2.3 ENUMERATION OF 3-CONFIGURATIONS (Part 2)

Combinatorial configurations (11_3) were first enumerated by Martinetti [M2] in 1887; using the method we shall describe in the next section, he found that $\#_c(11) = 31$. The enumeration of these configurations was independently carried out by Daublebsky [D1] in 1894; he used a variant of the remainders method. Diagrams supposed to show geometric realizations of all 31 combinatorial configurations (that is $\#_g(n) = 31$) were provided by Daublebsky in an appendix to [D2] in 1895 (shown in Figure 2.3.1 below; see also Figure 2.3.2).

Daublebsky states that all these combinatorial configurations can be realized as geometric configurations (that is, with points and straight lines) given by his diagrams, but does not give any justification beyond the intimation that he followed the method of Schroeter [S8]. An independent verification of the geometric realizability of all 31 configurations (11_3) was provided only nearly a century later, by Sturmfels and White [S23], [S24] in 1988 and 1990, with a different method; we shall discuss this method a little later. Sturmfels and White also proved that each of these configurations can be realized in the rational plane, in other words, one can always draw the configurations so that the vertices are at points of the integer lattice. The value of $\#_c(11) = 31$ was independently confirmed by Gropp (see [G8]) and by Betten *et al.* [B14], among others.

* * * * * The first enumeration of the combinatorial configurations (12_3) was carried out by Daublebsky [D2] in 1895, again using the method of remainder figures. He found that only 18 different remainder figures could possibly occur in such a configuration. Through various arguments (described only in general terms) Daublebsky arrived to the conclusion that these remainder figures could be combined to yield something like 1600 configurations (12_3). Then he "… drew a schematic diagram of each configuration on a separate piece of paper …" and determined for each the "remainder system", that is, a list of the different remainder figures occurring in the configuration. Finally, configurations with the same remainder system were investigated to see whether they are isomorphic. This turned out to be the case in most—but not all—cases

(see Exercises 2.3.4 and 2.3.5). Daublebsky presented the resulting 228 combinatorial configurations by their configurations tables (in the form he gave them, these take 23 pages!!!). He also gave some other data and provided drawings for geometric realizations of a few of the configurations. In a later paper [D3], Daublebsky gave results of his investigations of the groups of automorphisms of each of the 228 combinatorial configurations (12_3). However, not all of these are correct. The first independent enumeration of the combinatorial (12_3) configurations was carried out only in 1990, by Gropp (see [G8]). It showed that Daublebsky missed one, so that there are in fact $\#_c(12) = 229$ such configurations. Gropp published the configuration table of this additional configuration in [G13] and communicated it to me; the table can also be read off from the illustrations in the more readily available [D10] and [G25]. As with configurations (11_3), the 229 combinatorial configurations (12_3) have been independently enumerated (by two different methods) in [B14]. Even so, Dolgachev [D8] in 2004 still quotes $\#_c(12) = 228$.

The only published proof that all 228 combinatorial configurations (12_3) found by Daublebsky are geometrically realizable was given only recently, by Sturmfels and White [S23], [S24]. Sturmfels and White also proved that all these (12_3) configurations are realizable in the rational plane. In a private communication, B. Sturmfels showed that the "new" combinatorial configuration found by Gropp is also geometrically realizable, even in the rational plane; a diagram is shown in Dorwart – Grünbaum [D10] in 1992.

The numbers $\#_c(n)$ of combinatorial configurations for $13 \le n \le 19$ were determined by various computer programs. For $12 \le n \le 14$ these values were first found by Gropp [G8], for $n = 15$ by Betten and Betten, [B11]; the values for $16 \le n \le 18$ in Table 2.2.1 are from Betten, Brinkmann and Pisanski [B14]. The value $\#_c(19) = 7{,}640{,}941{,}062$ was determined by these authors and published in [B19] and [G46]. However, there is no information available about the possibilities of realization of the combinatorial configurations (n_3) for $n \ge 13$ by topological or geometric configurations, beyond individual examples — these will be discussed in the following sections. This is not very surprising in

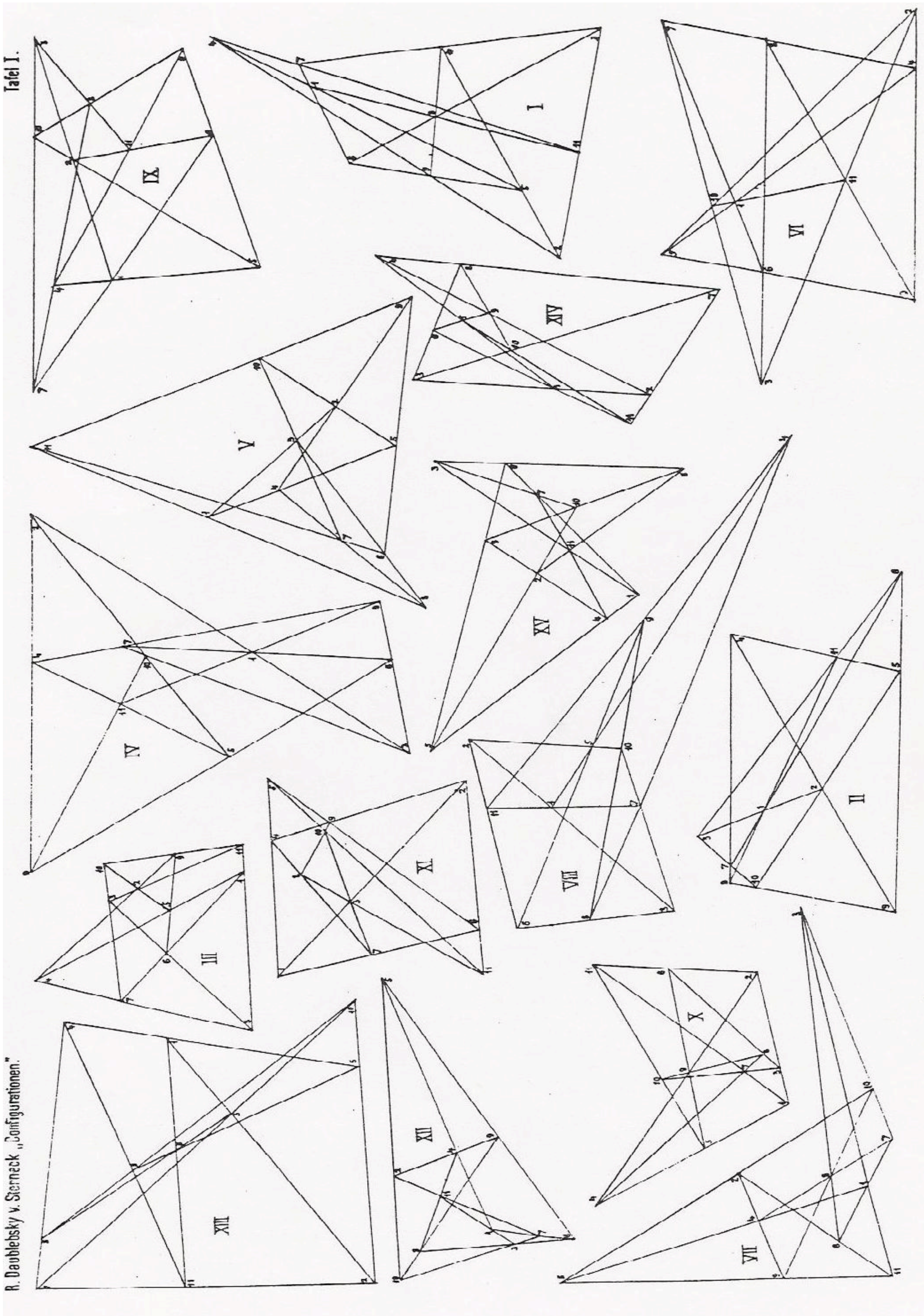

Figure 2.3.1. (first half).

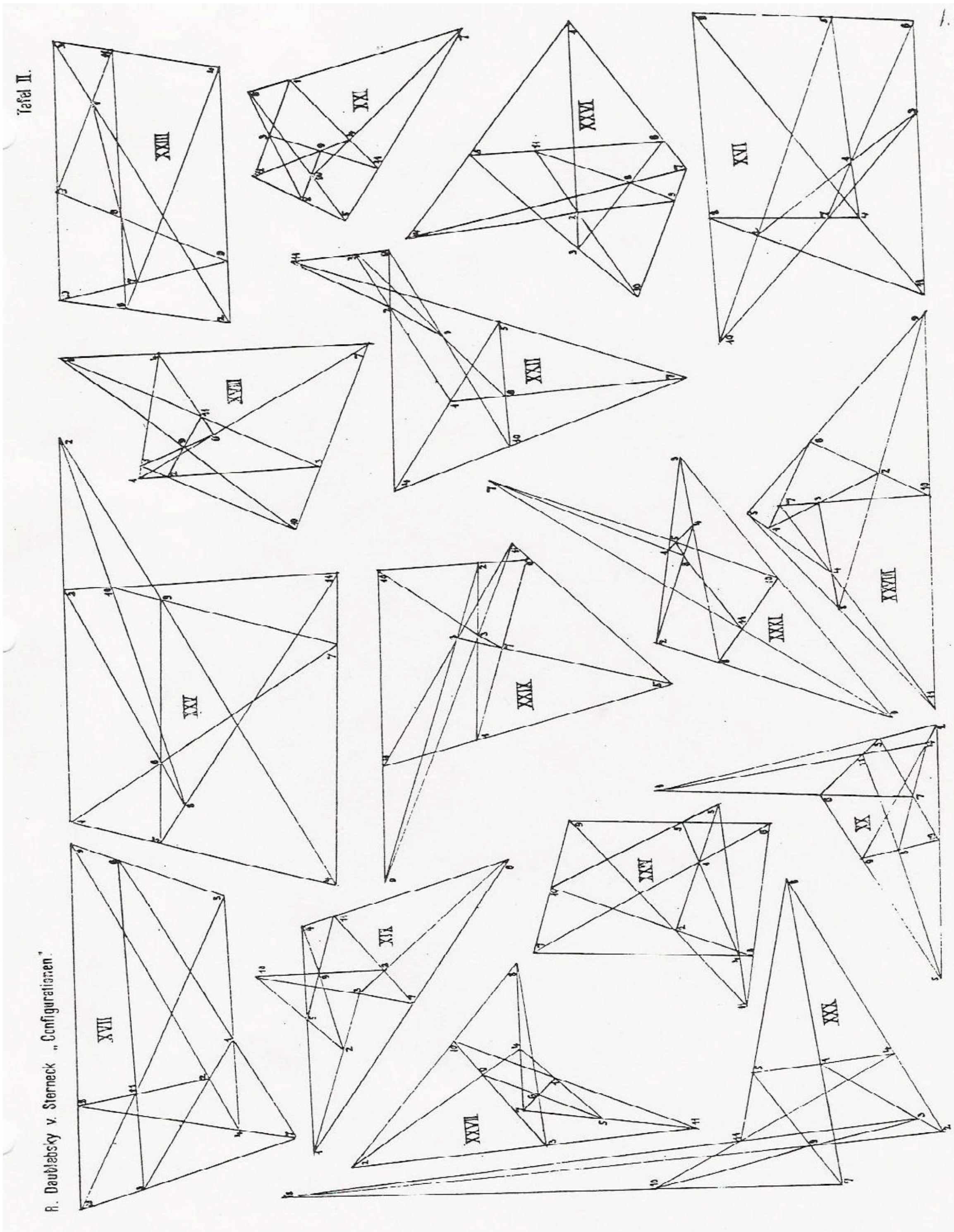

Figure 2.3.1 (second half). The diagrams of the (11_3) configurations, from Daublebsky [D2].

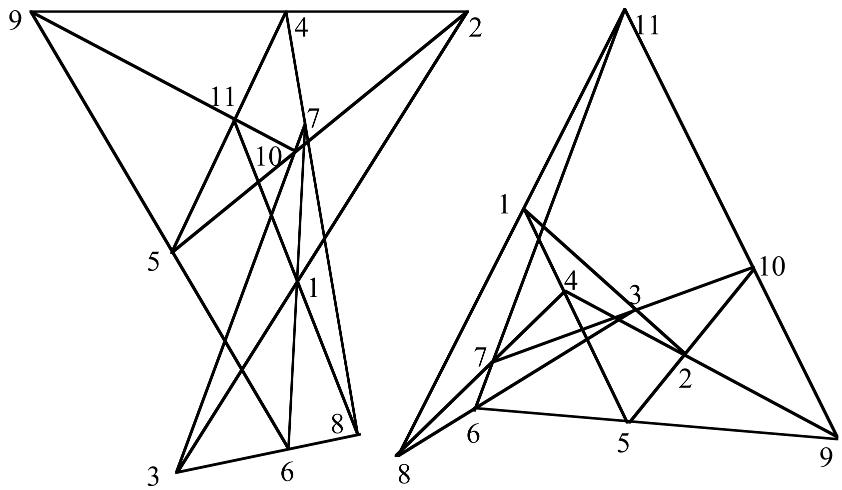

Figure 2.3.2. Diagrams of Daublebsky's configurations $(11_3)_4$ and $(11_3)_5$ redrawn for better visibility.

view of the number of combinatorial configurations. As shown in Table 2.2.1, this number is well above 2000 for n = 13, and increases by factors exceeding 10 for larger n.

This completes the discussion of the data in Table 2.2.1. The only additional information that is available is that $\#_c(n) > \#_t(n)$ for all $n \geq 14$, and that $\#_t(n) > \#_g(n)$ for all $n \geq 16$. The former happens due to the existence of **disconnected configurations** – that is, configurations that are disjoint unions of two or more configurations, between the elements of which there are no incidences.

As an example, consider the (14_3) which consists of two disjoint copies of the Fano configuration (7_3), or the (15_3) formed by disjoint copies of (7_3) and (8_3); the latter was implicitly recognized as disconnected by Betten and Betten [B11], the former is explicitly mentioned by Gropp [G7]. Since disconnected configurations arise as unions of smaller configurations, it is easy to determine the number of such configurations for all $n \leq 19$. Since the (7_3) and (8_3) set-configurations are not geometrically realizable, the smallest geometrically realizable disconnected configurations are the six arising as unions of two configurations (9_3). The same is true for topological configurations.

On the other hand, the inequality between the numbers $\#_t(n)$ and $\#_g(n)$ of topological and geometric configurations for $n \geq 16$ is a consequence of the existence of topological configurations of the kind illustrated by the scheme in Figure 2.3.3. Due to the theorem of Pappus, if this configuration scheme is rendered with straight lines instead of line segments, the points A_2, B_2, C_2, and F_3 are seen to be collinear. Hence this is a superfiguration and not a geometric configuration; clearly, this is not a problem if pseudolines are used. This example can be understood as arising by a "melding" of the Pappus configuration $(9_3)_1$ and the Fano configuration (7_3). (Note that the Fano part is missing one incidence, and this subfiguration is realizable by straight lines.) This construction can be modified in various ways. For example, instead of the Fano configuration one could use any (n_3) configuration, and instead of the Pappus configuration one could use Desargues' configuration $(10_3)_1$. This completes the proof of $\#_t(n) > \#_g(n)$ for all $n \geq 16$. It is not known whether $\#_t(n) = \#_g(n)$ for $n = 13, 14, 15$.

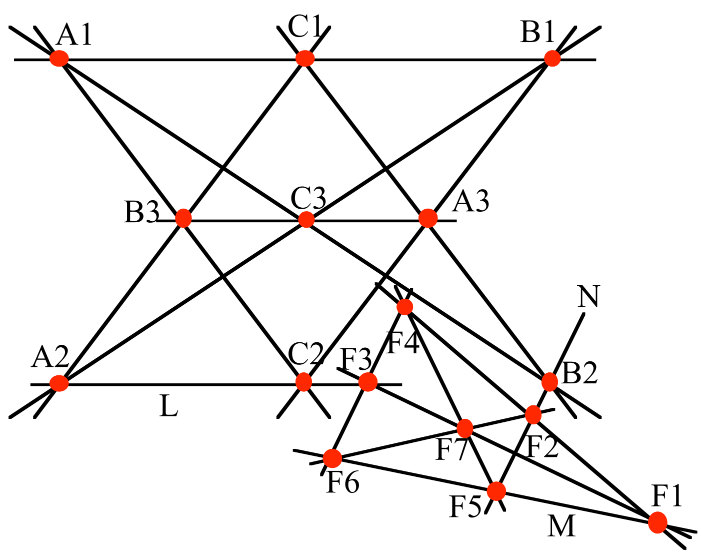

Figure 2.3.3. Pappus' theorem implies that the points A_2, B_2, C_2, and F_3 are collinear, hence this does not realize a configuration (16_3). It is obvious that using pseudolines the unwanted incidence can be avoided.

This ignorance is part of a larger open question. The single example establishing $\#_t(10) > \#_g(10)$ differs in one important respect from the examples just given with $n \geq 16$: The latter are only 2-connected, while the combinatorial and topological $(10_3)_4$ is 3-connected. The lack of any other 3-connected examples leads to

Conjecture 2.3.1. Every 3-connected topological configuration (n_3) with $n \geq 11$ is geometrically realizable.

* * * * *

The Schroeter constructions explained and illustrated above would nowadays be said to be *generic* constructions, the terminology supposing to indicate that it applies in run-of-the-mill situations. In fact, if understood literally — that all the choices can be made arbitrarily, with only the stated restrictions — the constructions may fail to lead to the *configurations* they are supposed to yield. Instead, *superfigurations* may result due to "accidental" incidences. This is illustrated in Figure 2.3.4.

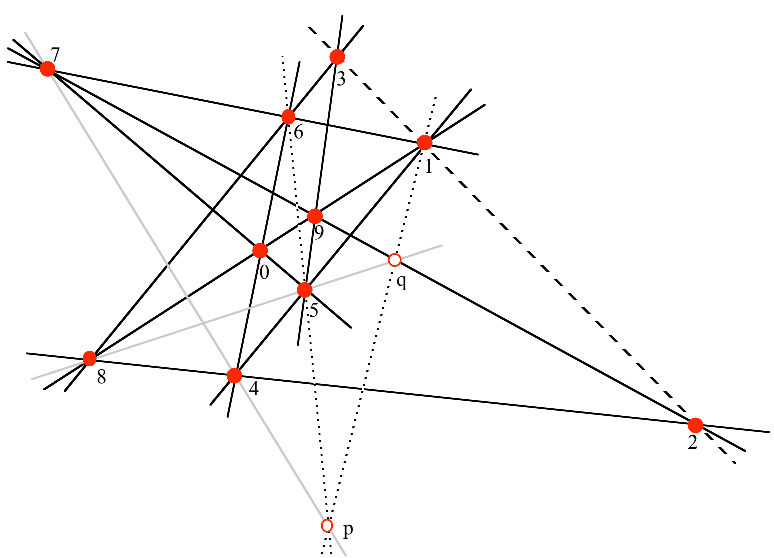

Figure 2.3.4. Failure of the Schroeder construction of the configuration $(10_3)_3$: The line 890 contains the point 1. Notation is the same as in Figure 2.2.5.

It is hard to understand that no publication on configurations during the classical period even mentioned the possibility of superfigurations arising in the construction of geometric configurations. This is astonishing since the study of accidental incidences in the Desargues configuration was already old hat at that time. In Figure 2.3.5 we show a Desargues superfigurations, with a line on four points and a point on four lines. The exceptional point and line are shown in contrasting color. In the paper [S22] Sternfeld *et al.* study possible superfigurations of (10_3) configurations (and more general incidence systems), both combinatorial and geometric. We conjecture:

Conjecture 2.3.2. Every geometric configuration (n_3) with $n \geq 10$ admits superfigurations with at least one pair of "accidental" incidences.

It is worth mentioning that the three (9_3) configurations do not have limiting positions that are superfigurations. On the other hand, the Pappus configuration $(9_3)_1$ has representatives in which an additional point incident with three lines, or a line incident with three of the points, or both, can be found. The last alternative is illustrated in Figure 2.3.6. It is not known whether many other configurations have this property.

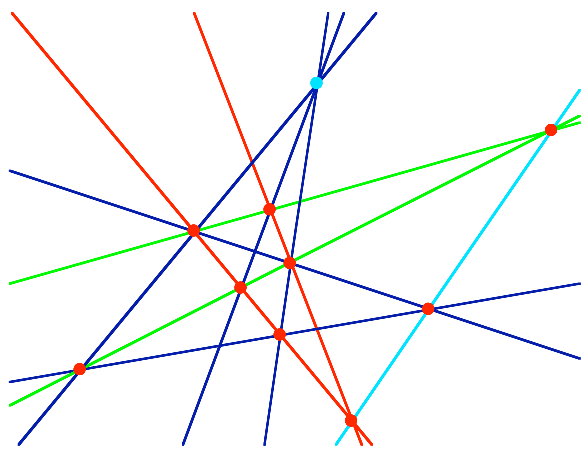

Figure 2.3.5. A superfiguration arising from the Desargues configuration $(10_3)_1$ through multiple incidences. The point and line of perspectivity are shown in teal.

1. Find the remainder systems of Daublebsky's configurations $(11_3)_4$ and $(11_3)_5$ shown in Figure 2.3.2, and use them to show that these are distinct configurations.

3. Find a superfiguration of the Desargues configuration that has three points and three lines incident with four elements of the other kind — or prove that such a configuration cannot exist.

4. Consider the two (12_3) configurations from Daublebski's paper [D2] shown in Figure 2.3.7, with their labels as given by Daublebski. Although they are tantalizingly similar, show that they are not isomorphic.

5. Determine whether the two (12_3) configurations in Figure 2.3.8 are isomorphic, and whether any is isomorphic with either of the configurations in Figure 2.3.7.

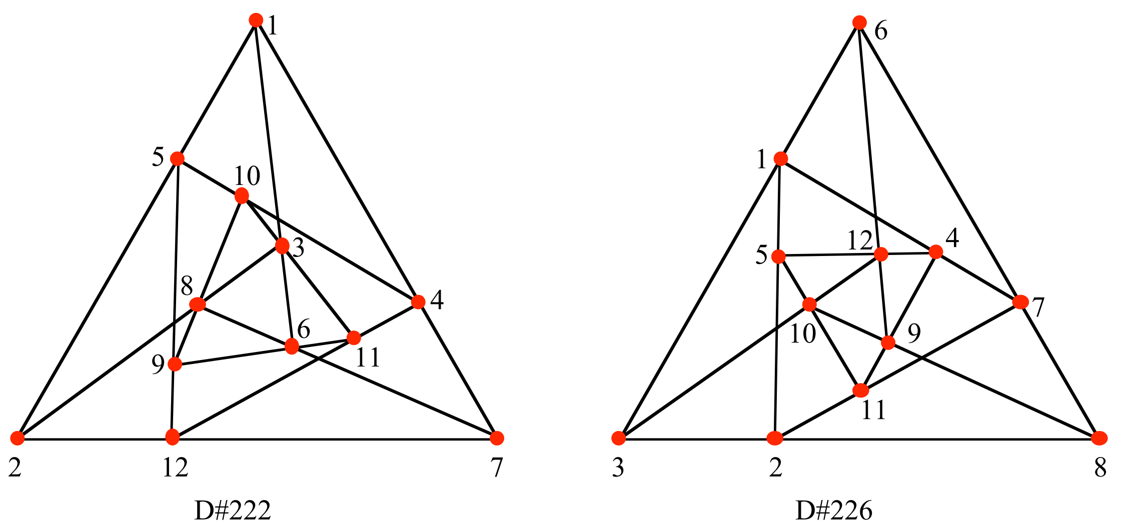

Figure 2.3.7. Two configurations (12_3) from [D2].

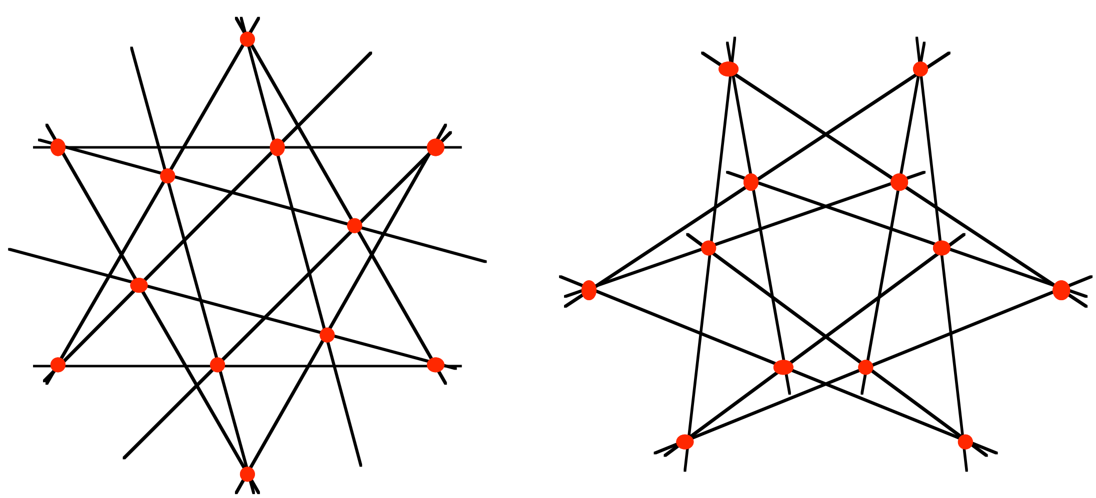

Figure 2.3.8. Two (12_3) configurations.