3.1 COMBINATORIAL 4-CONFIGURATIONS

The history of configurations (n_4) is much shorter than that of configurations (n_3), and more easily told.

The first explicit mention of such configurations seems to be in a paper [K11, p. 440] by Felix Klein in 1879, which deals with quartic curves in the complex plane. He noted that there is a family of 21 points and 21 lines with incidences that make it into a (21_4) configuration, in our terminology — albeit in the complex plane. Although this particular configuration continued to interest mathematicians (various references can be found in Coxeter [C10], and Burnside [B32] discovered it independently), it did not have any noticeable direct influence on the study of (n_4) configurations in general. However, it did play later a significant role in the theory of geometric configurations, which we shall discuss in Section 3.2.

The first slightly more general treatment of such configurations was by Georges Brunel (1856 – 1900) in [B31], a paper that seems to have escaped the attention of all writers on the topic of configurations (n_4) prior to [G46][1]. In an earlier paper [B30] Brunel followed an idea quite popular at that time: a polygon inscribed and circumscribed to itself (with sides understood as lines). Clearly, these are a special class of (combinatorial or geometric) 3-configurations, which we will discuss in Section 5.2. Aware of the need to distinguish between combinatorial and geometric configurations, in [B31] Brunel pursued this idea farther, by considering a "polygon *doubly* inscribed and circumscribed" to itself. In the current terminology we call such polygons "Hamiltonian circuits (or multilaterals)" of the configuration, and we will consider them in more detail in Sections 5.2 to 5.4. Each line of such a doubly self-inscribed and self-circumscribed "polygon" is incident, besides the two points (vertices of the polygon) that define it as a side of the poly-

[1] Biographical data on Brunel, and comments on his work, may be found in [B2] and, in great detail, in [D11]; see also [G29].

gon, with precisely two additional vertices of the polygon. Brunel determines that any combinatorial configuration (n_4) must satisfy $n \geq 13$, and gives two constructions.

In the first, Brunel presents a configuration table (that is actually an orderly configuration table, in the terminology of Section 2.5), and states that while the verification that this indeed determines a combinatorial configuration (35_4) is easy, the graphical representation requires some effort. (Unfortunately, the remarks in [D11, p. LXVIII] concerning the geometric realization of this configurations are, at best, misleading.) From Brunel's statement (especially in view of his later comments concerning the other construction) one may conclude that he had found a geometric realization of this configuration. In fact, this configuration turns out to be isomorphic to the geometric configuration (35_4) mentioned in [G50], communicated to the authors by Ludwig Danzer. (See also [G49].) Although no reasonable diagram of this configuration seems to be available, the configuration can be described easily enough by a construction of the kind used by Cayley and others in similar contexts a century and a half ago. In the case under discussion, start with seven points in general position in real 4-space; consider the 35 2-planes and 35 3-spaces they generate, and intersect this family by a 2-dimensional plane in general position to obtains the required geometric configuration (35_4). The absence of any reasonable geometric symmetry makes this configuration visually unattractive.

Brunel's second construction yields combinatorial configurations (n_4) on which a cyclic group operates transitively. This includes explicitly specified configurations for $13 \leq n \leq 16$. Unfortunately, the results Brunel presents are marred possibly by typos, but also by outright errors. Among the latter, in several cases Brunel lists isomorphic doubly selfinscribed and selfcircumscribed polygons as distinct. For example, in case $n = 13$ Brunel lists cyclic translates of {0,1,4,6} and {0,1,3,9} as the two polygons, although the permutation (0)(1)(2)(3,4)(5)(6,9,8,10,12,7)(11) maps the first polygon onto the second. Moreover, it is rather easy to prove that up to isomorphism, there can be only one such combinatorial configuration; this is completely analogous to the proof (in Section 2.2) that the configuration (7_3) is unique. But even allowing for these shortcomings, we see that Brunel anticipated the corresponding results of Merlin [M8], and even went a bit beyond them. A corrected list would show one cyclic configuration (or polygon) for n = 13 and 14, three for n = 15, and two for n = 16. This coincides with the recent list of cyclic configurations given by Betten and Betten [B13], to which we shall return soon. Brunel also noted that translates of $\{0,1,4,6\}$ yield a configuration (n_4) for all $n \geq 13$; this anticipated by nearly a century a result of Gropp [G8].

Merlin mentions in [M8] that configurations (n_4) have not been investigated systematically, although some isolated ones were discovered by F. Klein [K11], W. Burnside [B32], and others. Like Brunel, he constructs a combinatorial configuration (13_4); moreover, he proves its uniqueness and minimality. He also constructs a configuration (14_4) and proves it is unique. Merlin states that there are exactly **three** distinct configurations (15_4) which, however, are not presented. In fact, he is mistaken. As shown by Betten and Betten [B13], there are **four** different configurations (15_4), three of which are cyclic and coincide with the three doubly selfinscribed and selfcircumscribed polygons of Brunel (who did not comment on the possibility of noncyclic configurations (15_4), or (n_4) in general). In the same context, Merlin makes two additional errors:

(i) He claims that his three configurations (15_4) can be distinguished by the number of vertex-disjoint triangles present in them, which he claims to be 5, 1 and 0, respectively. In fact, all four configurations (15_4) have five such triangles, the maximal possible number.

(ii) He states that his configurations (13_4), (14_4) and (15_4) have orderly configuration tables; this is correct — see Section 2.5 — and has been proved by Steinitz in [S17] for all configurations (n_k). However, Merlin then claims that it follows that there is no Hamiltonian circuit for any of them — which is wrong. Steinitz's orderliness result has no such implications, and cyclic 4-configurations such as Brunel's explicit constructions in [B31] (of which Merlin is unaware) provide counterexamples to Merlin's claim.

By a construction analogous to the one devised by Martinetti (in [M2], see Section 2.3) for configurations (n_3), Merlin shows that for every $n \geq 30$ there are combinatorial configurations (n_4). In fact, it is easy to show that there are such configurations for all $n \geq 13$; for example, as noted by Brunel and mentioned above, for all $n \geq 13$ it is enough to consider cyclic translates of the "line" {0,1,4,6}.

Concerning the number N(n) of distinct combinatorial configurations (n_4), the only known values are those given by Betten and Betten [B13]: the old N(13) = N(14) = 1, and their new results N(15) = 4, N(16) = 19, N(17) = 1972, and N(18) = 971171. These new numbers seem not to have been independently verified, except for the value N(17) = 1972 (see [B29]).

The configurations (13_4) and (14_4) can be obtained as cyclic configurations with generating "line" {0,1,4,6}. The four configurations (15_4) can be characterized as follows: The three cyclic ones are generated by the "lines" {0,1,4,6}, {0,1,5,7} and {0,1,3,7}, given already by Brunel. The other three configurations given by Brunel yield isomorphic configurations (two to the first, and one to the second). Betten and Betten [B13] give other generators for the three cyclic configurations: {0,2,8,12}, {0,1,9,11}, and {0,1,9,13}, respectively; these are shown in [B13] by Levi incidence matrices (see Section 1.4), – but matrices that do not exhibit the *cyclic* character of the configurations. Their fourth configuration (n_4) is clearly illustrated in [B13] by a Levi incidence matrix shown in Figure 3.1.1(a). As it is the only non-cyclic configuration (15_4), it is necessarily selfdual. An incidence matrix exhibiting one of the selfdualities is shown in Figure 3.1.1(b); it is obtained by suitable permutations of the rows and columns of the matrix in Figure 3.1.1(a).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1				X	X	X	X								
2								X	X	X	X				
3												X	X	X	X
4	X			X				X				X			
5	X				X				X				X		
6	X					X				X				X	
7	X						X				X				X
8		X		X					X					X	
9		X			X			X							X
10		X				X					X		X		
11		X					X			X		X			
12			X	X						X			X		
13			X		X						X	X			
14			X			X			X						X
15			X				X	X						X	

(a)

	1	2	3	5	11	15	4	10	14	6	9	13	7	8	12
1				X			X			X			X		
2					X			X			X			X	
3						X			X			X			X
4	X						X							X	X
11		X						X					X		X
15			X						X				X	X	
5	X			X							X	X			
10		X			X					X		X			
14			X			X				X	X				
6	X							X	X	X					
8		X					X		X		X				
12			X				X	X				X			
7	X				X	X							X		
9		X		X		X								X	
13			X	X	X										X

(b)

Figure 3.1.1. (a) A Levi incidence matrix of the non-cyclic (15_4) configuration constructed by Betten and Betten [B13]. (b) A selfdual incidence matrix of this configuration.

Brunel's generating "lines" of the three cyclic configurations (15_4) given in [B31] have an advantage over the ones given by Betten and Betten [B13], even though they are isomorphic for the (15_4) configurations: Brunel's can serve as generating lines for combinatorial configurations (n_4) for **all** $n \geq 15$.

Concerning the (16_4) combinatorial configurations, it should be noted that the three generating lines of the cyclic (15_4) configurations listed above do serve to generate cyclic (16_4) configurations — but the three resulting configurations are isomorphic. There is one other configuration (16_4), also cyclic, specified in Betten and Betten [B13] by its generating line {0, 1, 6, 13}; Brunel renders the same configuration, but with a typo; when corrected, its generating line is {0,1,3,12}, or equivalently, {0,1,3,–4}. The generating lines {0, 1, 6, 13} or {0,1,3,12} do not yield a cyclic configuration for all $n > 16$; however, if the generating line is taken in the form {0,1,6,–3} or {0,1,5,–2}, which are equivalent for (16_4), then they works for all such n. Obviously, any generating line for a cyclic configuration is also a generating line for all sufficiently large n.

Besides the two cyclic configurations, Betten and Betten [B13] describe 17 non-cyclic combinatorial configurations (16_4); they state that these 19 are the complete list, but give no details of the determination of this claim. There seems to have been no independent confirmation of this list. As with all the listings in [B13], it seems that no attention was given to finding presentations of the configurations as symmetric as possible; in particular, there is no mention of duality or selfduality. Beyond the cyclic configurations already mentioned, and the (15_4) configuration in Figure 3.1.1(a), this is illustrated by one of the seventeen (16_4) configurations illustrated in Figure 8 of [B13]. This example is shown in Figure 3.1.2(a).

Betten and Betten [B13] state (or at least imply) that there are only two cyclic configurations (17_4); their generating lines given are equivalent to the ones mentioned above, {0,1,4,6} and {0,1,5,–2}.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	X										X	X	X			
2		X	X	X										X		
3					X	X	X								X	
4								X	X	X						X
5	X	X			X			X								
6	X		X			X			X							
7	X			X			X			X						
8		X				X					X					X
9		X								X		X			X	
10			X				X						X			X
11			X					X			X				X	
12				X	X							X				X
13				X					X				X		X	
14					X				X		X			X		
15						X				X			X	X		
16							X	X				X		X		

(a)

	1	14	15	16	11	12	13	8	6	4	9	2	7	5	3	10
1	X				X	X	X									
2		X								X		X			X	
3			X						X				X	X		
4				X				X			X					X
7	X									X			X			X
6	X								X		X				X	
5	X							X				X		X		
10				X			X						X		X	
9			X			X						X				X
14		X			X						X			X		
12				X		X				X				X		
15		X					X		X							X
11			X		X			X							X	
13			X				X			X	X					
16		X				X		X					X			
8				X	X				X			X				

(b)

Figure 3.1.2. (a) A (16_4) combinatorial configuration as illustrated in [B13] by its Levi incidence matrix. (b) A symmetric incidence matrix of the same configuration, illustrating its selfduality.

As we shall see in Section 3.2, except for one of the (17_4), none of the *combinatorial* configurations (n_4) with $n \leq 17$, is even *topologically* realizable (see Section 3.2). Merlin [M8] shows that the configurations (13_4), (14_4) and the three cyclic (15_4) are not geometrically realizable. But he also notes that *geometric* configurations (n_4) do exist for infinitely many values of n. His construction uses "stacks" of 3-configurations and vertical lines through their vertices to construct [4,3]-configurations, and then stacks of duals of the projections of these into the plane to construct 4-configurations. While this yields geometric configurations (n_4) for infinitely many values of n, there are infinitely many n that are not covered.

Much new information on the question of existence of topological and geometric 4-configurations has become available recently. We discuss it in the following sections.

Exercises and problems 3.1

1. Decide whether the (35_4) configuration of Brunel is cyclic or not.

2. Prove that the three cyclic configurations (15_4) generated by the "lines" $\{0,1,4,6\}$, $\{0,1,5,7\}$ and $\{0,1,3,7\}$, given by Brunel, are distinct (non-isomorphic).

3. Prove that the three cyclic configurations (15_4) generated by the "lines" $\{0,1,4,6\}$, $\{0,1,5,7\}$ and $\{0,1,3,7\}$ are isomorphic to the three generated by $\{0,2,8,12\}$, $\{0,1,9,11\}$, and $\{0,1,9,13\}$, respectively.

4. Investigate the duality properties of the three cyclic configurations (15_4).

5. Validate the claim that the three generating lines in Exercise 2 yield isomorphic configurations (16_4).

6. Show that the cyclic (16_4) configurations with starting lines $\{0,1,4,6\}$ and $\{0,1,6,13\}$ are not isomorphic.