2.4 GENERAL CONSTRUCTIONS FOR COMBINATORIAL 3-CONFIGURATIONS

From the start of investigations of configurations, the problem of constructing all configurations (n_3) for each value of n attracted considerable attention. Many of the results on that topic have been presented in Sections 2.2 and 2.3 for specific small values of n. However, already in 1887 Martinetti [M2] described an inductive procedure that can be used to generate the (n_3) configurations if all configurations with fewer points are known. He illustrated his method by determining all configurations (n_3) with $n \leq 11$, starting from the Fano configuration (7_3). As mentioned in Section 2.3, his enumeration of the 31 configurations (11_3) was correct. However, one of his claims was unfounded: He considered the enumeration of geometric configurations (n_3) to be the same as the enumeration of the combinatorial (n_3). This claim was also stressed in the review [L54] of [M2] by E. K. Lampe. As we have seen, $\#_c(n) = \#_g(n)$ for $n = 11$ and 12, but not for $n = 10$ and certainly $\#_c(n) \neq \#_g(n)$ for all $n \geq 14$. In the remaining part of this section, we consider *only* combinatorial configurations, even though we speak of "points" and "lines".

The central idea of Martinetti's construction is the following: Assume that in a combinatorial (n_3) configuration we have two "parallel" lines (that is, lines of the configuration that have no point of the configuration in common). If [A, A', A"] and [B, B', B"] are such lines and if A and B are on no line of the configuration, then we delete the two parallel lines and introduce a new point C, together with the three lines [A, B, C], [A', A", C], [B', B", C]. This is illustrated in Figure 2.4.1. Clearly, this leads to a combinatorial configuration $((n+1)_3)$. A configuration is called **reducible** if it can be obtained from a smaller one by the process just described; otherwise it is **irreducible**. Martinetti's main result is the claim that for each n there are very few irreducible (n_3) configurations, and he purports to give a complete description of all irreducible configurations. More precisely:

Martinetti's claim. A connected (n_3) combinatorial configuration is irreducible if and only if it is one of the following:

(i) The cyclic configuration $\mathscr{C}_3(n)$ with lines $[j, j+1, j+3]$ (mod n), for $n \geq 8$;

(ii) $n = 10m$ for some $m \geq 1$, and the configuration is the one described below and denoted $\mathsf{M}(m)$; $\mathsf{M}(1)$ is the Desargues configuration $(10_3)_1$.

(iii) $n = 9$, and the configuration is the Pappus configuration $(9_3)_1$.

(iv) $n = 10$, and the configuration is $(10_3)_2$ or $(10_3)_6$.

Martinetti's combinatorial configuration $\mathsf{M}(m)$ can best be explained as consisting of m copies of the family of the ten points indicated by solid dots in Figure 2.4.2, and the ten solid lines shown there. The j^{th} copy is joined to the $(j+1)^{st}$ by identifying A'''_j, B'''_j, C'''_j with A_{j+1}, B_{j+1}, C_{j+1}, respectively; all subscripts taken (mod n).

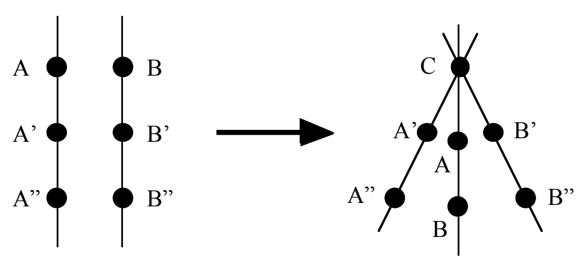

Figure 2.4.1. Martinetti's addition of a point and a line to a combinatorial configuration (n_3).

Martinetti's proof is, not surprisingly, involved and long. The result was quoted or mentioned many times over the next century; see, for example, Steinitz [S19, pp. 486-487], Steinitz-Merlin [S21, pp. 153 – 154], Gropp [G7], [G8], [G25], [G30], Carstens *et al.* [C1]. In some of these it was noticed that Martinetti should have included connectedness among the requirements of his claim. Moreover, in lecture notes for my configurations courses in 1999 and 2002 I wrote:

> I have not checked the details, and I do not know it as a fact that anybody has. The statement has been accepted as true for these 115 years, and it may well be true. On the other hand, Daublebski's enumeration of the (12_3) configurations was also considered true for a comparable length of time ...

As it turned out, my suspicion has been vindicated by Boben [B16], [B17]. He showed that Martinetti's list of irreducible configurations is incomplete. The problem in Martinetti's proof arises as follows. When constructing $\mathsf{M}(m)$, we attach to each other m copies of the "module" in Figure 2.4.2 as indicated above, but stop before attaching $\mathsf{M}(m)$ to $\mathsf{M}(1)$. Martinetti formed that attachment "straight", by identifying A'''_n with A_1, and similarly for the B's and C's, thus obtaining $\mathsf{M}(m)$. However, as shown by Boben, that attachment can be done in "twisted" ways as well; two such attachments yield irreducible configurations which we may denote by $\mathsf{M}^*(m)$ and $\mathsf{M}^{**}(m)$. These are obtained by identifying A'''_n with C_1, B'''_n with B_1, and C'''_n with A_1 for the former, and A'''_n with C_1, B'''_n with A_1, and C'''_n with B_1 for the latter. A separate argument shows that the three resulting configurations are non-

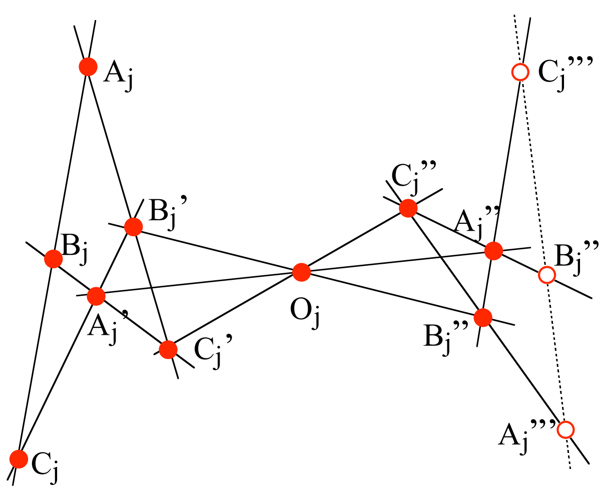

Figure 2.4.2. The "module" used in the Martinetti construction. Only the ten solid dots and the ten solid lines form one module.

isomorphic for every m. With this modification, we have the following corrected version of Martinetti's result:

Theorem 2.4.1. (Boben [B16], [B17]) A connected (n_3) combinatorial configuration is irreducible if and only if it is one of the following:

(i) The cyclic configuration $\mathscr{C}_3(n)$ with lines $[j, j+1, j+3]$ (mod n), for $n \geq 7$;

(ii) $n = 10m$ for some $m \geq 1$, and the configuration is one of $\mathsf{M}(m)$, $\mathsf{M}^*(m)$ or $\mathsf{M}^{**}(m)$ described above. For $m = 1$ these are the configurations $(10_3)_1$, $(10_3)_2$ and $(10_3)_6$ (in the notation used in Section 2.2).

(iii) $n = 9$, and the configuration is the Pappus configuration $(9_3)_1$.

A remarkable aspect of the situation is that all the irreducible configurations (n_3) with $n \geq 9$ are geometrically realizable by straight lines in the Euclidean plane. For the cyclic configurations we have seen this in the proof of Theorem 2.1.3. A different construction, involving cubic curves, was given by Schroeter [S6] in 1888. For the configurations $\mathsf{M}(m)$ the realizability is almost obvious from Figure 2.4.2, and can be proved in general.

Concerning configurations (n_3) for particular values of $n \geq 13$, there is very little specific information available in print. Gropp [G13] applied Martinetti's theorem to enumerate the combinatorial configurations with up to 14 points. He reports that there are 2036 combinatorial configurations (13_3), and 21,399 combinatorial configurations (14_3). These numbers were confirmed by [B14]; this paper reports the numbers $\#_c(n)$ of combinatorial configurations (n_3) for $n \leq 18$, see Table 2.2.1. The number $\#_c(19)$ was reported in [B19] and [G46].

One of the combinatorial configurations (14_3) consists of two disjoint copies of the (7_3) configuration, and is therefore not geometrically realizable. It is not known whether the other (13_3) and (14_3) combinatorial configurations are geometrically realizable. Clearly, analogous disconnected and non-realizable configurations exist for all $n \geq 14$. However, even if considering only connected configurations, the statement in Steinitz [S19, p.490] and Steinitz-Merlin [S21, p.158] that for $n \geq 11$ all (n_3) combinatorial configurations are "probably realizable" is contradicted by the example in Figure 2.2.8 (from [D10], see also Gropp [G25]), which shows that the statement is invalid even if restricted to configurations that are "connected" and "realizable by pseudolines". (In the example in Figure 2.2.8, the left part of the figure has all but one of the incidences of the Pappus configuration, and therefore by Pappus' theorem the line L must be incident with the point P.) It is not known whether the (16_3) in Figure 2.2.8 is the smallest configuration with these properties. We shall discuss this and related question in Sections 2.5 and 2.6 dealing with a remarkable result of Steinitz.

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Levi [L3, p. 93] mentions the possibility of obtaining a combinatorial configurations $((n+1)_3)$ from the configurations (n_3). He achieves this by manipulating Levi incidence matrices in a way that is equivalent to the Martinetti method illustrated in Figure 2.4.1. However, Levi does not mention Martinetti, or irreducible configurations — nor does he claim that all $((n+1)_3)$ configurations are obtainable in this way.

Exercises and problems 2.4.

1. Prove that all the irreducible configurations with at least nine points specified in Theorem 2.4.1 are geometrically realizable by points and straight lines.

2. Decide whether the (12_3) configurations in Figures 2.3.7 and 2.3.8 are reducible or irreducible. If any is reducible, to which irreducible one does it ultimately reduce? Is it possible for one configuration to reduce to different irreducible configurations?

3. Investigate the reducibility of the cyclic configurations $\mathscr{C}_3(n,1,4)$.

4. Give a formulation of Theorem 2.4.1 that is valid for all 3-configurations.