2.10 DUALITY OF ASTRAL 3-CONFIGURATIONS

In this section we shall investigate the duality and polarity properties of the chiral astral configurations (n_3). It should be kept in mind that the presentation is based on the assumption that we know all such configurations although, in fact, we are certain only to the extent that the topic has been explored by numerical calculations. As we have seen in Section 2.7, to a symbol m#(b,c;d) correspond either two, or one, or no chiral astral configurations (n_3), where n = 2m. In the case of two configurations, by their very construction they are isomorphic. But more is true:

Theorem 2.10.1. Every chiral astral configuration m#(b,c;d) is selfdual.

Proof. From the definition given above of the labels of points and lines of such configurations, illustrated in Figure 2.10.1 (which is a copy of Figure 2.7.1), we see that the line L_j contains the points B_j, C_j, B_{j+b}, and the line M_j contains the points B_{j+d}, C_j, C_{j+c}. The resulting incidences can then be described by the following criteria:

$B_j \in L_k \Leftrightarrow j - k \equiv 0$ or $b \pmod m$

$B_j \in M_k \Leftrightarrow j - k \equiv d \pmod m$

$C_j \in L_k \Leftrightarrow j - k \equiv 0 \pmod m$

$C_j \in M_k \Leftrightarrow j - k \equiv 0$ or $c \pmod m$.

From these relations there follows at once that for every configuration m#(b, c; d) the mapping δ determined by $\delta(B_j) = L_{-j}$, $\delta(C_j) = M_{-j-d}$, $\delta(L_j) = B_{-j}$ and $\delta(M_j) = C_{-j-d}$ is a selfduality. □

Another consequence is:

If two distinct configurations have the same symbol m#(b, c; d) then they are dual to each other.

This follows from the fact that they are isomorphic. But even more is true:

Theorem 2.10.2. If two distinct configurations have the same symbol m#(b, c; d) then they are polars of each other.

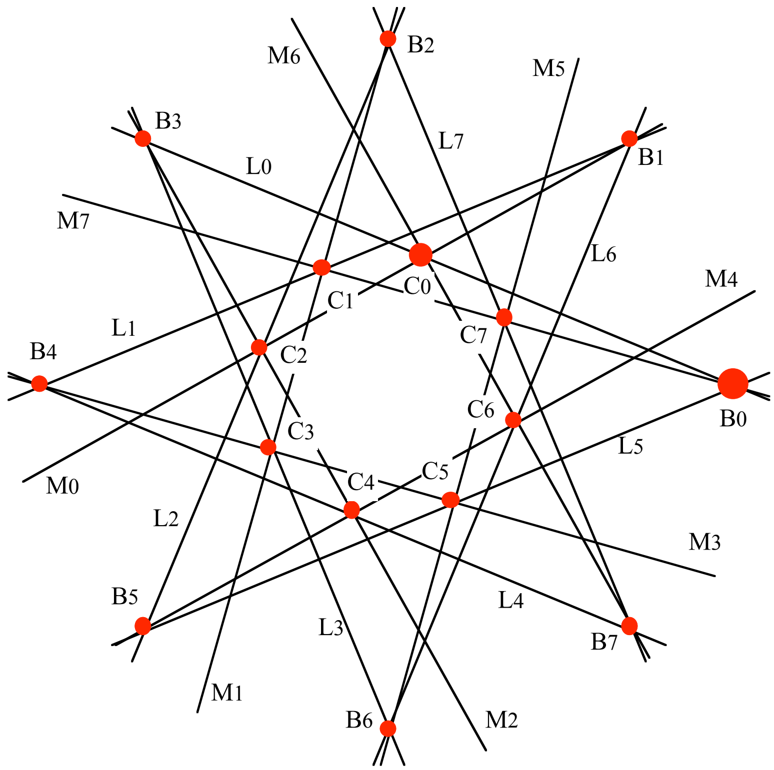
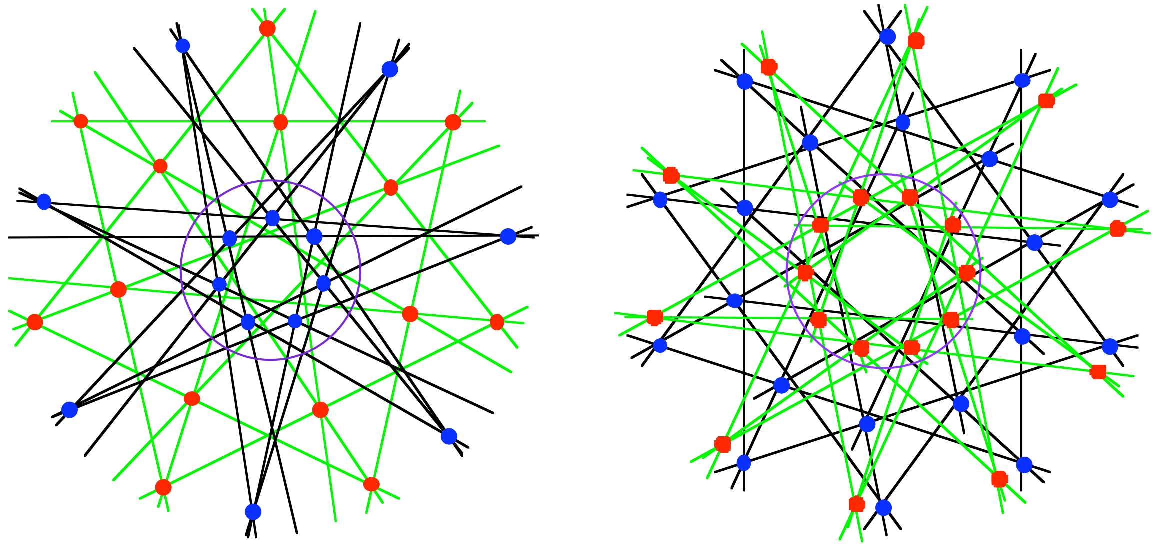


Figure 2.10.1. The labeling of the configuration 8#(3, 2; 1) explained in the text.

Proof. Indeed, polars are combinatorially dual to each other, and the only combinatorially dual astral configuration of an astral configuration m#(b, c; d) is either the configuration itself, or the other one with the same symbol. Since there are two configurations m#(b, c; d), neither is polar to itself, but each is polar to the other. □

This fact is illustrated in Figure 2.10.2.

It is almost selfevident that in general there are other duality maps from a configuration to its dual. For example, Figure 2.10.3 presents the same pair of configuration as Figure 2.10.2(a), with a labeling that shows that the map ε from the red configuration to the black one is a duality different from the duality δ described above.



(a) (b)

Figure 2.10.2. (a) The configuration 7#(3,2;1)' (red points and green lines) and its polar 7#(3,2,1)" (blue points and black lines). Polarity is with respect to the purple circle. (b) The same for 10#(4,3;2)' and 10#(4,3;2)".

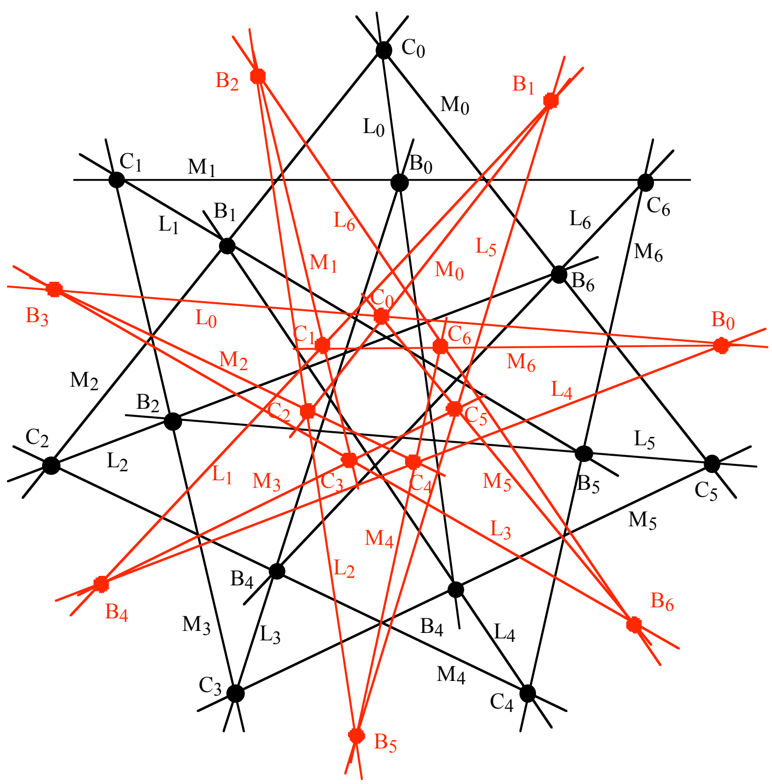
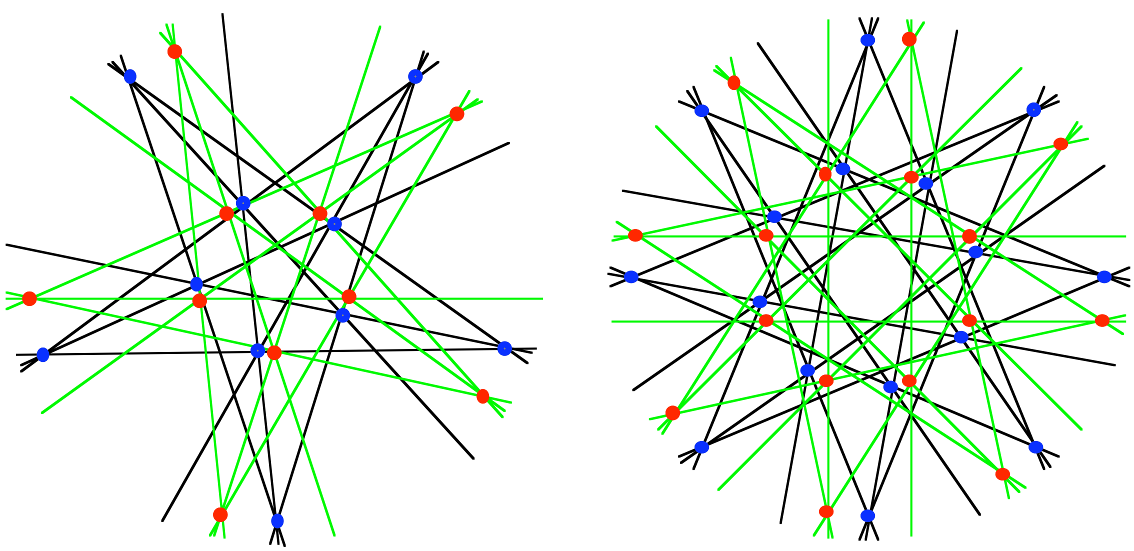


Figure 2.10.3. The dual configurations of Figure 2.10.2(a) illustrate a duality map ε.

In case that only a single configuration m#(b,c;d) exists (that is, if b + c = 2d, or if b = c), the configuration is not only selfdual, but selfpolar. The map δ is applicable to all selfdual configurations, and is concordant with selfpolarity. The polars (in an appropriate circle) are *congruent* to each other, but only after a reflection in a suitable mirror.

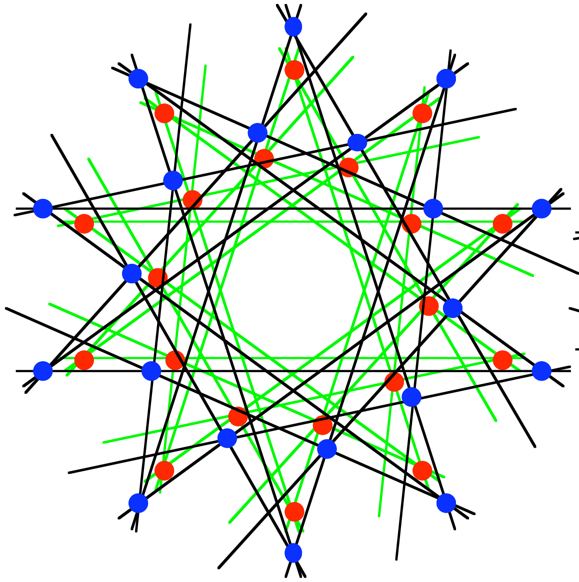
For configurations of this type, the map δ and its rotates are the *only* maps compatible with the polarity. We say that these configurations are **oppositely selfpolar**. This happens for the selfpolar configurations with symbol m#(b,b;d). Examples are shown in Figure 2.10.4.

Other configurations, called **directly selfpolar configurations**, have symbols of type m#(b,c;d) with 2d = b+c. Here the polar pairs are congruent without reflection. There are two subtypes: In the first, both b and c are even, in the second they are both odd. In the former case the polars actually coincide with each other, while in the latter they are related by reflection in the common center (that is rotation through 180°). The two subtypes are illustrated in Figures 2.10.5 and 2.10.6.



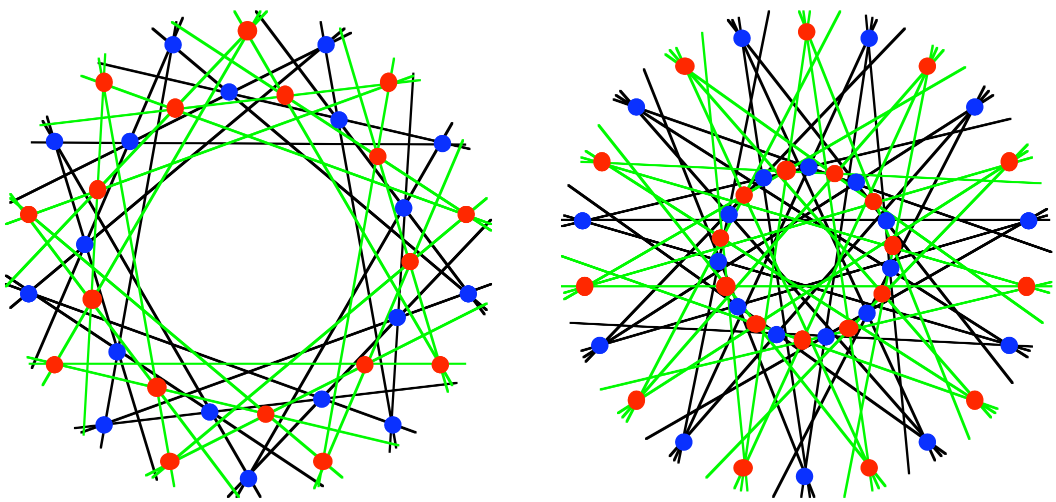
(a) #(2,2;1) (b) 8#(3,3;2)

Figure 2.10.3. Two examples of oppositely selfpolar configurations, characterized by symbols of the type m#(b,b;d).



(a) 10#(4,2;3) (b) 13#(6,4;5)

Figure 2.10.4. Two examples of directly selfpolar configurations m#(b,c;d) with b and c even. In this subtype the polars may coincide (for an appropriate circle). In the illustration the circle was chosen to yield different sizes, in order to improve intelligibility.



(a) 9#(3,1;2) 11#(5,1;3)

Figure 2.10.5. Two examples of directly selfpolar configurations m#(b,c;d) with b and c odd. In this subtype the polars are congruent but coincide only after reflection in the common center (that is, a rotation of 180°). We also say that these configurations are **selfpolar*.**

Exercises 2.10.

1. Verify that the correspondence δ is a duality. Determine whether this correspondence establishes a selfduality.

2. Describe the duality introduced by the polarity, for the polar configuration in Figure 2.10.2(b); use the labels on the two configurations that are given by their isomorphism.

3. Label the selfpolar configurations in Figures 2.10.4 and 2.10.5 to show that they are selfdual.

4. Verify that the Cremona-Richmond configuration (15_3), shown in Figure 1.1.1 and mentioned in Exercise 2.9.7, is selfdual. Is it selfpolar, and if it is, what is its type?

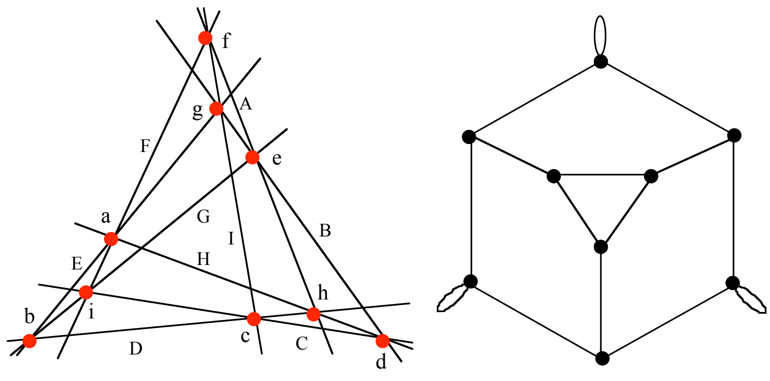
5. Find criteria for dual pairs of configurations of the various kinds discussed in Sections 2.8 and 2.9.

6. R. Artzy [A1] considers selfdual configurations and for a given selfduality δ describes a RLG ("reduced Levi graph" — this is not the same concept we are using throughout the book!) by identifying each element B with its image δ(B). This clearly depends on the selfduality chosen, but in each case the original Levi graph can be retrieved in a unique way. As observed by Artzy, the RLG may contain loops, this occurs in case B and δ(B) are incident. Artzy illustrates the use of RLGs by investigating special cases of the Desargues configuration. (On this topic see also Killgrove *et al.* [K10].) Assign labels to the RLG in Figure 2.10.6b to show that it corresponds to the Pappus configuration in Figure 1.10.6a, with the selfduality δ indicated by the upper and lower case letters.

7. Find a selfduality δ of the Desargues configuration in Figure 2.10.7a that leads to the RLG in Figure 2.10.7b.

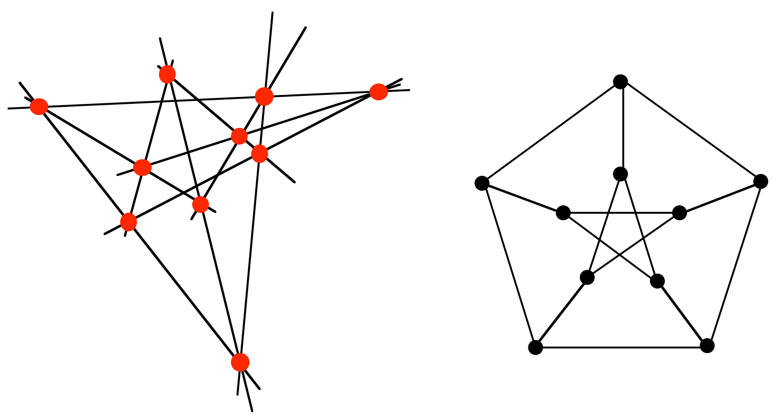
8. Is there a meaningful extension to all polar pairs of astral 3-configurations of the distinction between directly and oppositely selfpolar ones?

9. Describe the polars of the configurations BB(m; s, t), and determine whether there are any selfpolar ones among them.



(a) (b)

Figure 2.10.6. (a) A version of the Pappus configuration (9_3), with a selfduality indicated by upper- and lower-case letters. (b) An RLG corresponding to the selfduality in (a).



(a) (b)

Figure 2.10.7. (a) A version of the Desargues configuration (10_3). (b) An RLG of (a).

10. (Refresh your memories of elementary geometry.) Given a pair of astral configuration for which it is claimed that they are polar to each other with respect to a circle – how do you find the circle that justifies the assertion? Practice your solution on the selfpolar configurations in Figures 2.10.3, 2.10.4 and 2.10.5.