2.11 OPEN PROBLEMS (AND A FEW EXERCISES)

Many unsolved problems and open question have been mentioned in the preceding sections. While some of these may be challenging and others may hold interest for some people, there are a few problems concerning 3-configurations that seem to be of a fundamental nature; these problems exhibit the paucity of our understanding of what makes geometric configurations work. Some of the problems are related to Steinitz's geometric theorem of Section 2.6.

1. The first problem concerns geometric realizations of connected combinatorial configurations. By Theorem 2.6.1 we know that a (geometric) prefiguration representation is always possible if one incidence is disregarded. As shown by the examples of the (7_3) and (8_3) configurations, even allowing pseudolines it is not possible to achieve the last incidence. However, it is well possible that all connected (n_3) configurations with $n \geq 9$ admit realizations as topological *con*figurations, or even (for $n \geq 11$) realizations as geometric *pre*figurations. On the other hand, it may well be that already for $n = 13$ some counterexamples can be found for either version of the question. A subsidiary question is to determine the maximal number $t(n)$ of "lines" in a topological configuration (n_3) that may need to be non-straight pseudolines in each *realization* of the configuration in question. It seems that $t(n) \geq c\, n$ for some $c > 0$.

2. The second problem deals with obstructions to geometric realization of 2-connected 3-configurations with $n \geq 11$ lines. All known examples that include unwanted incidences (superfigurations) contain either a Pappus or a Desargues subfiguration (one incidence of the configuration is missed), or several such subfigurations. Are there any other obstructions to the geometric realizability, or is the presence of at least one of these two a characterization of 3-configurations with unwanted incidences?

3. The third problem, simply stated, is this: Is the combinatorial configuration $(10_3)_4$ using the notation in Section 2.2, the **only** 3-connected configuration (n_3) with $n \geq 9$ that does not have a geometric realization? A negative answer may appear at any time — if somebody hits upon an appropriate example — possibly even with $n = 13$. On the other

hand, a positive solution would seem to require several breakthroughs in directions for which we are not even dimly aware of how to start. These would have to include the elimination of *super*figurations (unwanted incidences) as well as *sub*figurations (missing incidences, as in Steinitz's theorem). As a possible example of a negative solution consider the abstract configuration (14_3) derived from the geometric configuration in Figure 2.11.1 on replacing the existing incidences of points A and B with the lines a and b, and insisting instead that A be incident with a, and B with b.

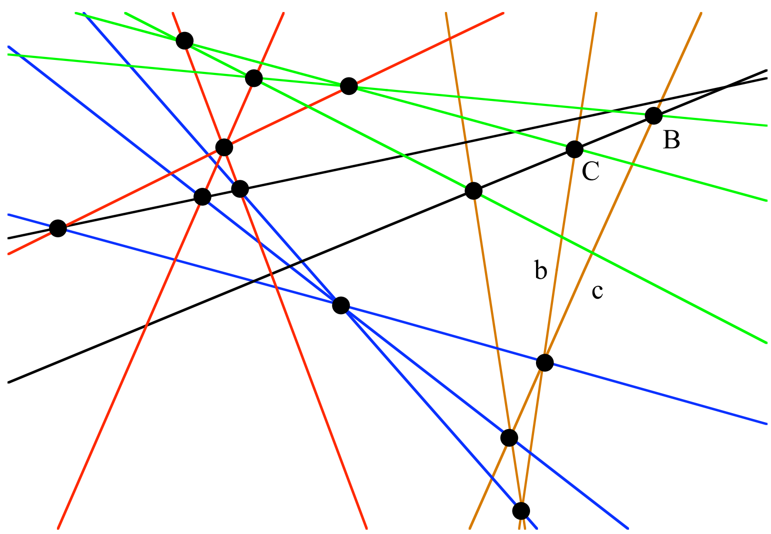

Figure 2.11.1. Is there a geometric realization of the combinatorial configuration (14_3) obtained from the above by keeping all indicated incidences except that C is to be incident with c (and not with b) and B be incident with b?

4. Is it true that if a 3-configuration admits a geometric realization in the Euclidean plane then it admits a realization in the rational plane? Or is it (at least) true that every geometrically realizable 3-configuration can also be realized in a plane over a quadratic extension of the rational field? In contrast, it is easy to verify that the #2-superfiguration shown in Figure 1.3.4 is realizable in the Euclidean plane but not in the rational plane.

5. For the various classes of very symmetric 3-configurations (such as astral, 3-chiral, k-dihedral, BB, ...) determine the precise range of the parameters for such configurations.

6. For connected astral configurations m#(b,c;d), is m = 12 the only case in which various superfigurations occur?

7. Is there any relation between the automorphism group of a configuration and the symmetries of its possible realizations? In particular, if the automorphisms act transitively on the points (or lines, or flags), does there have to exist a realization with non-trivial symmetry?

8. The object in Figure 2.11.2 is not a configuration, but the labeling clearly indicates that it is selfdual; the same can be said for the superfiguration in Figure 1.3.4. These seem to be interesting objects, analogous to configurations in the sense used in this book — but without any systematic framework to support their investigation. A formal proposal to consider such "generalized configurations" was made in [Z9] by K. Zindler as long ago as 1889. An example described by Zindler (as well as in the review [S11] of [Z9] by H. Schubert) is shown in Figure 2.11.3. However, it seems that Zindler's general challenge has never been met.

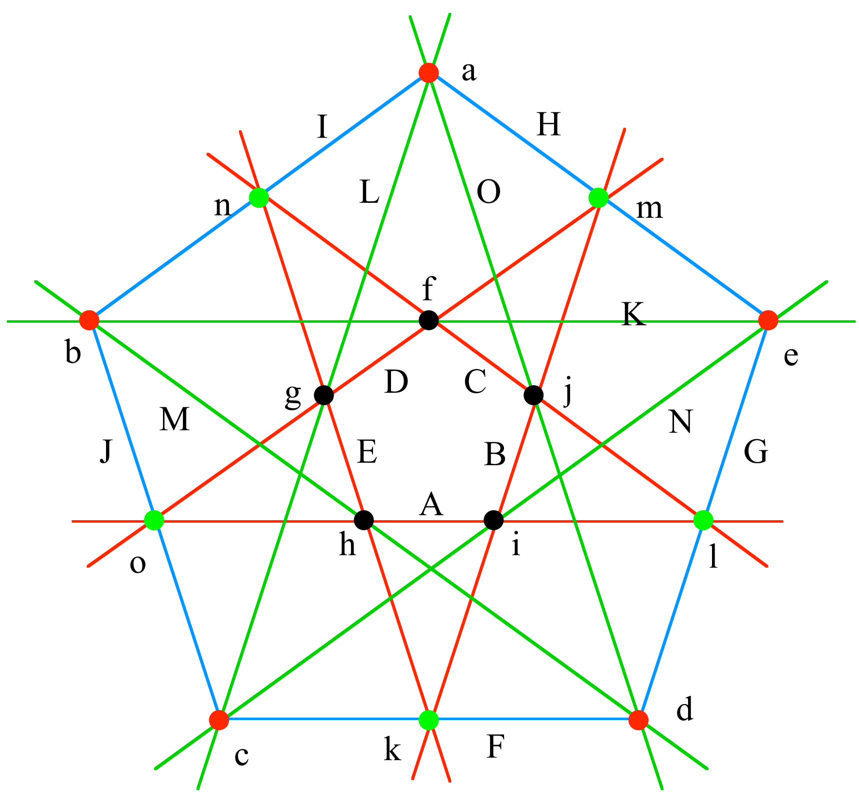

Figure 2.11.2. An intriguing selfdual collection of 15 points and 15 lines.

9. Decide whether the selfduality of the superfiguration in Figure 2.11.2 is a selfpolarity.

10. Prove that the incidences claimed in Figure 2.11.3 are valid.

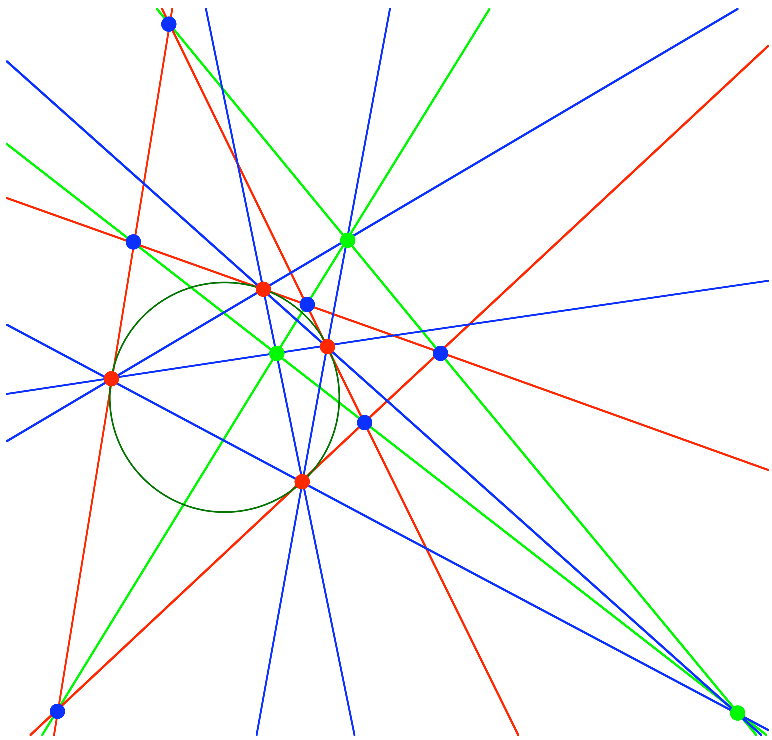

Figure 2.11.3. A "generalized configuration" of 13 points and 13 lines from Zindler [Z9]. It consists of four concyclic points (red) that determine a complete quadrangle (six blue lines) and its three "diagonal points" (green). The four tangents (red) to the circle at the four concyclic points are a complete quadrilateral that determines the six blue points and the three "diagonal lines (green). The selfpolar "configuration" has six points incident with three lines each, and seven points incident with four lines each, and dually for lines.