2.2 ENUMERATION OF 3-CONFIGURATIONS (Part 1)

We turn now to the presentation of the results known about the number of non-isomorphic configurations (n_3) of the three kinds, for each value of n — as far as these numbers are known. As we shall see, this is not very far. Moreover, we have to discuss some other questions concerning these enumerations.

For the purposes of this section, we shall denote by $\#_c(n)$, $\#_t(n)$ and $\#_g(n)$ the number of non-isomorphic combinatorial, topological or geometric configurations (n_3), respectively. We begin with:

Theorem 2.2.1. The complete list of known numbers $\#_c(n)$, $\#_t(n)$ and $\#_g(n)$ is given in Table 2.2.1.

We start by presenting the proofs of the enumerations for $n \leq 8$. Following this we shall first discuss the case $n = 9$, then the rather unexpected situation for $n = 10$, and finally the cases of $n \geq 11$. Some general considerations will be explained next, with exercises and problems to follow.

From Section 2.1 we already know that all three numbers are 0 for $n \leq 6$. Now we first show that each of the (combinatorial) configurations (7_3) and (8_3) is unique. This follows easily from the consideration of the formation of their configuration tables. For (7_3), starting with the three lines that contain 1 and then continuing by using the freedom of assigning labels to previously uncommitted points in the only possible way, we obtain the unique configuration table shown in Table 2.2.2. For (8_3) we first note that since each point is connected (by a line of the configuration) to six other points, it is fails to be connected to a unique point. Designating the unconnected pairs by {1,5}, {2,6}, {3,7}, and {4,8}, a similar procedure leads to the unique configuration table shown in Table 2.2.3. The uniqueness of these configurations has been known since early in the study of configurations. The uniqueness of (8_3) seems to have been established first by Kantor [K3], while $\#_c(7) = 1$ was proved by Martinetti [M2, pages 3,4] ; for other proofs see, for example, Levi [L3, pp. 94, 98], Hilbert–Cohn Vossen [H4]).

n	$\#_c(n)$	$\#_t(n)$	$\#_g(n)$
≤ 6	0	0	0
7	1	0	0
8	1	0	0
9	3	3	3
10	10	10	9
11	31	31	31
12	229	229	229
13	2,036		
14	21,399		
15	245,342		
16	3,004,881		
17	38,904,499		
18	530,452,205		
19	7,640,941,062		

Table 2.2.1. The numbers of non-isomorphic configurations (n_3) of the three kinds, for each n. All known values are shown.

1	1	1	2	2	3	3
2	4	6	4	5	4	5
3	5	7	6	7	7	6

Table 2.2.2. A configuration table of the unique combinatorial configuration (7_3).

1	1	1	2	2	3	3	5
2	4	7	4	5	4	6	6
3	6	8	7	8	5	7	8

Table 2.2.3. A configuration table of the unique combinatorial configuration (8_3).

Similar arguments can be applied to the determination of the different combinatorial configurations (9_3). Easier to carry out is an application of the "remainder figures" method described in Section 1.4. First comes the observation that each point fails to be connected to precisely two other points. Drawing an edge (segment) between any two unconnected points, we see that the unconnected pairs form one or more circuits. (This is the *deficiency graph* of this configuration introduced in Section 1.4.) Since a circuit has to have at least three points, there are four potential sets of circuits: A single 9-circuit, a 6-circuit and a 3-circuit, a 5-circuit and a 4-circuit, and three 3-circuits. It is obvious that different sets of circuits imply that the configurations are not isomorphic, since any isomorphism preserves connected pairs of points, hence also disconnected pairs. Similarly to the earlier cases, it is possible to show that each case corresponds to a (unique) configuration, except that the case of one 5-sided and one 4-sided circuits corresponds to no configuration. The reason for this is the following: Assume that it is possible, and consider the lines incident with the vertices of the 4-circuit. Two lines correspond to the "diagonals" of the 4-circuit, while each of the four vertices has to be on two additional lines, all distinct and different from the earlier two; this would require the existence of at least 10 lines. Hence such a possibility cannot lead to a configuration. The result, using these or other arguments, appears in Kantor [K3], Martinetti [M2], Schroeter [S6], and again in Levi [L3, p. 103], Hilbert–Cohn Vossen [H4], Gropp [G23].

Configuration tables of the three combinatorial configurations (9_3) are shown in Tables 2.2.4, 2.2.5, and 2.2.6. All three of these configurations can be geometrically realized; this was first proved by Kantor [K3], and more thoroughly analyzed by Schroeter [S6]. Representative examples of such realizations are shown in Figure 2.2.1, and the same representatives are shown in Figure 2.2.2 with the circuits formed by non-connected pairs of points.

1	1	1	2	2	2	3	3	3
4	5	6	4	5	6	4	5	6
7	8	9	8	9	7	9	7	8

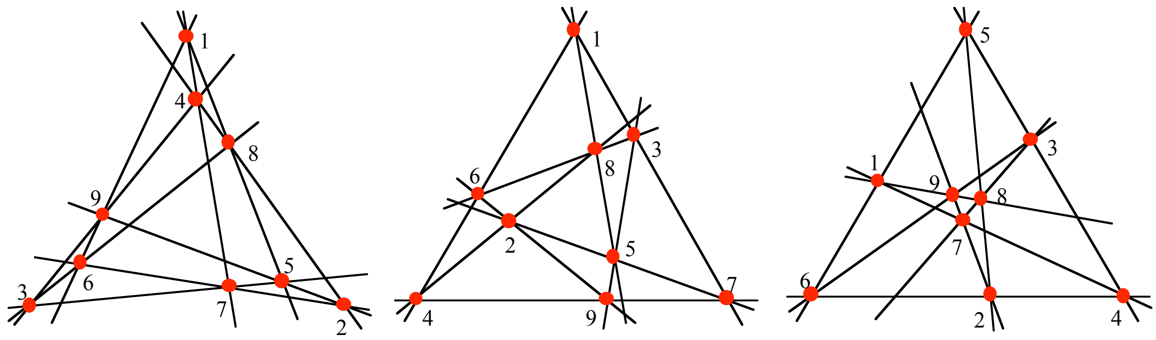
Table 2.2.4. A configuration table for the configuration $(9_3)_1$.

1	1	1	2	2	2	3	3	4
3	4	5	4	5	6	5	6	7
7	6	8	8	7	9	9	8	9

Table 2.2.5. A configuration table for the configuration $(9_3)_2$.

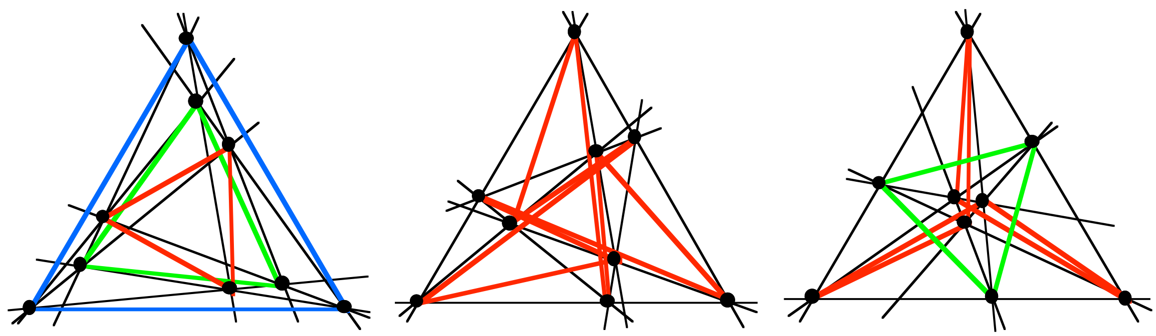
1	1	1	2	2	2	3	3	3
4	5	8	4	5	7	4	6	7
7	6	9	6	8	9	5	9	8

Table 2.2.6. A configuration table for the configuration $(9_3)_3$.



$(9_3)_1$ $(9_3)_2$ $(9_3)_3$

Figure 2.2.1. Examples of the three types of geometric configurations (9_3). The claim by Steinitz [S19, p. 489] that H. A. Schwarz [S12] found the form of the configuration $(9_3)_3$ with 3-fold rotational symmetry is not correct.



$(9_3)_1$ $(9_3)_2$ $(9_3)_3$

Figure 2.2.2. The circuits formed by the non-connected pairs in the three types of geometric configurations (9_3).

Concerning the (10_3) configurations, we start by presenting in Table 2.2.7 configuration tables for all ten combinatorial configurations. The existence of precisely ten non-isomorphic combinatorial configurations (10_3) has been established repeatedly, by more-or-less brute force enumeration; historical details and references will be given below. In order to prove that these configurations are distinct, we shall use the concept of "remainder figures" which was introduced in Section 1.4: For each vertex of the combinatorial configuration (10_3) consider the three vertices that are not on any of the lines passing through the given vertex. There are three possibilities concerning these three points: Either all three are on one configuration line, or they determine two lines of the configuration, or they determine three such lines (a *trilateral* or “triangle”). We shall denote the three possibilities by I, V, Δ. No other situations are possible. Indeed, if the three points were collinear, but on a line that is not in the configuration, there would be nine configuration lines through them, and three additional lines through the original vertex – while only ten lines are available. But if the three points are not collinear, they determine a triangle. If none of the three lines determined by the points were a configuration line, the configuration would again have to have at least 12 lines. On the other hand, if just one of the lines determined by the sides of the triangle were a line of the configuration, then there would have to be present in the configuration at least 1 + 2 + 2 + 3 + 3 = 11 lines. Thus, the three cases listed earlier are the only ones possible.

The above arguments that the remainder figure in case of (10_3) must be one of I, V, Δ are taken from Schroeter [S8], together with his notation. By very exhaustive and exhausting argumentation one can show that only ten combinations of the different remainder figures (listed in Table 2.2.8) can occur in a combinatorial configuration (10_3), and that each corresponds to a unique isomorphism type of combinatorial configurations, represented by one of the ten configurations in Table 2.2.7. The detailed discussion of the possible combinatorial configurations depending on the kind and number of the remainder figures is spread over 22 pages in [S8]. It leads to the conclusion that each column in Table 2.2.8 corresponds to one and only one combinatorial configuration (10_3).

As far as geometric realizations go, in Figure 2.2.3 are shown sketches similar to the ones in the first enumeration of the (10_3) configurations by Kantor [K4]; our Figure 1.2.2 is a copy of one of the Kantor diagrams. The diagram of $(10_3)_1$ is easily checked to be an illustration of the Desargues configuration. We shall encounter $(10_3)_{10}$ in Section 2.6 as the astral configuration denoted 5#(2,2;1).

$(10_3)_1$

1	1	1	8	2	3	2	3	4	5
2	4	6	9	4	5	6	7	6	7
3	5	7	0	8	8	9	9	0	0

$(10_3)_2$

1	1	1	8	2	3	2	3	4	5
2	4	6	9	4	7	6	5	6	7
3	5	7	0	8	8	9	9	0	0

$(10_3)_3$

1	1	1	8	2	3	2	3	4	5
2	4	6	9	4	6	7	5	6	7
3	5	7	0	8	8	9	9	0	0

$(10_3)_4$

1	1	1	8	2	3	2	3	4	5
2	4	6	9	4	6	5	7	6	7
3	5	7	0	8	8	9	9	0	0

$(10_3)_5$

1	1	1	8	2	3	2	4	3	5
2	4	6	9	4	7	5	6	6	7
3	5	7	0	8	8	9	9	0	0

$(10_3)_6$

1	1	1	8	2	3	2	5	3	4
2	4	6	9	4	7	6	7	5	6
3	5	7	0	8	8	9	9	0	0

$(10_3)_7$

1	1	1	2	4	6	5	3	7	2
2	4	6	8	8	9	7	5	3	4
3	5	7	9	0	0	8	9	0	6

$(10_3)_8$

1	1	1	3	5	7	2	6	4	2
2	4	6	8	8	9	7	5	3	4
3	5	7	9	0	0	8	9	0	6

$(10_3)_9$

1	1	1	2	4	6	5	3	2	3
2	4	6	8	8	9	7	5	7	4
3	5	7	9	0	0	8	9	0	6

$(10_3)_{10}$

1	1	1	3	2	7	5	6	4	2
2	4	6	8	8	9	7	5	3	4
3	5	7	9	0	0	8	9	0	6

Table 2.2.7. The ten non-isomorphic combinatorial configurations (10_3), in the notation of Schroeter [{S2]. They were first determined by Kantor [K4], using other methods and different notation and labeling.

	$(10_3)_1$	$(10_3)_2$	$(10_3)_3$	$(10_3)_4$	$(10_3)_5$	$(10_3)_6$	$(10_3)_7$	$(10_3)_8$	$(10_3)_9$	$(10_3)_{10}$
I	10	4	2	6	1	1	0	0	0	0
V	0	6	6	0	3	9	9	3	6	0
Δ	0	0	2	4	6	0	1	7	4	10
	X	VIII	V	II	I	IX	III	VI	IV	VII
	B	G	D	C	H	F	E	J	K	A

Table 2.2.8. The number of occurrences of the different remainder figures in the ten combinatorial configurations (10_3). The last two rows give the notation by Martinetti [M2] and Kantor [K4

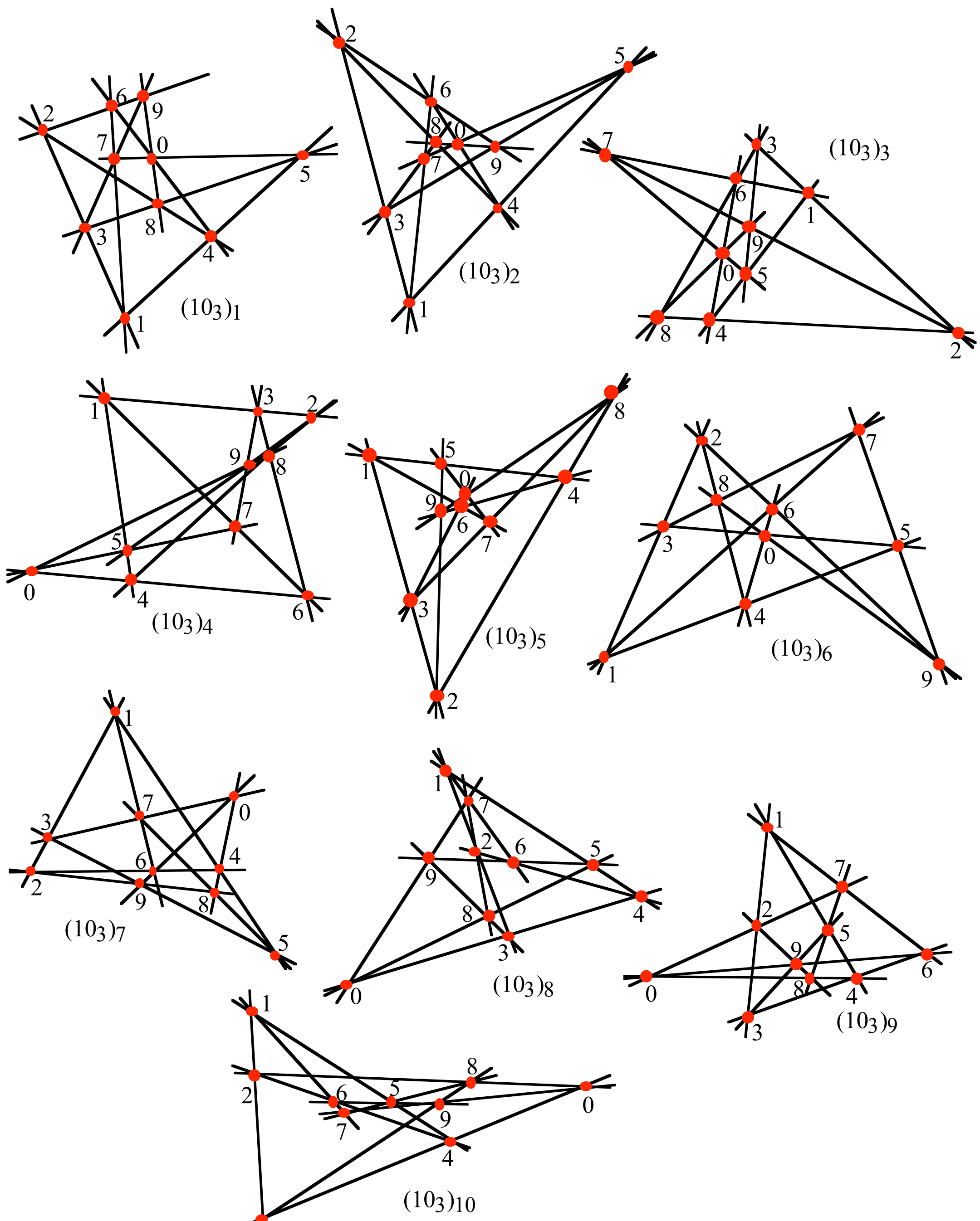


Figure 2.2.3. Sketches of configurations (10_3), analogous to the ones presented in Kantor's paper [K4].

One of the most striking features exhibited by Table 2.2.1 is the inequality $\#_t(10) = 10 > \#_g(10) = 9$. This arises because the diagram in Figure 2.2.4, which *appears* to show a (10_3) geometric configuration (which is a cleaner drawing of the configuration

$(10_3)_4$ in Figure 2.2.3), cannot in fact be realized by straight lines—although the diagram clearly indicates that a topological realization is possible.

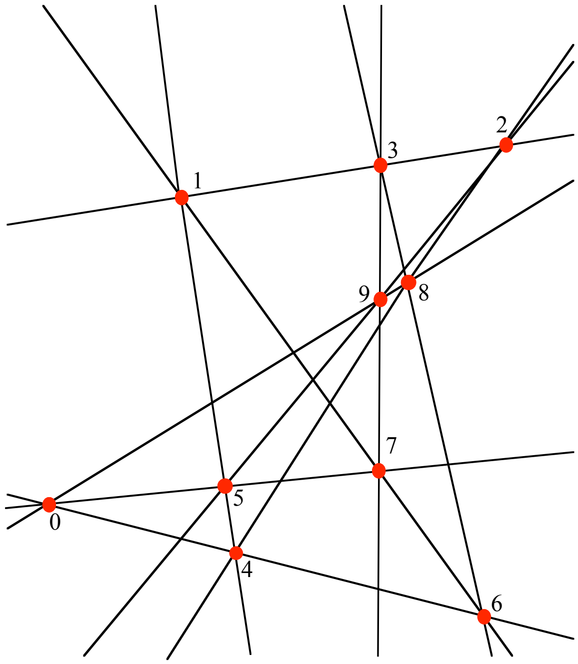


Figure 2.2.4. An apparent geometric realization of the combinatorial configuration $(10_3)_4$, which was also shown in Figure 1.2.2. However, both diagrams are misleading. This configuration is not isomorphic to any configuration of points and (straight) lines. On the other hand, the "lines" are (very mildly curved) pseudolines, hence a topological realization of this configuration is possible.

The impossibility of a geometric realization of $(10_3)_4$ can be established as follows.

The complete quadrangle 2,3,8,9 contains the three pairs of opposite sides

23-1	28-4	29-5
89-0	39-7	38-6

while the complete quadrangle 6,7,9,0 contains the three pairs of opposite sides

67-1	60-4	70-5
90-8	97-3	96-*

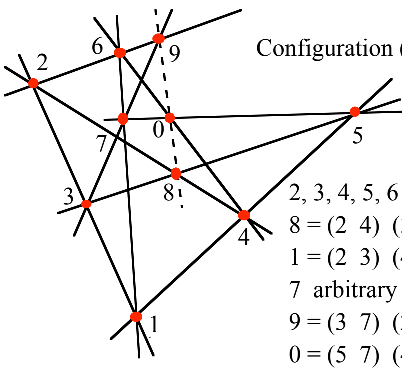
By a basic theorem of projective geometry, the three pairs of lines of each quadrangle intersect the line 145 in three pairs of points of an involution. But these involutions must coincide, since two of the pairs coincide: 1 and $890 \cap 145$, and 4 and $379 \cap 145$. Then the point paired with 5 in the involution must be the intersection point of the three lines 145, 368, 96*. This cannot be 6, since then 145 would contain four points; the only alternative is that 368 and 96* coincide – but then 368 would contain the fourth point 9. Hence the configuration $(10_3)_4$ cannot be realized geometrically.

This proof of the impossibility of a geometric realization of the configuration $(10_3)_4$ is due to Schroeter [S8]. The difference between topological and geometric realizability seems to have been taken as a challenge by many people, leading to a variety of proofs of geometric non-realizability, or at least mention of it; see Carver [C2], Laufer [L1], van de Craats [V1], Glynn [G2], Killgrove *et al.* [K10], Sternfeld et al. [S22], and others. Zacharias [Z5] is not aware of the earlier works and attempts to enumerate all the (10_3) configurations. There are several errors in [Z5] (as well as in the review [T1] by Togliati); corrections appear in [Z7]. Some other publications discuss just the enumeration of combinatorial (10_3) configurations; for example, we may mention Betten and Schumacher [B15].

This situation makes it even more important to make sure that the remaining nine combinatorial configurations (10_3) are geometrically realizable. Following Schroeter [S8], we present here a method of stepwise construction for each of the nine *geometric* configurations (10_3). The method leads to several important conclusions; among them are: the number of parameters needed to determine each of these configurations (that is, the number of "degrees of freedom"), the possibility of constructing each of them using only an unmarked ruler, and the possibility of realizing each in the rational plane (or, equivalently, with all vertices at points of the integer lattice). Since these are quite non-trivial results, which can be found in few of the more recent publications, Schroeter's constructions are shown in Figure 2.2.5. Naturally, each of these constructions requires justification, which is given in the paper; examples follow.

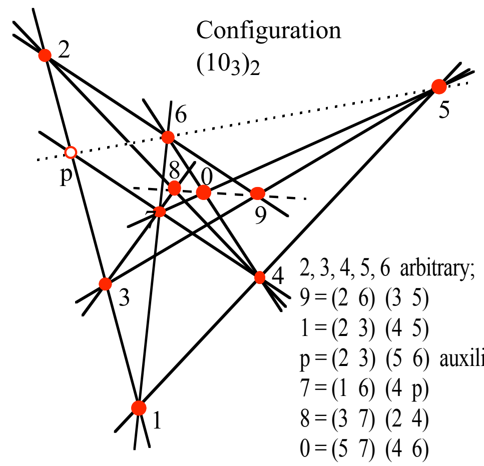
From the configuration table for $(10_3)_1$ we see that the intersections (24,35) = 8, (26, 37) = 9 and (46, 57) = 0 are collinear, hence by the Desargues' theorem the triangles 246 and 357 are in perspective from a point – which in the table is identified as 1. This justifies the construction in Figure 2.2.5. Moreover, it enables one to find out how many degrees of freedom are there in the construction (precisely 11), and that the construction is linear. By this is meant that only systems of linear equations need to be solved, and hence that it can be carried out in the rational plane (the plane in which only points with rational coordinates are considered).

Configuration $(10_3)_1$



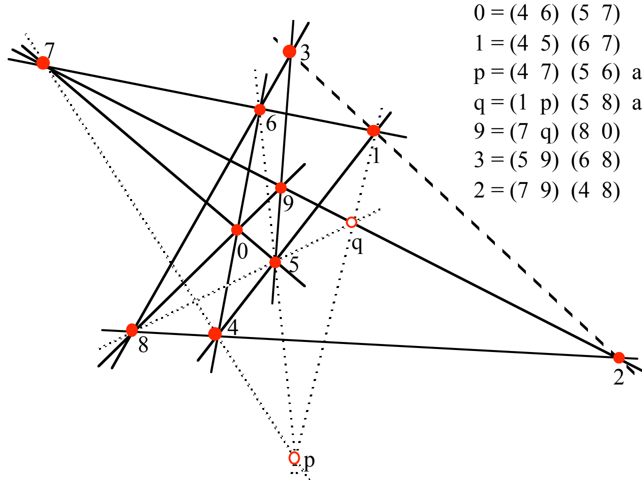
2, 3, 4, 5, 6 arbitrary;
8 = (2 4) (3 5)
1 = (2 3) (4 5)
7 arbitrary on (1 6)
9 = (3 7) (2 6)
0 = (5 7) (4 6)

Configuration $(10_3)_2$



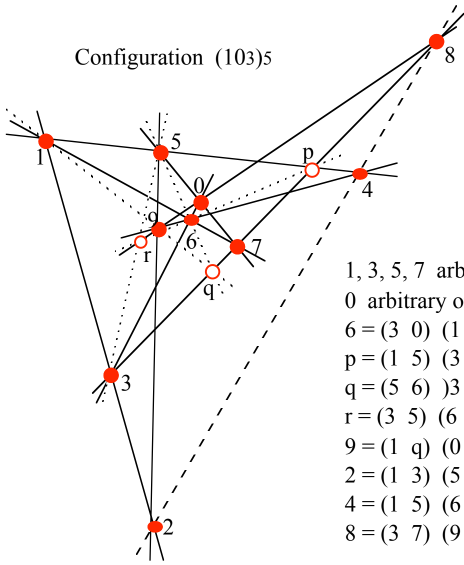
2, 3, 4, 5, 6 arbitrary;
9 = (2 6) (3 5)
1 = (2 3) (4 5)
p = (2 3) (5 6) auxiliary point
7 = (1 6) (4 p)
8 = (3 7) (2 4)
0 = (5 7) (4 6)

Configuration $(10_3)_3$



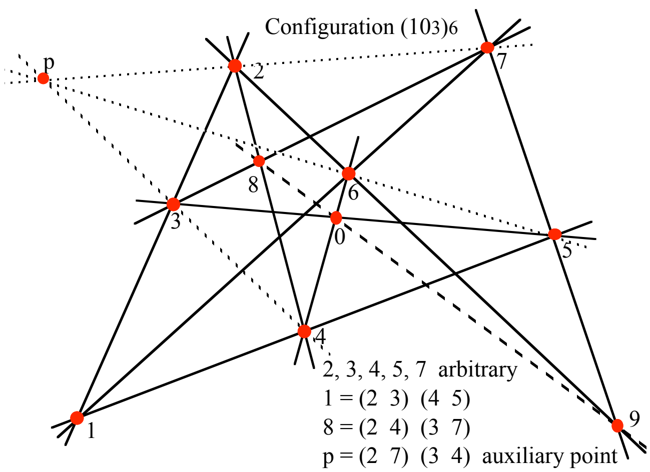
4, 5, 6, 7, 8 arbitrary
0 = (4 6) (5 7)
1 = (4 5) (6 7)
p = (4 7) (5 6) auxiliary point
q = (1 p) (5 8) auxiliary point
9 = (7 q) (8 0)
3 = (5 9) (6 8)
2 = (7 9) (4 8)

Configuration $(10_3)_5$



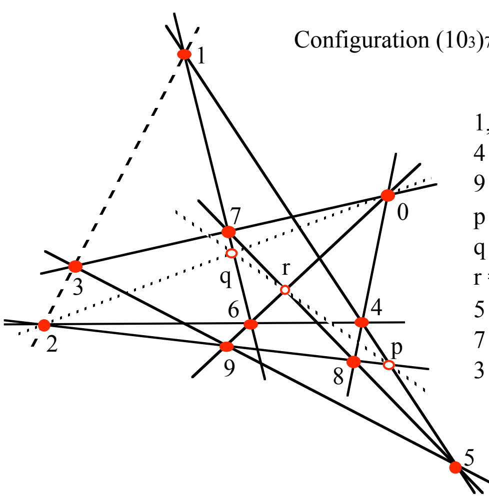
1, 3, 5, 7 arbitrary
0 arbitrary on (5 7)
6 = (3 0) (1 7)
p = (1 5) (3 7) auxiliary point
q = (5 6))3 7) auxiliary point
r = (3 5) (6 p) auxiliary point
9 = (1 q) (0 r)
2 = (1 3) (5 9)
4 = (1 5) (6 9)
8 = (3 7) (9 0)

Configuration $(10_3)_6$



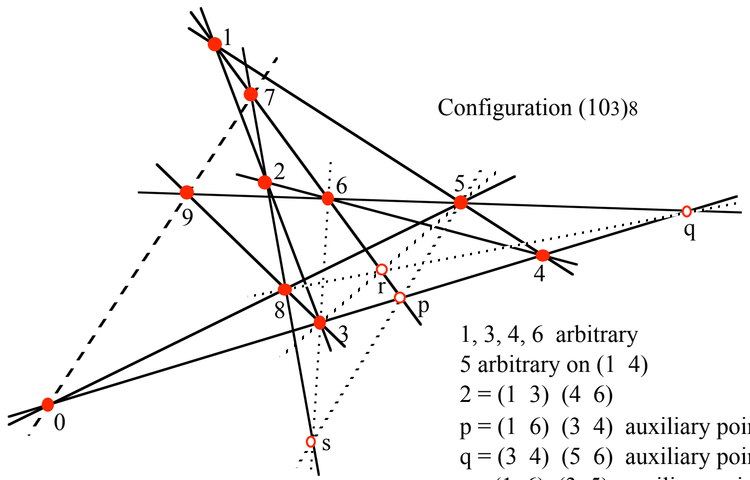
2, 3, 4, 5, 7 arbitrary
1 = (2 3) (4 5)
8 = (2 4) (3 7)
p = (2 7) (3 4) auxiliary point
6 = (1 7) (5 p)
9 = (2 6) (5 7)
0 = (4 6) (3 5)

Configuration $(10_3)_7$



1, 2, 6, 8, 0 arbitrary
4 = (2 6) (8 0)
9 = (2 8) (6 0)
p = (1 4) (2 8) auxiliary point
q = (1 6) (2 0) auxiliary point
r = (6 0) (p q)
5 = (1 4) (8 r)
7 = (1 6) (8 r)
3 = (5 9) (7 0)

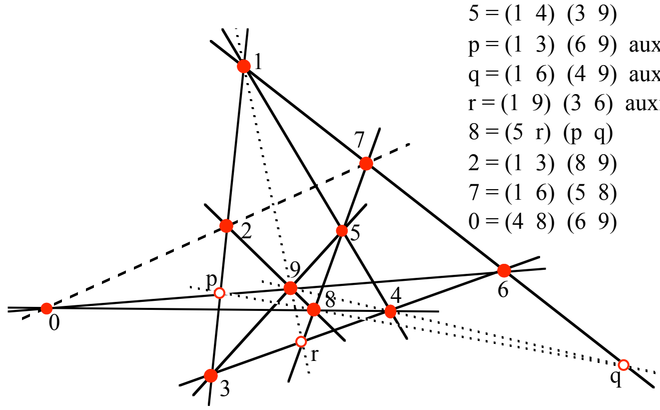
Configuration $(10_3)_8$



1, 3, 4, 6 arbitrary
5 arbitrary on (1 4)
2 = (1 3) (4 6)
p = (1 6) (3 4) auxiliary point
q = (3 4) (5 6) auxiliary point
r = (1 6) (3 5) auxiliary point
s = (3 6) (5 p) auxiliary point
8 = (2 s) (q r)
7 = (1 6) (2 8)
9 = (3 8) (5 6)
0 = (3 4) (5 8)

Configuration $(10_3)_9$

1, 3, 6, 9 arbitrary
4 arbitrary on (3 6)
5 = (1 4) (3 9)
p = (1 3) (6 9) auxiliary point
q = (1 6) (4 9) auxiliary point
r = (1 9) (3 6) auxiliary point
8 = (5 r) (p q)
2 = (1 3) (8 9)
7 = (1 6) (5 8)
0 = (4 8) (6 9)



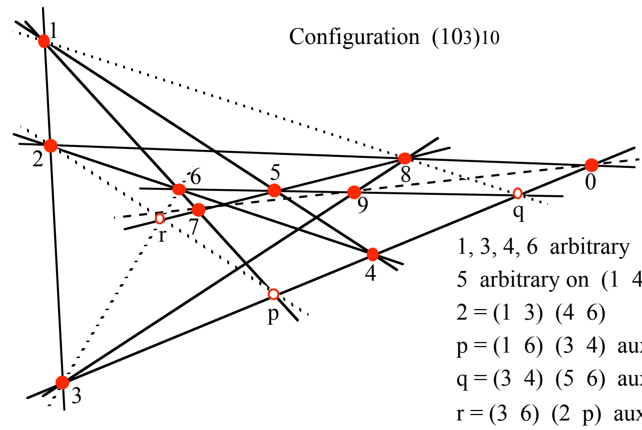


Figure 2.2.5. The construction of the nine geometric configurations (10_3) following Schroeter [S8].

For the combinatorial configuration $(10_3)_2$ we see that the triangles 246 and 357 are again perspective from point 1, but the sides of the triangles 246 and 375 (in that order!) intersect in the collinear points 8, 9 and 0, hence by Desargues they must be perspective from some point. This is a point p that is not a point of the configuration. The construction now follows. Note that in this case there are only 10 degrees of freedom.

Arguments of similar kinds can be made in the seven remaining cases. They are explained in detail in Schroeter [S8]. The steps outlined with each construction enable the determination of the degree of freedom of each configuration. The result -- after taking into account that projective transformations account for eight degrees of freedom, which are not deemed essential in the present context – is shown in Table 2.2.9.

$(10_3)_1$	$(10_3)_2$	$(10_3)_3$	$(10_3)_4$	$(10_3)_5$	$(10_3)_6$	$(10_3)_7$	$(10_3)_8$	$(10_3)_9$	$(10_3)_{10}$
3	2	2	-–	1	2	2	1	1	1

Table 2.2.9. The number of degrees of freedom beyond projective transformations, for each of the geometric configurations (10_3).

Exercises and problems.

1. Since the configuration (7_3) is combinatorially unique, the configuration in Table 2.1.1 for n = 7 must be isomorphic with the configuration in Table 2.2.2. Find the mapping that transforms one into the other. The same task for Table 2.1.1 for n = 8 and Table 2.2.3.

2. Verify the entries for $(10_3)_2$, $(10_3)_4$, and $(10_3)_9$ in Table 2.2.8.

3. Justify the numbers in Table 2.2.9.

4. Explain and justify Schroeter's construction of the configurations $(10_3)_5$ and $(10_3)_6$.

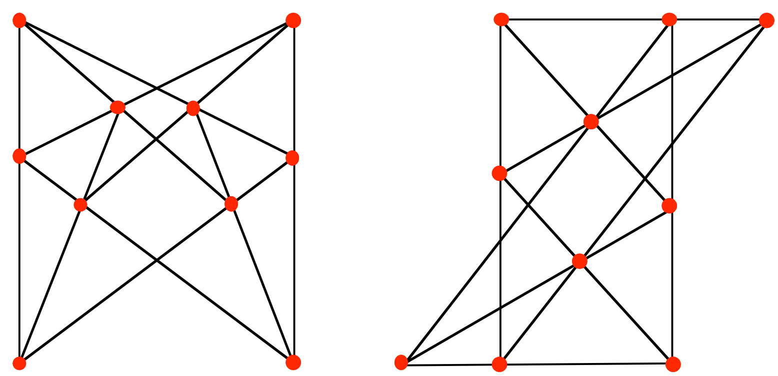


Figure 2.2.6. Two (10_3) configurations.

5. For each of the two configurations in Figure 2.2.6 decide whether it is a "fake". If not, find the coordinates of its points, and determine with which of the configurations in Figure 2.2.5 it is isomorphic.

6. Use the configurations tables of the (10_3) configurations to find the automorphism group of each.

7. Is there a topological realization of the $(10_3)_4$ configuration that has a nontrivial symmetry?

8. In Section 1.7 we demonstrated that the configuration $(10_3)_9$ has no geometric realization with nontrivial symmetry. What about the configuration $(10_3)_5$?