2.7 ASTRAL 3-CONFIGURATIONS WITH CYCLIC SYMMETRY GROUP

We have seen in Section 1.5 that a 2-astral 3-configuration must have two orbits of points and two orbits of lines. By the convention introduced there we simplify the expressions and call such configurations **astral**, for short.

Lemma 2.7.1. If an astral 3-configuration has one orbit of points at infinity, it must have reflective symmetries.

Proof. If such a configuration has no reflective symmetries, then the orbit of points in the finite plane has to coincide with the vertices of a regular polygon; the only alternative would be that they are the vertices of an isogonal polygon – but their equivalence requires reflection. Each of the points at infinity is on three lines, two of which are in the same orbit. Even if these two are related by a rotational (halfturn) symmetry, they must be parallel and of the same length, and by the rotational symmetry the third line parallel to them must pass through the center of the polygon. Thus all lines come in triplets of parallel lines, the middle one serving as mirror for the other two; these mirror lines are spaced at equal angles, hence they are mirrors of the configuration. Hence we again are led to reflective symmetries. □

As a consequence of Lemma 2.7.1 we see that astral configurations (n_3) that have a cyclic group of symmetries are necessarily configurations in the Euclidean plane. Astral 3-configurations with a cyclic group of symmetries and no mirrors will be called **chiral**. (Note that this does not mean that all astral configurations contained in the Euclidean plane have a cyclic group of symmetries. We shall consider those with dihedral symmetry in Section 2.8.) The points of a chiral astral configuration are at the vertices of two concentric regular polygons with m = n/2 vertices each; the polygons clearly have different sizes. As we shall show next, such 3-configurations depend (up to similarity) on three additional integer parameters. The notation we shall use for these configurations is m#(b,c;d); a detailed explanation follows, and an illustration is given in Figure 2.7.1.

The lines of one geometric transitivity class are the diagonals of one of the polygons, those of the other class are diagonals of the other polygon; each line of the configuration contains two points of one polygon and one of the other. The numbers of edges of the polygons bridged (spanned) by the diagonals are the integers b and c of the symbol; usually we shall follow the convention that $m/2 > b \geq c > 0$, but the relative size of b and c is not of intrinsic importance and it is sometimes convenient to disregard the convention. The corresponding points (vertices) are accordingly called the b-points resp. c-points, and the lines are b-lines and c-lines. Starting from an arbitrary b-point denoted B_0, and proceeding in an arbitrary orientation we label the other b-points consecutively $B_1, \dots, B_{m-1}$. Each b-line is then of the form $L_i = \mathrm{aff}(B_i, B_{i+b})$, and it contains a c-point which we label C_i. The c-line that passes through C_0 determines the labeling of the c-lines. In the orientation of the c-points which is induced by the orientation chosen for the b-points, the earlier point of that c-line is C_0, and the later accordingly is C_c. The remaining c-points are then labeled in the obvious way; the c-lines are labeled by $M_i = \mathrm{aff}(C_i, C_{i+c})$, . Here and throughout, all subscripts are to be understood mod m. From a given (n_3) configuration of the kind considered, the values of m, b and c can be read off instantly. Now we can find a tentative determinations of the symbol d in the notation m#(b,c;d) for the configuration. We consider the b-point that is incident with the c-line $M_0 = \mathrm{aff}(C_0, C_c)$; like all b-points, it already carries a label. This label we take as the value of d in the *preliminary* symbol of the configuration.

The value of d in the final symbol requires a comparison of two possibilities. One is what we have just described, and the other is obtained in the same way but going in the opposite orientation around the b-polygon. As the *final* symbol m#(b,c;d) for the configuration we shall generally choose that one of the alternatives which has the smaller value of d. As is easily verified, the two values of d add up to b+c; hence we may assume that $d \leq (a+b)/2$, which means that in fact only one of the determinations has to be carried out. If it yields such a value of d we take it, otherwise we subtract it from b+c to get the correct value of d. This is illustrated by the examples in Figure 2.7.2.

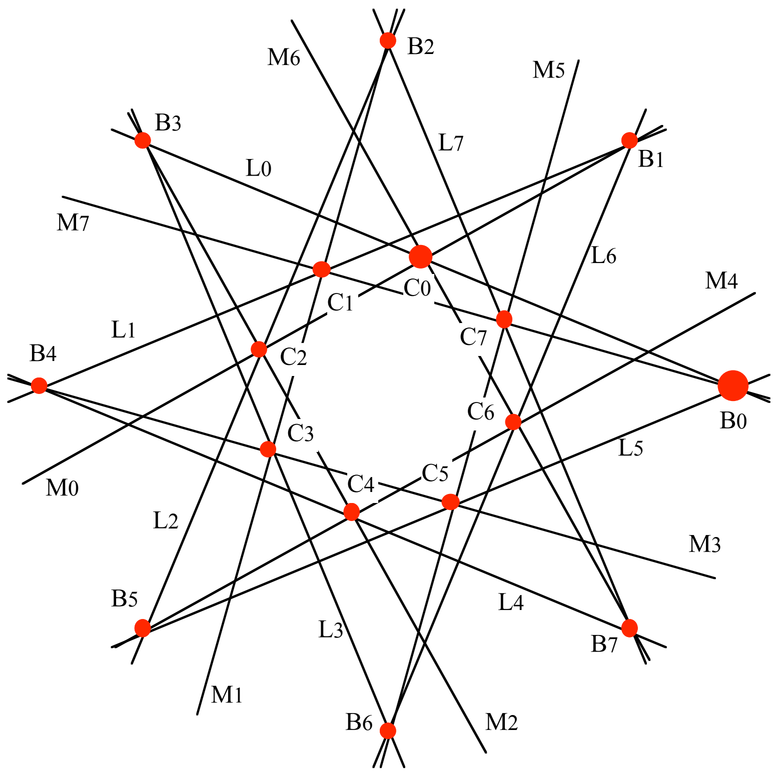


Figure 2.7.1. An example of the labeling and designation of a chiral astral configuration. Following the explanations in the text, this is configuration 8#(3,2;1).

While our conventions assign a unique symbol to each astral (n_3) configuration, the converse is not valid. In general, **two** configurations are represented by the same symbol m#(b c; d). They differ by the ratio of the radius of the circle of c-points to that of the b-points; the one with smaller ratio is denoted by a single tag ', the other one by double tags ". This is illustrated in Figure 2.7.3. Another way of distinguishing the two configurations is by specifying the ratio in which the point C_0 divides the segment B_0B_b; this information is very useful for drawing the configuration, as well as for determining which symbols are possible.

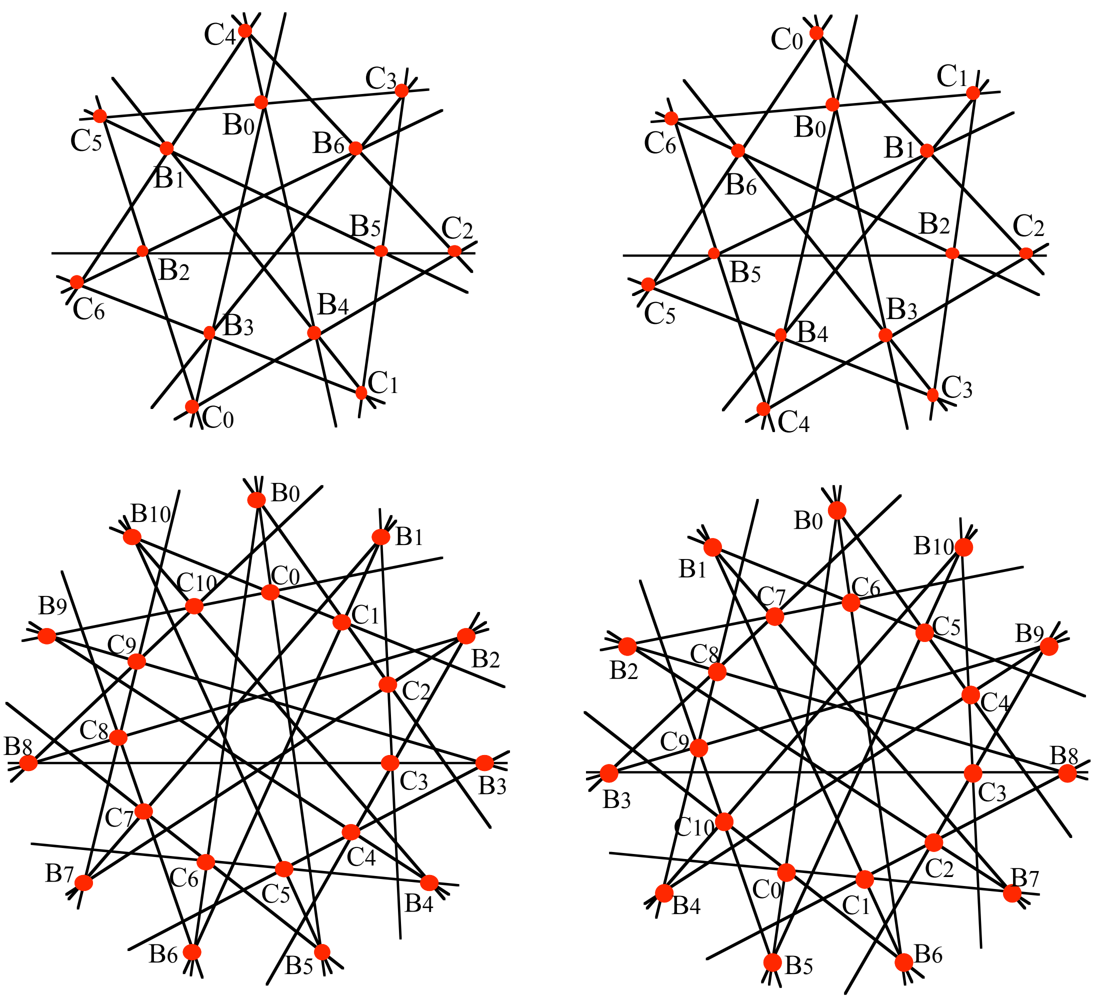


Figure 2.7.2. Additional examples of labeling astral (n_3) configurations. The one in the upper row has symbols 7#(3, 2; 4) and 7#(3, 2; 1), so the latter is the one conventionally accepted. The configuration in the bottom row has symbols 11#(5, 1; 10) = 11#(5, 1; –1) since all subscripts can be taken (mod n) and 11#(5, 1; 7). Hence the conventional symbol is 11#(5, 1; –1).

However, in cases in which either $b = c$ or $2d = b + c$ the symbol m#(b, c; d) represents only a single configuration. Examples of these situations are shown in Figure 2.7.4, for the symbols 6#(2, 2; 1) and 11#(5, 1; 3).

If the highest common factor of m, b, c, d is $f > 1$, then the configuration m#(b,c;d) is not connected, but consists of f copies of the configuration m/f#(b/f, c/f; d/f) . However, exceptions to all the above happen when there are additional “accidental”

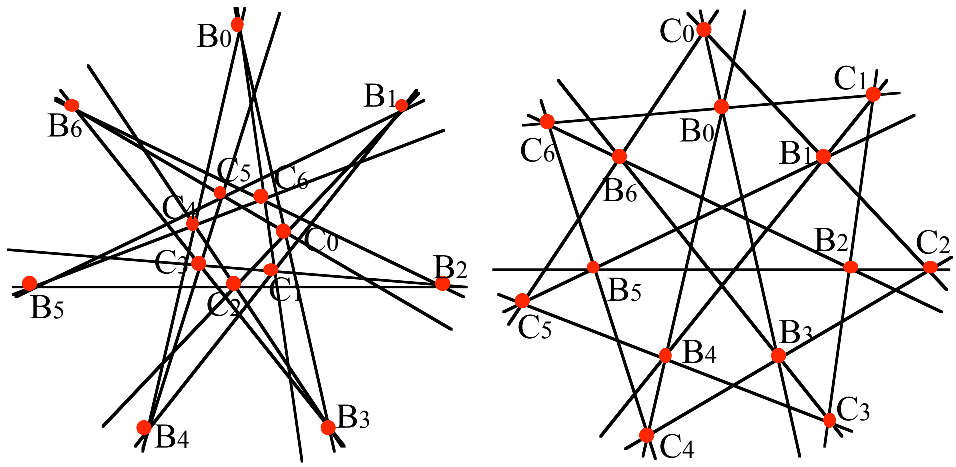


Figure 2.7.3. The two astral configurations with common symbol 7#(3, 2; 1). The one on the left is specified by 7#(3, 2; 1)', the other one by 7#(3, 2; 1)".

incidences. For example, an attempt to draw the configuration 12#(5, 1; 3) leads to the superfiguration shown in Figure 2.7.5a; it has additional incidences and is, in fact, a configuration (24_4). In Figure 2.7.5b we show how sensitive the situation is with respect to correctly drawing the configurations – a seemingly legitimate configuration does not really exist. On the other hand, Figure 2.7.5a can be interpreted as a *representation* of the configuration 12#(5,1;3), as well as a representation of configurations 12#(5,1,-1), 12#(4,4;1) and 12#(4,4;2). Figure 2.7.5b serves to illustrate a *topological* realization of the configuration 12#(5,1;-1).

A different type of unintended incidences is illustrated by the example in Figure 2.7.6. Here the result is a collection of points and lines which is not a configuration under the definitions we adopted at the beginning, since some lines (but not all) are incident with four points, and some points with four lines.

Disregarding the possible presence of unintended incidences, how does one get from the symbol to a drawing, and how does one decide whether a symbol corresponds to any configuration? For the answer to both parts of the question, we can proceed either algebraically or geometrically.

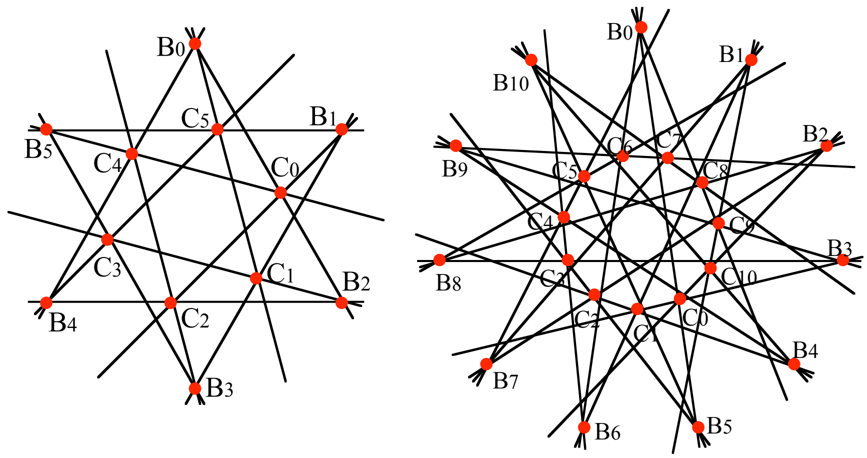
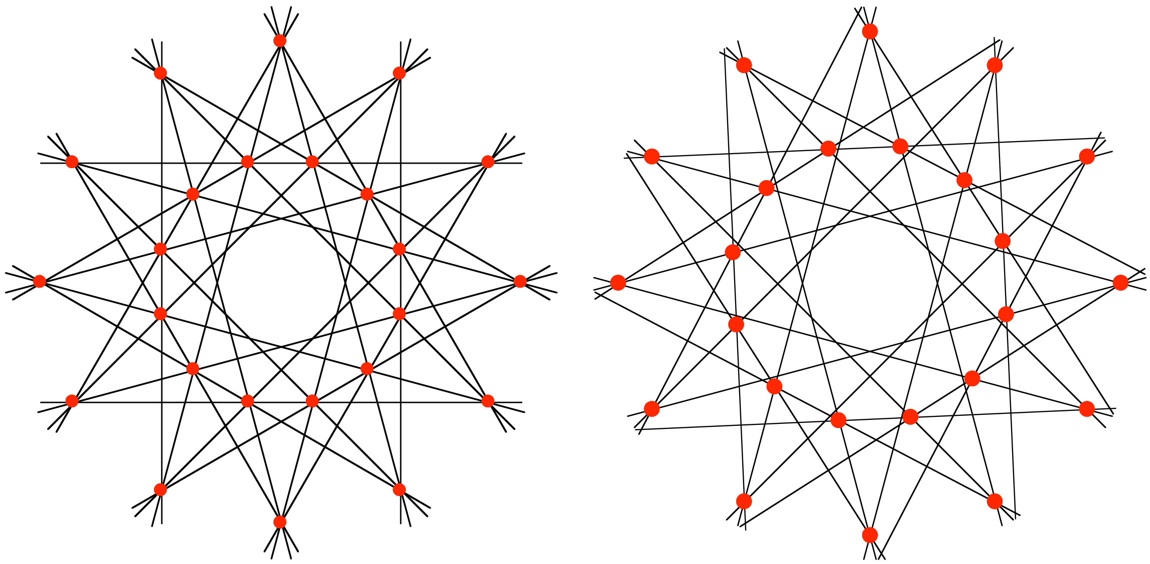


Figure 2.7.4. Astral configurations which are examples of the case in which only a single configuration corresponds to its symbol, here 6#(2, 2; 1) and 11#(5, 1; 3).

In the algebraic approach, given a symbol m#(b, c; d) we start with the vertices of a regular m-gon, and draw all diagonals of span b (or their extensions, if needed). Points of the other orbit will be the vertices of another regular m-gon, situated on the diagonals of the first one. Their location is determined by the ratio which, in the notation of Figure 2.7.1, is given by the still undetermined ratio of lengths $\lambda = B_0C_0/B_0B_b$. The position of C_c is determined by the same ratio, since $\lambda = B_cC_c/B_cB_{b+c}$. Now, the line C_0C_c contains the point B_d of the first orbit. Hence, writing the collinearity condition in terms of a determinant, involving the variable λ and the known coordinates of the B points, yields a quadratic equation for λ. Depending on whether there are two, one, or no solution in real numbers we obtain the pair of isomorphic configurations, a single configuration, or no configuration at all. Thus the complete characterization of possible symbols is, in principle, determinable by the non-negativity of the discriminant of that quadratic equation. In any particular case, the software used (various versions of Mathematica® on different Macintosh computers) had no problem finding the value(s) of λ, and then drawing the configuration(s). However, no amount of effort, on the computer or manually, was successful in explicitly describing the necessary and sufficient conditions on the integer parameters m, b, c, d for the existence of the configurations. The



(a) (b)

Figure 2.7.5. The diagram in (a) is supposed to show the configuration 12#(5,1;3); however, additional incidences turn it into an astral (24_4) configuration which, by the conventions we shall specify in Section 3.5, has the symbol 12#(5,4;1,4). Note that the same (24_4) configuration results when drawing any of 12#(5,1,-1), 12#(4,4;1) and 12#(4,4;2). The first of these is illustrated by the pseudoline configuration in (b). Note that these are actually straight lines, but that their incidences are faked (ever so slightly). For a different presentation of these cases see Figure 5.8.1 and the explanations given there.

best I could do is to deduce several necessary conditions from many specific cases, and from an argument to be described below. In any case, the known conditions for a symbol m#(b, c; d) are as follows (this includes the notational conventions introduced earlier):

$$0 < c \le b < m/2$$

$$2[(b + c - m)/2] \le c - b + 1 \le 2d \le b + c$$

$$0 \ne d \ne c$$

$$2 \cos(b\pi/m) \cos(c\pi/m) \le 1 + \cos((b + c - 2d)\pi/m)$$

While the use of calculational and graphic capabilities of appropriate software (Mathematica, Mathlab, Maple, and others) enables one to find out whether a symbol leads to a configuration, it is of same interest to note that geometric means can yield the same result. In fact, if the vertices of a regular m-gon are given, the configurations m#(b, c; d) can be drawn with just the classical Euclidean tools. (Naturally, the construction of the regular m-gon may or may not be possible with Euclidean tools, depending on the value of m.) Here is how the construction proceeds, illustrated for m#(b,c;d) = 11#(5,1;2) by the steps in Figure 2.7.7.

- (a) Draw the lines determined by the diagonals of span b = 5; this yields a regular polygon P of type {m/b}.
- (b) Construct the isosceles triangle T determined by two vertices V_1 and V_2 of P, that are separated by span c = 1, and the center O of P.
- (c) Construct the circumcircle C of the triangle T described in (b).
- (d) Label the sides of the polygon P. We label "0" the two lines of P that touch T at V_1 and V_2, but do not go through the interior of T. The other lines of

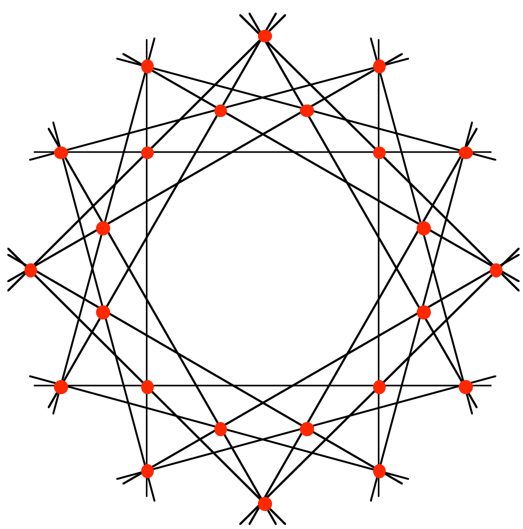


Figure 2.7.6. A drawing of the astral configuration 12#(3, 3; 1) shows unintended incidences. The resulting family of points and lines is not a *configuration* according to our definitions; it is a *superfiguration*. In fact, by ignoring some incidences, it could be interpreted as a *representation* of the astral configuration 12#(3, 3; 1).

P are numbered by their sequence at the central convex m-gon determined by these lines. The sides closer to the center of C are labeled "1", "2", ... in order, the ones farther from the center of C are labeled "-1", "-2", ...

- (e) Find the intersection points of the lines of P with the circle C.

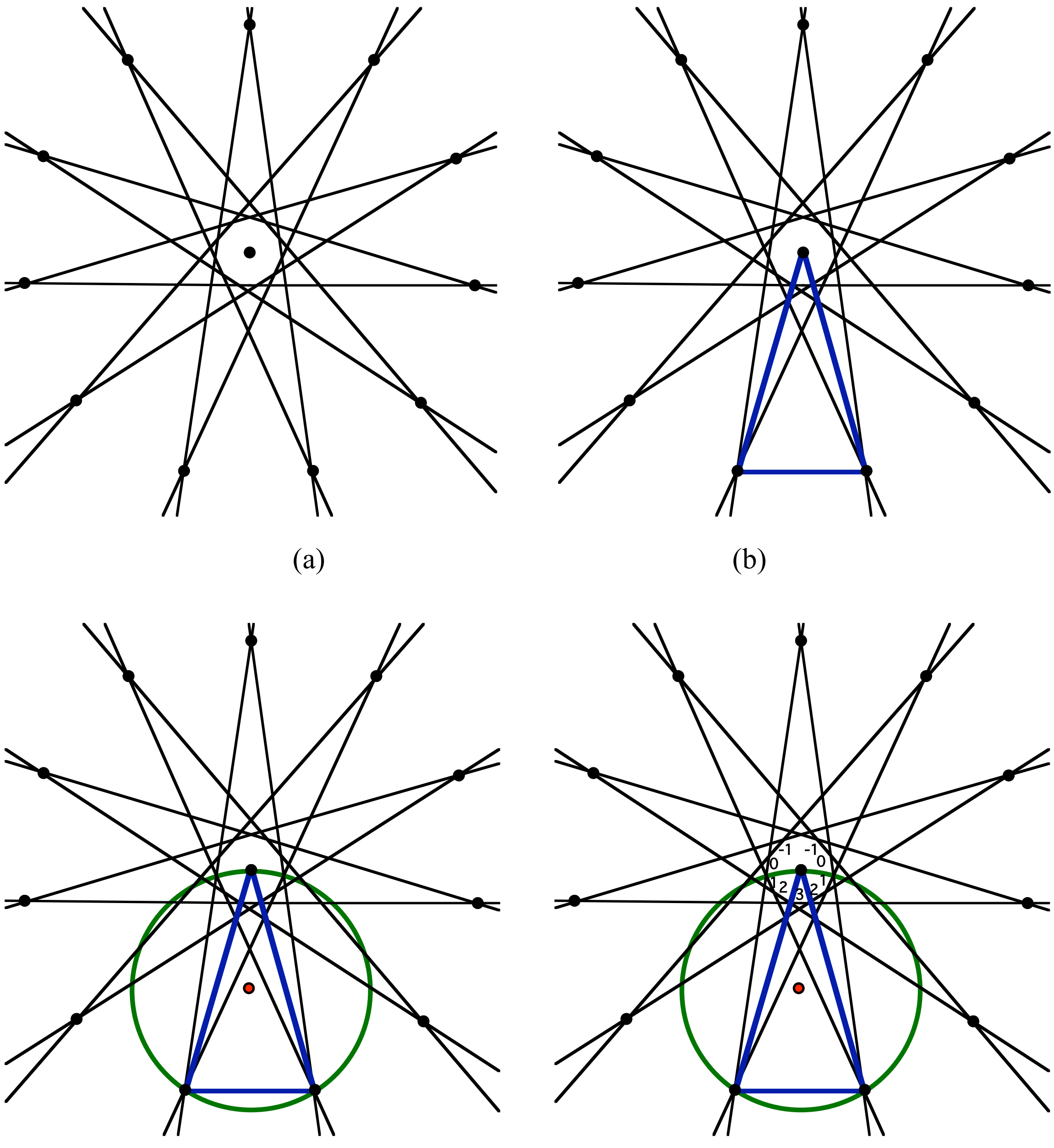


Figure 2.7.7 (first part). The geometric construction of the configuration 11#(5,1;2).

- (f) Label these intersection points by the labels of the lines.
- (g) Select one of the points labeled d = 2, and draw the line connecting it with one of the points V_1, V_2.
- (h) Rotate through all the multiples of $2\pi/m = 2\pi/11$ the point chosen in (g) and the line constructed there. A configuration $m\#(b,c;d) = 11\#(5,1;2)$ is obtained.

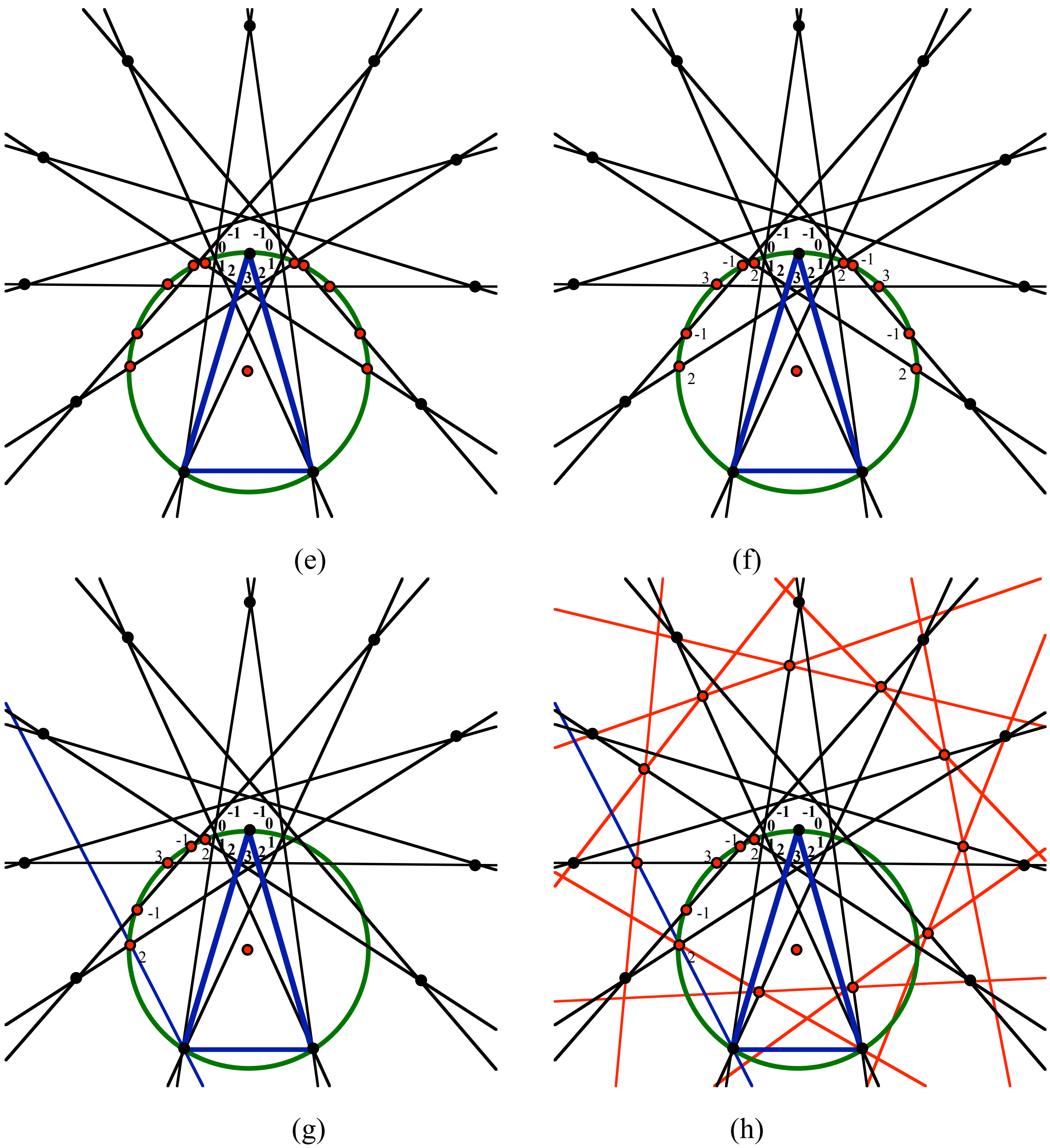


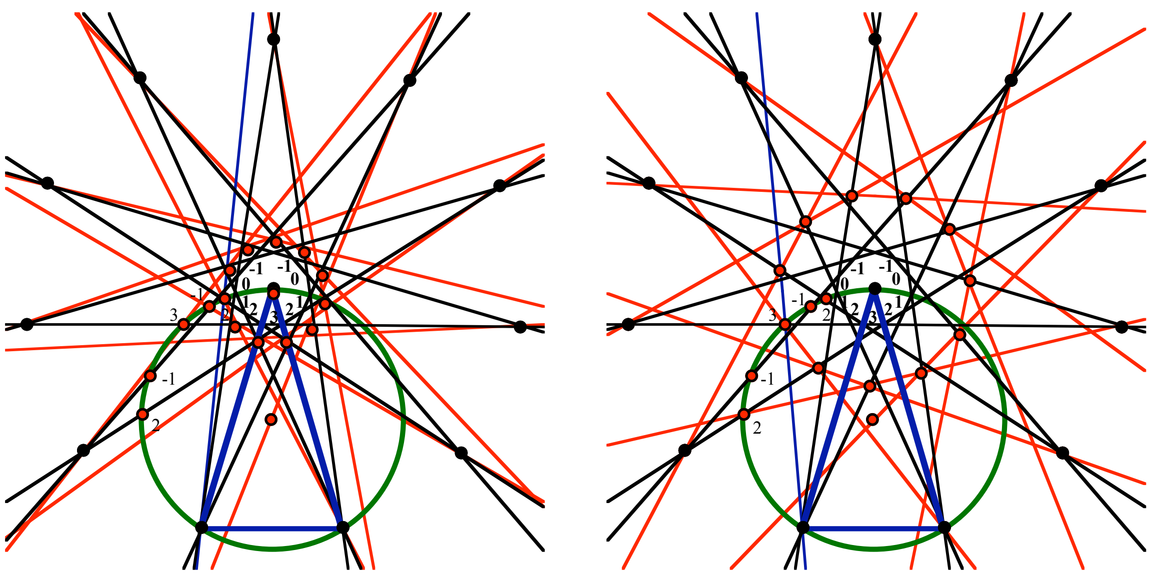
Figure 2.7.7. (second part)

- (i) The same construction as in (g) and (h), but with the other point labeled d = 2, yields the other configuration 11#(5,1;2). The remaining possibilities of pairing a point labeled "2" with the other V_j yield configurations congruent to the ones in (h) and (i).

- (j) An analogous construction but with a point labeled "3" yields the configuration 11#(5,1;3). As we shall see in Section XY this configuration is selfpolar.

Naturally, these constructions need justification, which we shall provide below. However, it is appropriate to recall that establishing results by using graphical means can be rigorously justified; see, for example, [M19].

The reasoning follows the above method of construction, and is illustrated in Figure 2.7.8 by the example of the configuration 11#(4,3;2).



(i) (j)

Figure 2.7.7. (third part).

The triangle O V2 V1 is isosceles. The angle O V(2) V1 equals the angle O V2 V1 since both are peripheral angles over the same arc O V1. Let X be the point on the ray V(2) V1 such that the angle V(2) O X equals the angle V2 O V1. Then the triangle O X V(2) has correspondingly equal angles with the triangle O V1 V2, hence is similar to it.

Therefore it is also isosceles, so OX has the same length as OV(2) and is thus on the circle centered at O and with radius OV(2). As the angle V(2) O X is the same as the angle V2 O V1, which spans a diagonal of span c = 3 of the m-gon (m = 11), it follows that follows that V(2) X spans the same diagonal on the m-gon determined by the rotates of V(2). The existence of the configuration m#(b,c;d) = 11#(4,3;2) is established. □

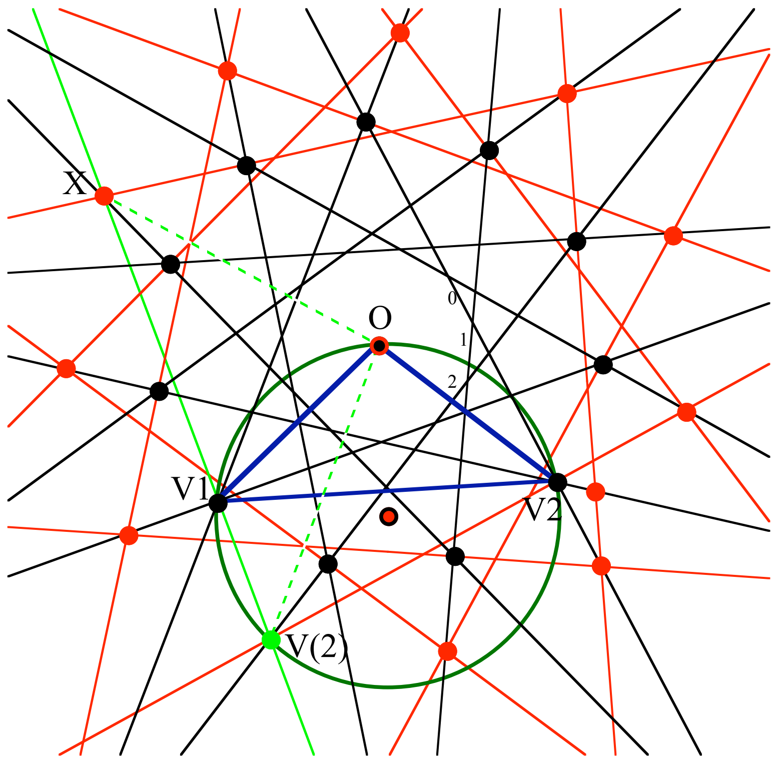


Figure 2.7.8. Starting with the {11/4} polygon (black points and lines), the configurations 11#(4,3;2) is constructed by the method described above.

Using the description of the determination of the symbol m#(b,c;d) of an astral configuration (n_3) it is immediate that the reduced Levi graph is as shown in Figure 2.7.9. The simplicity of the reduced Levi graph of such a configuration can be interpreted as the source of the usefulness of such graphs, but it also serves to indicate that the encoding of such an astral configuration by our symbol is natural and not arbitrary.

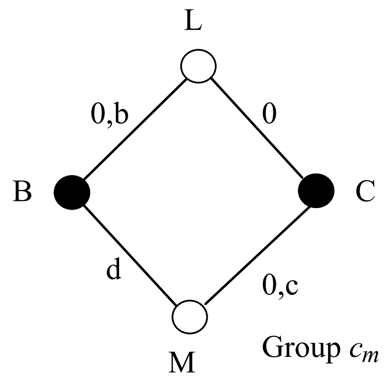


Figure 2.7.9. The reduced Levi graph of an astral configuration m#(b,c;d); the notation is analogous to the one used in Figure 2.7.1.

Astral 3-configurations of were first defined in [G39], but isolated examples occur in earlier publications. The first seem to be by Zacharias [Z4]; he shows examples of astral (10_3), (12_3) and (14_3) configurations and comments on their star-like appearance, but reaches no general conclusions or constructions. Similarly, van de Craats [V1] shows the astral (10_3) and notes various interesting properties associated with it; he also shows an astral (14_3), and mentions that analogous astral (n_3) can be found for all $n = 2m+2$, where $m \geq 2$. Several other examples can be found in [B19], as well as in [G46].

Exercises and problems 2.7.

1. Derive explicitly the quadratic equation for λ mentioned in the text in the case of 9#(4,2;3), and use this to draw this configuration using suitable software.

2. Derive explicitly the quadratic equation for λ in the general case m#(b,c;d), and try to find criteria on these parameters that will imply that the solutions of the equation are real.

3. Use the geometric construction to draw the configuration 9#(4,2;3).

4. Show that the configurations 12#(5,1;2) and 12#(5,1;4) are congruent. Explain this, and generalize.

5. The configuration 5#(2,2;1) has a cyclic automorphism group that acts transitively on its points and lines. Describe this group, and determine whether it acts transitively on the flags of the configuration.

6. The automorphisms group of the astral chiral configurations 5#(2,2;1) is transitive on its points. Find other astral chiral configurations with this property. Can you characterize all such configurations?