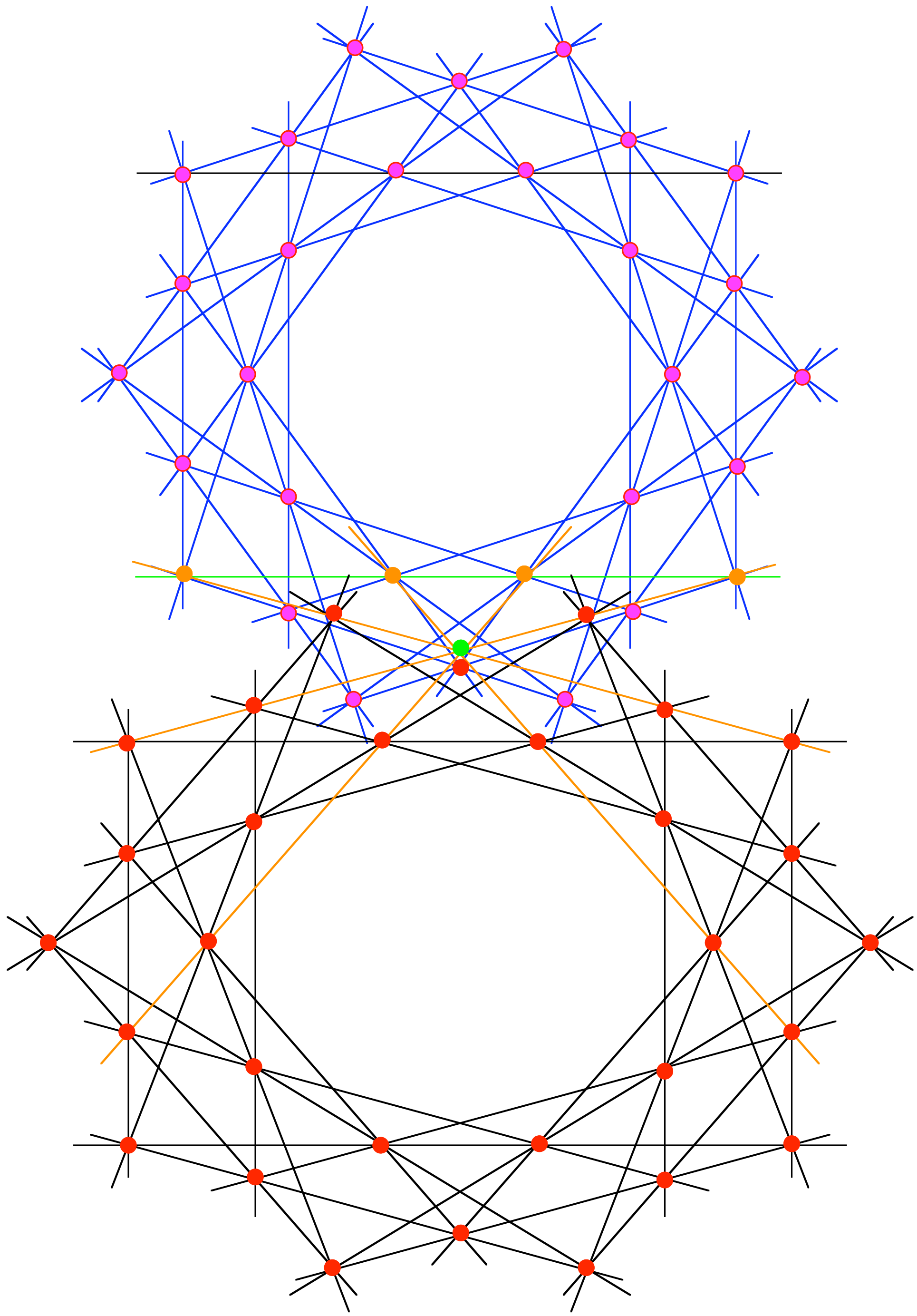


Figure 3.3.17. (59_4) from two copies of (30_4) using (DU-1). p+q = 8

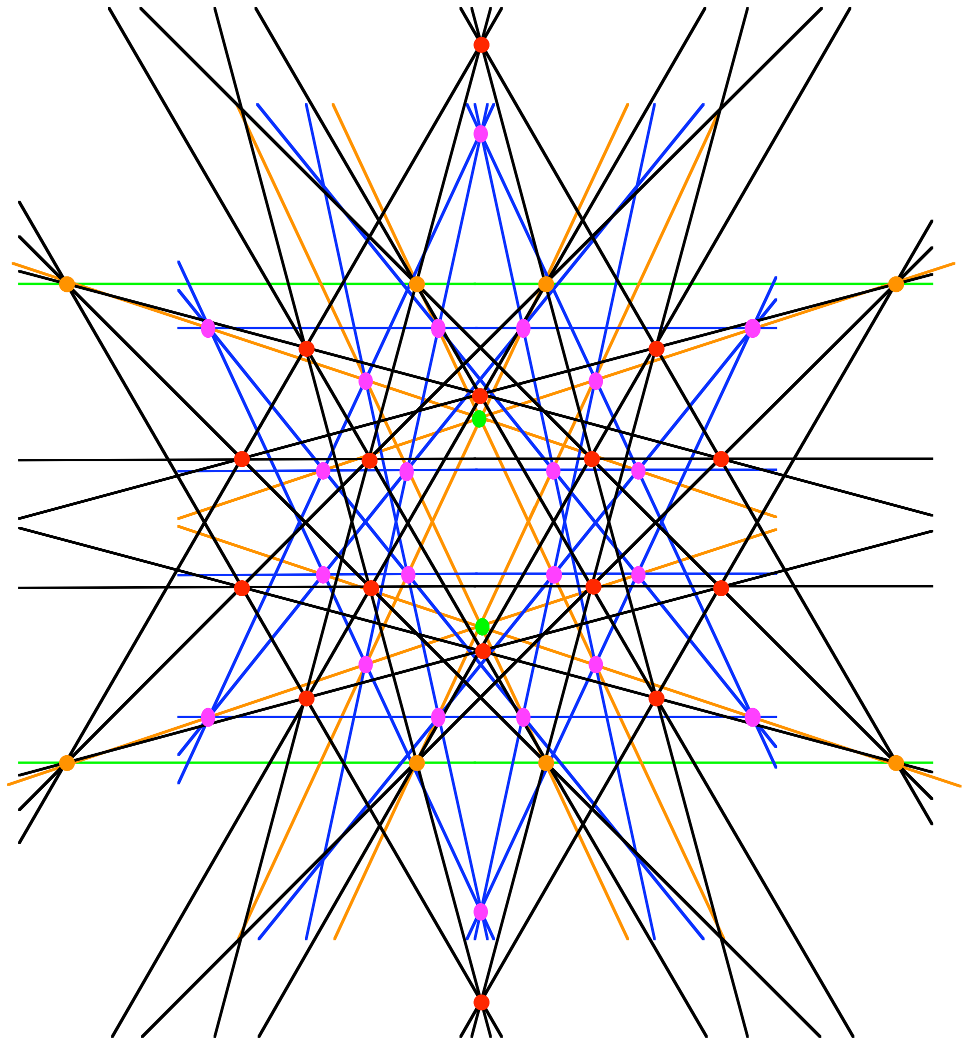


Figure 3.3.18. (46_4) from two copies of (24_4), using (DU-2).

Exercises and problems 3.3

1. Carry our the construction of the (35_4) configuration described in Figure 3.3.3.

2. Determine whether any of the three configurations (25_4) in Figure 3.3.13 are isomorphic.

3. Devise a general proof for the validity of the (4m) construction, as detailed in the text.

4. Formulate the analog of the (3m+) construction that leads from 3-configurations to 3-configurations. Illustrate by a simple example.

5. The (6m) construction is applicable to regular star-polygons as well. Explore the case of a pentagram, and of one of the regular star-heptagons.

6. Explain why the (DU-1) construction cannot be applied to get a (43_4) from (20_4) and (24_4).