3.2 EXISTENCE OF TOPOLOGICAL AND GEOMETRIC 4-CONFIGURATIONS

As mentioned in Section 3.1, both Brunel [B31] in 1897 and Merlin [M8] in 1913 discussed *geometric* 4-configurations in the real Euclidean plane, and were clear about the distinction between combinatorial and geometric configurations. However, neither did actually show a drawing of any geometric configuration.

The first published diagram of a geometric configuration (n_4) appeared only in [G50], published in 1990. It is reproduced here as Figure 3.2.1. As it happens, it is a realization of Klein's configuration (21_4), introduced in [K11] and mentioned in Section 3.1. The paper [G50] marked the beginning of research of *geometric* configurations (n_4); the results of these investigations form the topic of the remaining part of Chapter 3. The results are intimately connected to the study of topological configurations (n_4), and we shall first describe the known facts concerning these configurations.

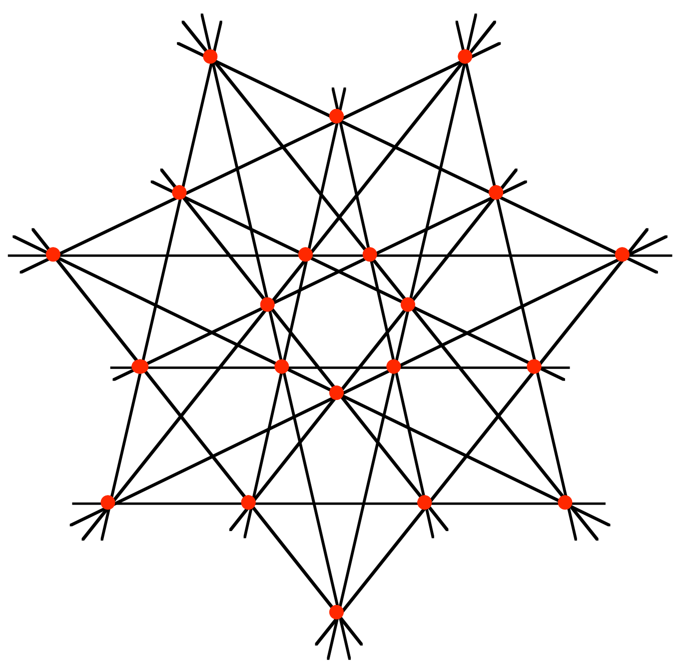

Figure 3.2.1. A geometric configuration (21_4).

The arguments given by Merlin [M8] to establish the non-existence of geometric configurations (n_4) for $n \leq 15$, do not carry over to topological configurations. However, we have:

Theorem 3.2.1. (Bokowski and Schewe [B23]) For $n \leq 16$ there are no topological configurations (n_4).

This is the best possible, since we also have

Theorem 3.2.2. (Bokowski, Grünbaum and Schewe [B22]) Topological configurations (n_4) exist for every $n \geq 17$.

In contrast to this situation, we have:

Theorem 3.2.3. (Bokowski and Schewe [B24]) For $n \leq 17$ there are no geometric configurations (n_4).

Theorem 3.2.4. (Bokowski and Schewe [B24]) There exist geometric configurations (n_4) for all $n \geq 18$ except possibly for $n = 19, 22, 23, 26, 37, 43$.

Theorems 3.2.3 and 3.2.4 demonstrate how the understanding of the (n_4) configurations has developed during the past twenty years. In [G50] it was conjectured that there are no geometric configurations (n_4) with $n \leq 21$ other than the configuration in Figure 3.2.1. Similar conjectures were repeated in various other publications, such as [G41], [G42], [G43]. However, the recent discovery (see [G47]) of a (20_4) configuration led to a modified conjecture, that geometric configurations (n_4) exist only for $n \geq 20$. But this was also short-lived, and was resolved in the negative by the discovery of a geometric (18_4) configuration by J. Bokowski and L. Schewe [B24]. Thus Theorems 3.2.3 and 3.2.4 settle the 20-years quest for the smallest geometric configuration (n_4).

The history of the Theorem 3.2.4 illustrates the rapid improvement in the understanding of configurations (n_4). The first version, in [G47], established that connected (n_4) configurations exist for every $n \geq 21$ except possibly if $n = 32$ or $n = p$ or $n = 2p$ or $n = p^2$ or $n = 2p^2$ or $n = p_1p_2$, where p, p_1, p_2 are odd primes and $p_1 < p_2 < 2p_1$. The number of exceptional cases was soon reduced (in [G41]) to a finite number: There are (n_4) configurations for all $n \geq 21$ except possibly if n has one of the following thirty two values: 22, 23, 25, 26, 29, 31, 32, 34, 37, 38, 41, 43, 46, 47, 49, 53, 58, 59, 61, 62, 67, 71, 77, 79, 89, 97, 98, 103, 113, 131, 178, 179. Newly found construction methods [G43] reduced the list of possible exceptions to the following ten values: 22, 23, 26, 29, 31, 32, 34, 37, 38, 43. All this was while the general belief was that 21 is the smallest number of points in an (n_4) configuration. After Theorem 3.2.3 was established, and additional constructions found, the result became that connected (n_4) configurations exist if and only if $n \geq 18$, except possibly if n has one of the eight values 18, 19, 22, 23, 26, 34, 37, 43. Finally, the discovery of a (18_4) configuration led to the result stated above [B24].

The proofs of Theorems 3.2.3 and 3.2.4 will be given in the next section; here we shall give outlines, and some details, of the proofs of Theorems 3.2.1 and 3.2.2.

The proof of Theorem 3.2.1 given in [B23] is easy for $n \leq 15$. The case $n = 16$ is much more complicated, and forms the bulk (six pages) of that paper. It follows a large number of *a priori* possible topological subconfigurations, and in each case leads to a contradiction. We have to refer the reader to the original paper. In contrast, the case $n \leq 15$ is easily explained, and for fixed k is applicable to all combinatorial configurations (n_k) with n sufficiently small. We present the proof from [B23] with only minor adaptations.

Assume that a combinatorial (n_k) configuration is realized by pseudolines in the projective plane. Due to the possibility of locally perturbing pseudolines at points that are not vertices of the configuration, we may assume that in the *arrangement* (see Appendix A2) generated by the perturbed pseudolines each vertex of the arrangement is incident with either k or 2 pseudolines. Since each of the former accounts for $k(k-1)/2$ pairwise intersections of pseudolines, the total number of vertices of the modified arrangement is $f_0 = n + n(n-1)/2 - nk(k-1)/2 = n(n - k^2 + k + 1)/2$. Similarly, the number of edges of the modified arrangement is $f_1 = n(n - k^2 + 2k - 1)$. From Euler's theorem for the projective plane it follows that the number of cells (faces) of the arrangement is $f_2 = f_1 - f_0 + 1 = n(n - k^2 + 3k - 5)/2$. On the other hand, arrangements of pseudolines have no digons, hence counting incidences of edges and cells yields $3f_2 \leq 2f_1$. Therefore we have $f(n) = -n^2 + nk^2 + nk - 5n + 6 \leq 0$ as a necessary condition for the existence of a topological realization of a combinatorial (n_k). For fixed k, this function $f(n)$ of n has its only maximum for $n = (k^2 + k - 5)/2$ and is decreasing for all larger n. Simple checking shows that for $k = 4$ we have $(4^2 + 4 - 5)/2 < 8$ and $f(15) = 6 > 0$, hence (n_4) is not topologically realizable for $n \leq 15$. Since $f(16) = -10 < 0$, this criterion is not applicable for $n = 16$. On the other hand, this result shows that there are no topologically realizable configurations (n_5) for $n \leq 24$, nor are there any topological (n_6) for $n \leq 36$.

Turning now to Theorem 3.2.2, the first thing to observe is that geometric configurations are, obviously, examples of topological configurations. Hence, assuming that Theorem 3.2.4 can be proved without reliance on Theorem 3.2.2 (as is in fact the case), we need only provide examples of topological configurations for those values of $n \geq 17$ for which there are no known geometric configurations. These values are n = 17, 19, 22, 23, 26, 37, 43. We shall now show such examples, together with a few others that we find appropriate for various reasons. Most of these examples are modified from [B22].

In Figure 3.2.2 we show a topological configuration (17_4) that is a realization of the configuration given by Table 3.2.1. This is taken from [B22], where a proof is outlined according to which this combinatorial configuration (17_4) is the only one admitting a topological realization. It should be noted that this realization has 4-fold rotational symmetry in the extended Euclidean plane. It is not known whether there are realizations with any symmetry in the Euclidean plane proper, or whether there are additional combinatorial automorphisms. Since the configuration is the only topologically realizable (17_4)

configuration, it is necessarily self-dual. (Although it seems not well-known, topological configurations in the projective plane do have dual configurations. This can be inferred from results in [G6].)

1	1	1	1	2	2	2	3	3	3	4	4	4	8	9	10	10
2	5	8	11	5	6	7	5	6	7	5	6	7	13	13	11	12
3	6	9	12	8	9	11	12	8	9	11	10	12	15	14	14	16
4	7	10	13	14	15	16	15	16	17	17	13	14	17	16	15	17

Table 3.2.1. A configuration table of the only (17_4) configuration that admits a topological realization.

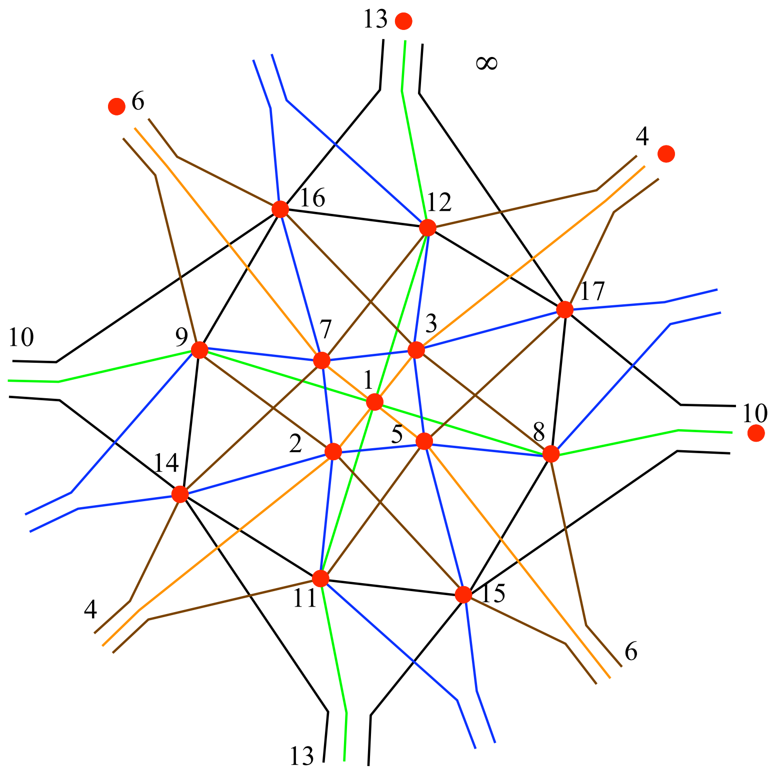

Figure 3.2.2. A topological configuration (17_4). It is a realization of the unique combinatorial configuration (17_4), specified in Table 3.2.1, that has a topological realization.

A topological configuration (18_4) is shown in Figure 3.2.3. This configuration is not isomorphic to the geometric configuration (18_4) we shall see in the next section, and it is not known whether is can be realized geometrically. On the other hand, it has a six-fold rotational symmetry.

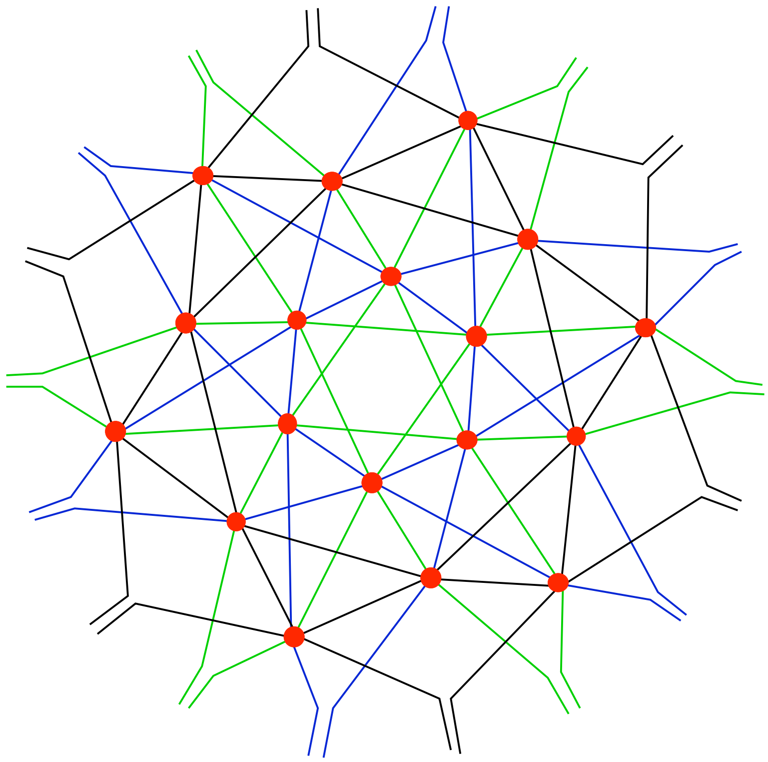

Figure 3.2.3. An example of a topological configuration (18_4) with six-fold rotational symmetry in the Euclidean plane. Adapted from [B22].

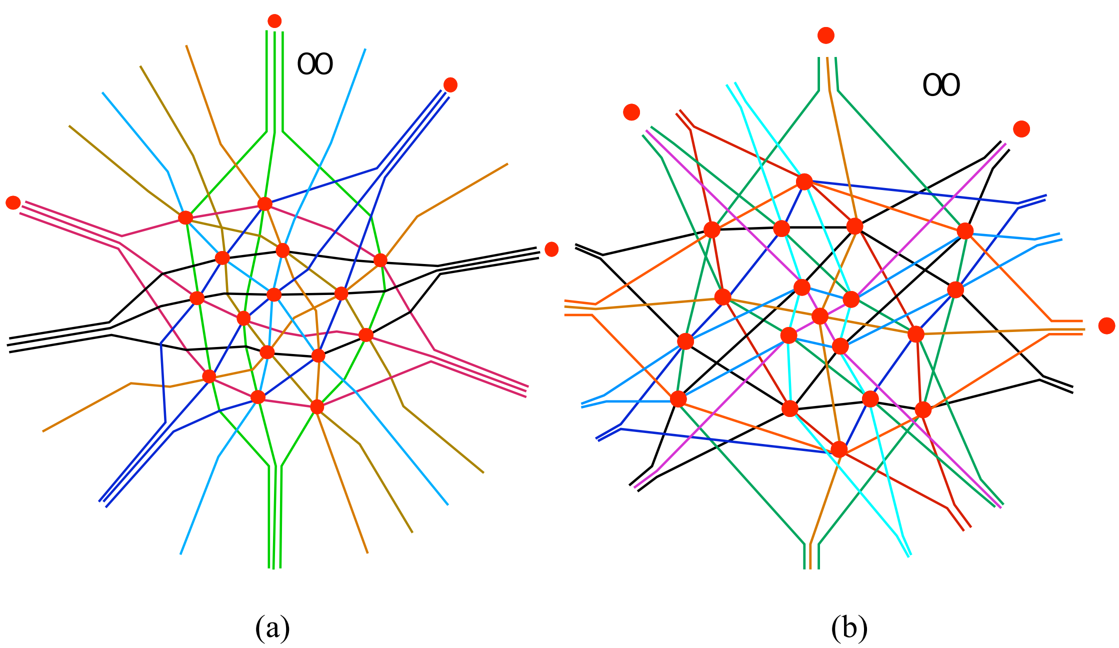

Figure 3.2.4. Topological configurations (a) (19_4) and (b) (23_4). Adapted from [B22].

In contrast, all known topological configurations (19_4) and (23_4) have only trivial symmetry groups. Examples of these configurations are shown in Figure 3.2.4.

The examples of topological configurations presented so far have been *ad hoc*, obtained essentially through (lots of) trial and error. Their rather ungainly appearance is a reminder of their genesis. In contrast, the examples of configurations (22_4) and (26_4) shown in Figure 3.2.5, are members of a systematic family: they are *topological* examples of *astral* configurations; the *geometric* members of the family will be studied in detail in several sections, starting with Section 3.5. The two examples in Figure 3.2.5 are representatives of configurations (n_4) possible for all even $n \geq 22$. In the terms of astral configurations we shall discuss in Sections 3.5 and 3.6, these configurations have *spans* 4 and 5; other possibilities exist, increasing in number with increasing n. Additional information will be given in the discussion of geometric astral configurations, and in Section 5.8.

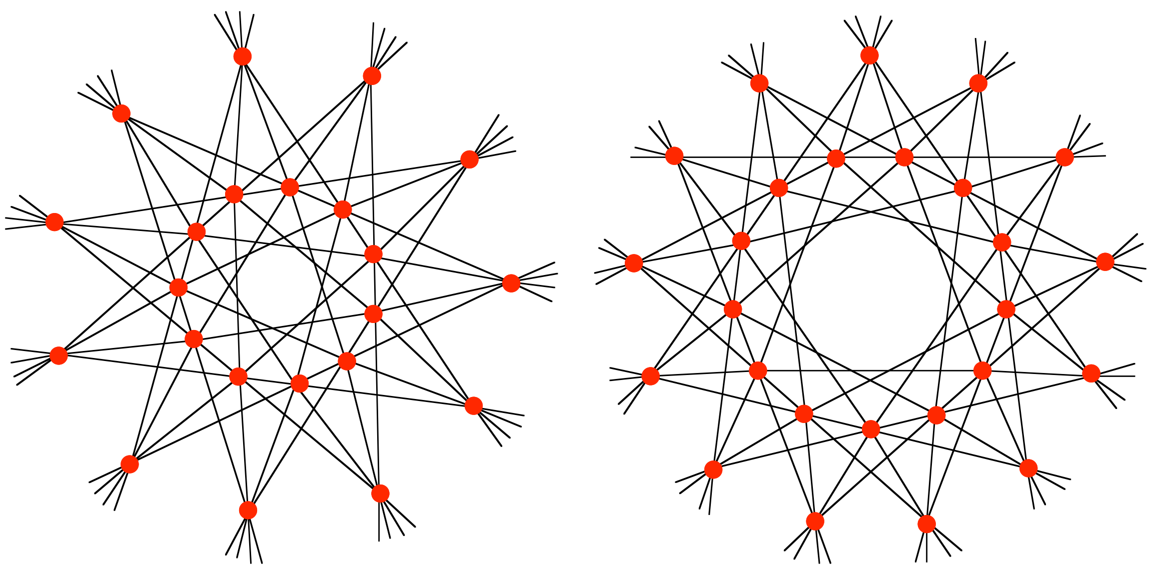

Figure 3.2.5. Topological configurations (22_4) and (26_4), with 11-fold resp. 13-fold dihedral symmetry. They are typical of topological astral configurations (n_4) possible for all even $n \geq 22$.

The examples we provide for (37_4) and (43_4) are special cases of a much more general construction, that is actually very simple. Assuming we have a (p_k) topological configuration, and a (q_k) topological configuration, for some $k \geq 2$, we can construct an (n_k) topological configuration, where $n = p + q - 1$, in the following way: Delete one pseudoline from the former configuration, and a point from the latter, and make the k pseudolines that are now incident with only $k-1$ points, pass through the k points that are incident with only $k-1$ pseudolines. In Figure 3.2.6 is shown the case of (24_4) and (20_4) geometric configurations, leading to a (43_4) topological configuration; the significant points and lines are shown in red. Another (43_4) configuration could be obtained by pairing in the same way an (18_4) configuration with a (26_4). The same kind of construction with (20_4) and (18_4) configurations (either geometric or topological) yields the last of the required topological configurations, (37_4); alternatively, the topological (17_4) could be paired with the geometric (21_4). A different topological configuration (37_4) is shown in [B22]. We shall revisit the same idea for construction of geometric configurations in the next section.

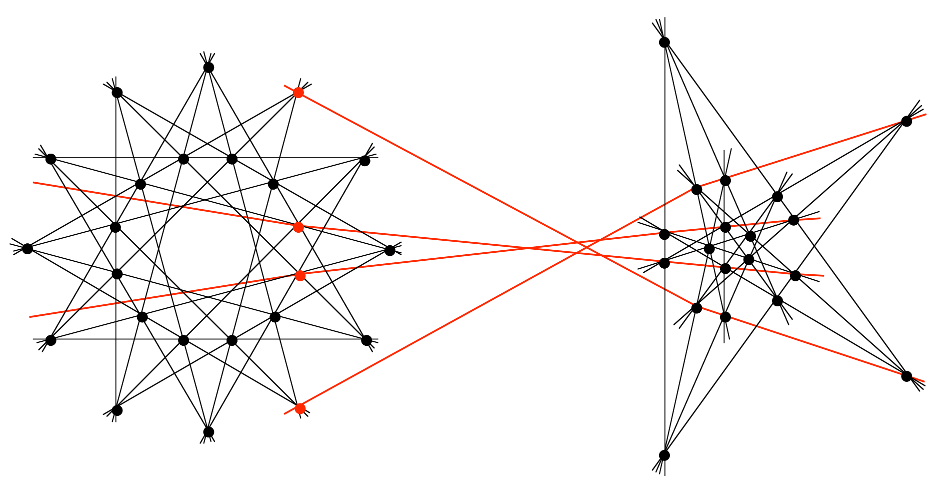

Figure 3.2.6. A topological configuration (43_4). The red points are collinear on the deleted line, the red (pseudo)lines were concurrent at a deleted point of the (20_4) configuration.

Since the combinatorial (n_4) configurations, for n = 13, 14, 15, 16, as well as a large majority of such configurations for $n \geq 17$, cannot be realized by topological configurations, two kinds of questions arise naturally.

First, what relaxation of incidence requirements would be sufficient to enable the construction of topological near-configurations realizing these combinatorial ones?

Second, what are the obstructions preventing topological realizations of some of the combinatorial configurations? For the smallest combinatorial configurations (such as (7_3), (13_4), (21_5), (31_6), (49_8), ...) the existence of *ordinary points* in any family of pseudolines not all of which pass through the same point (see Lemma 2.1.1) can be interpreted as such an obstruction. Indeed, it implies that these configurations cannot have topological realizations since all intersections of pairs of pseudolines would have to be "used up" in points incident with multiple pseudolines, leading to an absence of ordinary points.

The inequality $-n^2 + nk^2 + nk - 5n + 6 \leq 0$ mentioned above as a necessary condition for the existence of an (n_k) topological configuration is another kind of obstruction. It shows that combinatorial configurations (n_k) with $n \leq k^2 + k - 5$ cannot be topologically realized. Since $n \geq k^2 - k + 1$ in all cases, that shows that for each k certain values of n lead to topologically non-realizable configurations (n_k). However, it must be noted that for quite a few of the relevant pairs n, k there exist no combinatorial configurations either – and there is no necessary and sufficient criterion for their existence.

Exercises and problems 3.2.

1. Construct the configuration table dual to the one in Table 3.2.1, and show that it is realized by the configuration in Figure 3.2.2.

2. Prove that the topological (18_4) configuration in Figure 3.2.3 is not isomorphic to the geometric (18_4) configuration shown in Figure 3.3.4 and 3.3.5.

3. Find a topological (18_4) configuration that is dual to the one in Figure 3.2.3.

4. Construct a topological (37_4) configuration.

5. There seems to be no *a priori* reason that would preclude the existence of topological (19_4) or (23_4) configurations with halfturn symmetry. Do any exist?

6. Determine how many topological configurations (26_4) with dihedral symmetry d_{13} exist.

7. Which multiastral combinatorial configurations (n_4) have topological realizations?