2.6 STEINITZ'S THEOREM – THE GEOMETRIC PART

We turn now to a consideration of the geometric part of Steinitz's claim. We recall from Section 2.5 the claim from Steinitz' PhD thesis [S17]:

Steinitz's claim. Every connected combinatorial 3-configuration has a geometric realization by points and lines as a #1-subfiguration in the Euclidean plane; moreover, the point and line of the ignored incidence can be arbitrarily chosen.

Recall from Section 1.3 that a *#1-subfiguration* of a combinatorial configuration is a family of points and (straight) lines that satisfies all the incidence requirements except possibly one that is ignored, and has no additional incidences.

By our definition, which coincides with the definition generally used, a **geometric configuration** (n_k) is a family of n points and n (straight) lines such that each point is incident with k lines, and each line is incident with k points. The intention of this definition is that each of the points and lines is incident with **precisely** k objects of the other kind. Even though this requirement often was not explicitly stated, in many instances it was stated and it has been taken as self-understood by all nineteenth century writers on configurations.

However, the following situation does arise: We start with a combinatorial configuration and find a set of points and a set of straight lines which fulfill the incidence requirements of the combinatorial configuration. In other words, every combinatorial incidence corresponds to a geometric incidence. However, it is possible that the points and lines we found have **additional** geometric incidences, not specified in the combinatorial configuration. As mentioned in Section 1.3, in such a case we shall say that the points and lines form a **representation** (or **superfiguration**, or **weak realization** are terms also used) of the combinatorial configuration. It may happen that a different choice of points and lines will result in a geometric configuration without additional incidences; if it is necessary to stress this fact, we shall say that we have a **realization** (or, if there is need for a more specific expression, a **strong realization**). But, as is easy to see, some combinatorial configuration admits **only** superfigurations.

For example, consider the combinatorial configuration given by Table 2.6.1. A superfiguration is shown in Figure 2.6.1, in which the line H passes through the point m although they are not combinatorially incident according to Table 2.6.1. (This configuration is isomorphic to the one in Figure 2.2.6.) The reason that **every** geometric presentation of this configuration by points and lines is a *representation* (and not a *realization*) lies in the fact that the hexagon abcdef has vertices that alternate on the two lines A and B, and therefore the three points g, h, m are collinear by the Pappus theorem. More complicated examples can have several unintended geometric incidences — in fact, there is no upper bound on the possible number of such incidences. In Figure 2.6.2 we show a topological configuration (18_3) such that each of its representations in a #2-superfiguration. This possibility is also ignored in the Wikipedia article [W5] on Ernst Steinitz (as of February 4, 2008).

A more detailed analysis of this topic will be presented in [B18].

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
e	d	h	m	f	a	c	g	j	k	l	p	b	i	n	o
c	f	d	e	g	h	b	i	o	n	m	j	a	p	l	k

Table 2.6.1. A combinatorial configuration (16_3).

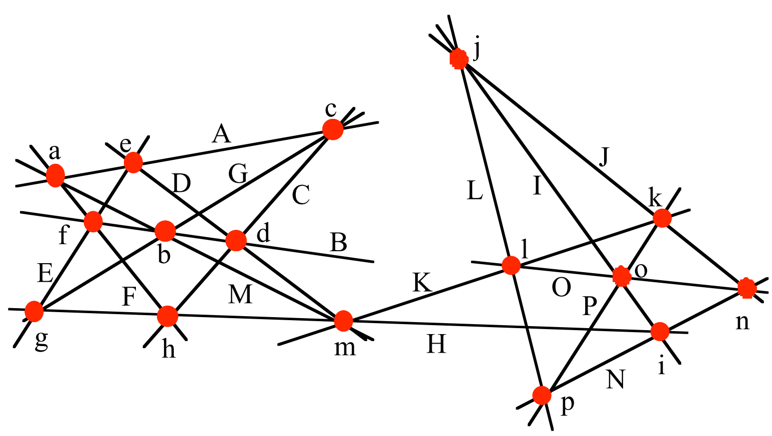

Figure 2.6.1. A *representation* of the combinatorial configuration (16_3) given by Table 2.6.1.

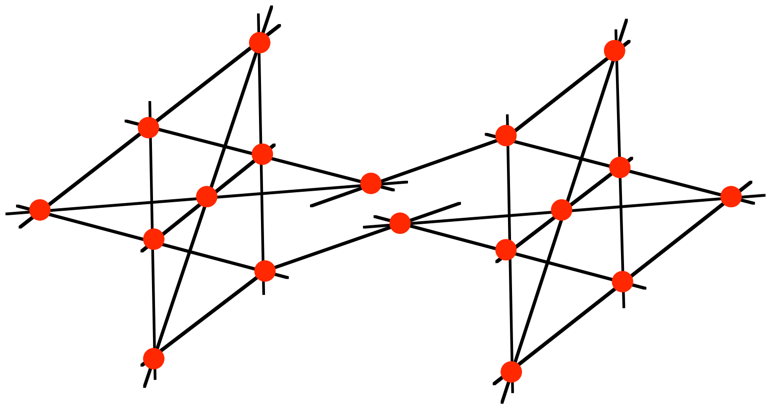

Figure 2.6.2. A topological configuration which can be realized by straight lines only as a #2-superfiguration.

The above preamble to the geometric part of Steinitz's theorem about geometric "realizations" of combinatorial 3-configurations was needed to set the stage for dealing with this rather remarkable result and its history. The following result is what Steinitz actually proved:

Theorem 2.6.1. For every connected combinatorial 3-configuration and every choice of one incident point-line pair, there is a selection of distinct points and (straight) lines which realize all the incidences except possibly the incidence of the chosen line with the chosen point.

In other words, every connected 2-configuration has a **near-representation** in the Euclidean plane, in which the point and line of the ignored incidence can be arbitrarily chosen. As shown by the example in Table 2.6.1 (and illustrated in Figure 2.6.1) there are combinatorial configurations for which no near-representation is a near-realization. (The "near" part of these terms is meant to convey that one incidence is disregarded.)

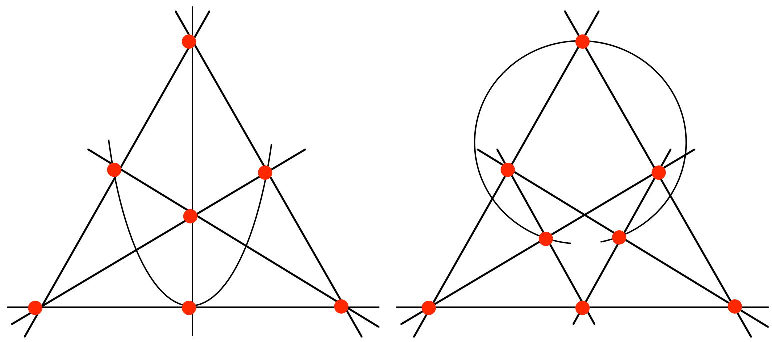

Figure 2.6.3. Frequently shown illustrations of Steinitz's theorem, by 1-subfigurations of the Fano (7_3) and Möbius-Kantor (8_3) configurations.

Note that, since the chosen line is incident with only three points (one of which is the chosen point), it is in every case possible to find a curve of degree at most two (even a circle, unless there is a straight line), incident with these three points.

There is no indication in any of the writings of Steinitz or other mathematicians during the 20th century that they were aware of the fact that Steinitz's claim as formulated by him is not valid, since it pretends to prove strong "near-realizability". The first indication (known to me) of the awareness that the theorem actually *established* by Steinitz has to be formulated in terms of weak "near-realizations" — that is, near-representations, as we did above — was in a talk by T. Pisanski [P3], at a meeting in Ein Gev (Israel) in April 2000. The fact that some combinatorial 3-configurations have only geometric representations and no geometric realization was also noticed by W. Kocay and R. Szypowski [K12] in 1999 and by Glynn [G3] in 2000; neither work mentions Steinitz.

Proof of Theorem 2.6.1. Starting from a connected combinatorial configuration (n_3) given by a configuration table, the first step is to convert the table into an orderly table. This is possible by Theorem 2.5.1. Next, the rows are permuted so that the exceptional point of the exceptional line is in the first row. Note that since the table was orderly, this yields a correspondence (possibly different from the one we started with) in which each point is associated with a line that contains it. As mentioned earlier, and as is easily seen, by possibly interchanging the order of the columns (that is, lines) the orderly configuration table can be rearranged to show the multilateral decomposition in such a way that the lines of each constituent multilateral occur consecutively. We rearrange the columns in such a way that the multilateral that contains the chosen line is placed last, and the selected line is chosen as the last line in the multilateral. If the multilateral is Hamiltonian, this part of the proof is completed. Otherwise, since the configuration is connected, at least one of the points of this multilateral must be associated to (that is, be in the first row of) a line which is not in the multilateral. Choose the multilateral containing this line to be the next-to-last, and the line in question to be its last line. Then some point of this multilateral must be associated with another multilateral not used so far, and we continue in the same way. At the end we reach what we called an **arranged** configuration table.

As an illustration, we show in Table 2.6.2 an orderly configuration table of a configuration (14_3). Choosing as the exceptional elements the point e and the line G, and rearranging the columns so as to make the multilateral decomposition visible, we obtain Table 2.6.3.

A	B	C	D	E	F	G	H	J	K	L	M	N	P
c	k	n	a	q	b	r	h	m	p	e	d	g	f
d	m	a	q	k	h	b	r	e	c	n	f	p	g
b	a	r	p	g	n	e	k	c	f	d	q	m	h

Table 2.6.2. An orderly configuration table of a connected combinatorial configuration (14_3).

A	M	P	N	K	B	J	L	C	D	E	F	H	G
b	q	h	m	f	a	c	d	r	p	g	n	k	e
c	d	f	g	p	k	m	e	n	a	q	b	h	r
d	f	g	p	c	m	e	n	a	q	k	h	r	b

Table 2.6.3. A rearranged configuration table of the (14_3) configuration of Table 2.6.2, in which the lines of each multilateral appear as consecutive columns. The point e of line G was chosen as the exceptional point, so its row is the first one. The line G is the last line of its multilateral, which is the last multilateral. Each multilateral is specified by rows 2 and 3 of the table.

In an abbreviated form, using just the names of the vertices (second entries in the list of points in each column) and the lines to which they belong, this multilateral decomposition can be written as:

c A d M f P g N p K | k B m J e L n C a D q E | b F h H r G (1)

With line G and point e chosen as the exceptional elements, we make the final rearrangement of the columns. The multilateral containing G is placed last, and G is placed as the last line of it. The entry b of the first column in this multilateral is the first point which is associated with a line not in the multilateral. Since b is associated with the line A of the first multilateral, this multilateral is placed next-to-last, and A is placed at its last column; then M is the first line, and the corresponding first point is d. Therefore the multilateral preceding it has L as its last line (column). The rearranged multilateral decomposition (1) has now the following representation:

$$\mathrm{n}\ \underline{\mathrm{C}}\ \underline{\mathrm{a}}\ \underline{\mathrm{D}}\ \underline{\mathrm{q}}\ \underline{\mathrm{E}}\ \underline{\mathrm{k}}\ \underline{\underline{\mathrm{B}}}\ \underline{\mathrm{m}}\ \underline{\mathrm{J}}\ \underline{\mathrm{e}}\ \underline{\underline{\mathrm{L}}}\ \Big|\ \underline{\mathrm{d}}\ \underline{\underline{\mathrm{M}}}\ \underline{\mathrm{f}}\ \underline{\mathrm{P}}\ \underline{\underline{\mathrm{g}}}\ \underline{\underline{\mathrm{N}}}\ \underline{\underline{\mathrm{p}}}\ \underline{\underline{\mathrm{K}}}\ \underline{\underline{\mathrm{c}}}\ \underline{\underline{\mathrm{A}}}\ \Big|\ \underline{\mathrm{b}}\ \underline{\underline{\mathrm{F}}}\ \underline{\mathrm{h}}\ \underline{\underline{\mathrm{H}}}\ \underline{\underline{\mathrm{r}}}\ \underline{\underline{\underline{\mathrm{G}}}} \qquad (2)$$

From this decomposition (2) it is obvious that each element (point or line) is incident with at most two elements that come before it, except that the last line (G in the present case) is incident with three of the preceding points. (For the other elements, the situation is indicated by the single or double underline of the symbols.) This means that elements incident with no previous element can be chosen completely arbitrarily in the plane, those incident with one previous element can be chosen freely as a point on a line or as a line through a point, while those incident with two earlier ones are determined without any freedom of choice. The last triplet may be collinear but need not be — in which case a second degree curve can be passed through it. ¨

For clarity, the final rearranged configuration table is shown in Table 2.6.4.

C	D	E	B	J	L	M	P	N	K	A	F	H	G
r	p	g	a	c	d	q	h	m	f	b	n	k	e
n	a	q	k	m	e	d	f	g	p	c	b	h	r
a	q	k	m	e	n	f	g	p	c	d	h	r	b

Table 2.6.4. The multilateral decomposition (2) in configuration table form.

The geometric subfiguration that resulted from a set of particular choices is shown in Figure 2.6.4. Figure 2.6.5 illustrates the possibility of making choices which happen to satisfy the last incidence as well, hence yield a proper geometric realization of this configuration (14_3).

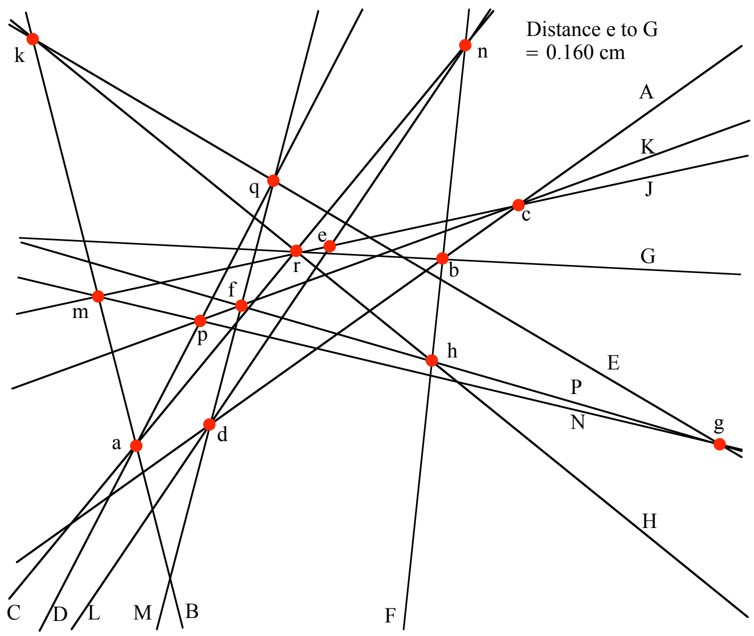

Figure 2.6.4. A "near-realization" following Table 2.4.4 of the configuration (14_3) of Table 2.6.1, in which all the incidences except the one of point e and line G are satisfied.

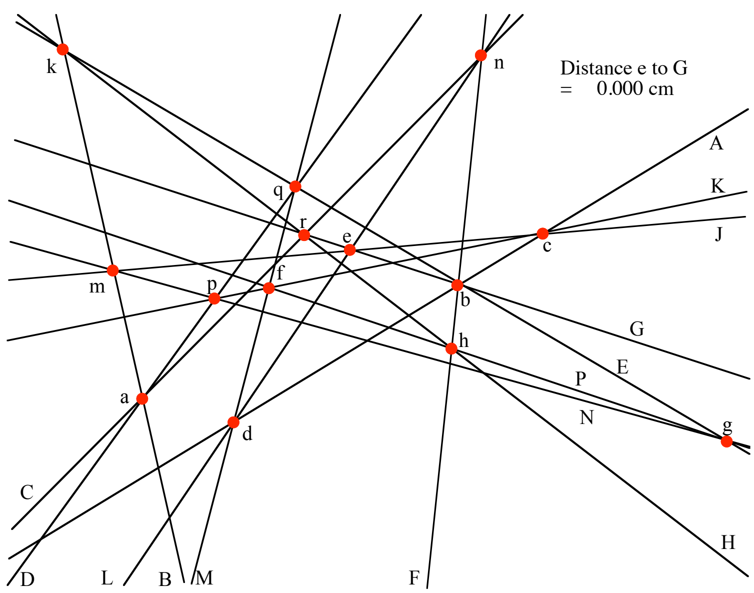

Figure 2.6.5. A *realization* of the configuration (14_3) of Table 2.6.4, in which *all* the incidences are satisfied.

As is clear from the above proof, at no point has there been made any claim that the lines and/or points we introduce have no additional incidences. It has been tacitly assumed that the construction is not undermined by (mis)using the freedom of choice to intentionally place points on already present points or lines they are not supposed to be incident with, and analogously for the selection of lines. However, as shown at the beginning of this section, there are circumstances in which unintended incidences cannot be avoided. The remarkable fact that this possibility has been ignored for a century is nearly incomprehensible.

One still remaining mystery in this context is the fact that all known instances in which unwanted incidences in geometric realizations of combinatorial 3-configurations are unavoidable deal with configurations that are connected (hence 2-connected) but not 3-connected. It is possible that the following holds:

Conjecture 2.6.1. Every 3-connected combinatorial 3-configuration admits geometric realizations by points and straight lines with no incidences except the required ones.

Steinitz's theorem 2.6.1 was proved in [S17] in 1894. In 1999 it was independently discovered by Kocay and Szypowski [K12] in a different setting, and in 2000 by Glynn [G3]. A presentation of the above material and other aspects of Steinitz's theorem appears in [G46].

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The realization in Figure 2.6.5 was obtained by utilizing continuity: In the near-realization shown in Figure 2.6.4 the point e is above the line G, while by choosing some other appropriate positions for the points used in the construction a near-realization can be obtained in which the point e is below the line G. This is a situation that *seems* to be quite general. Steinitz made the same observation in [S17]. Steinitz devoted more than half the dissertation [S17] (24 pages) to a consideration of ways in which one could *guarantee* that the final step in the above proof can be made using a straight line instead of a curve of degree 2. While this might be another interesting result, I have not been able to follow the exposition in [S17]. (In fact, I know of nobody who claims to have understood and verified this part of Steinitz's thesis.) The opaqueness of the exposition can best be seen from the last two sentences of Steinitz's introduction to this part of the work (see [S17, p. 22]):

> ... Without any particular assumptions about the configurations, a method will be presented below following which one can reach a linear presentation. However, for each configuration to which we want to apply this method, an additional investigation is necessary since the method becomes *illusory in certain cases*. [My translation and italics]

In mentioning [S17] in the survey [S19, p. 490], Steinitz is equally uninformative. Stating that his method is an extension of Schroeter's approach in [S6], [S8], he ends the explanation by stating:

> Schroeter's method can be generalized so that it is applicable to *most* configurations n_3. [My translation and italics]

It seems that the "method of Schroeter" is rooted in arguments due to Möbius in the early part of the nineteenth century, in particular in [M20].

However, even if the proof in [S17] is valid, and if somebody were to make the exposition understandable — this would prove only that every connected configuration has *representations*. It would not be a proof of Conjecture 2.6.1 for *realizations*, as claimed by Steinitz. Indeed, we know from examples such as the one in Figure 2.6.1 that some representable configurations are not realizable, hence Conjecture 2.6.1 cannot be generally valid for realizations.

Exercises.

1. Find a geometric construction analogous to the one given above, for the combinatorial configuration of Table 2.5.1, but with the choice of point q and line D as the exceptional elements.

2. Find the analogous construction for the combinatorial configuration which has as its table the first three rows of the orderly table obtained in the worked example in Section 2.5, and with point a and line A as exceptional elements. Using suitable software, see whether this configuration has a proper realization.

3. Apply the methods of construction we used here to the configurations $(10_3)_3$ and $(10_3)_4$ of Table 2.2.7.

4. Find a connected combinatorial configuration for which every representation is a #3-subfiguration, that is, contains at least three unwanted incidences.

5. Show that there are connected combinatorial configurations (n_3) for which in every representation the number of unwanted incidences is at least $c \cdot n$, for some constant $c > 0$. Open problem: What is the best possible c?