2.1 EXISTENCE OF 3- CONFIGURATIONS

Among the questions attacked and solved in the early years of the study of configurations is the one we formulated as question (A) in Section 1.1. We can now formulate it more specifically as follows:

Determine all values of n *such that there exists a combinatorial, or a topological, or a geometric configuration* (n_3).

The complete answer to this problem is given by:

Theorem 2.1.1. Combinatorial configurations (n_3) exist if and only if $n \geq 7$.

Theorem 2.1.2. Topological configurations (n_3) exist if and only if $n \geq 9$.

Theorem 2.1.3. Geometric configurations (n_3) exist if and only if $n \geq 9$.

To prove Theorem 2.1.1 we note that the inequalities of Section 1.3 imply, for $k = 3$, that $n \geq 7$. Hence we only have to show that for each $n \geq 7$ there exist a combinatorial configuration (n_3). Of the various ways of fulfilling this task, probably simplest is the listing of a configuration table for a *cyclic* (n_3), as illustrated in Table 2.1.1. We shall encounter this configuration repeatedly, and we reserve the symbol $\mathscr{C}_3(n)$ for it. Besides, the existence of topological and geometric configurations (n_3) for $n \geq 9$ implies the existence of the corresponding combinatorial ones.

1	2	3	4	.....	n–3	n–2	n–1	n
2	3	4	5	.....	n–2	n–1	n	1
4	5	6	7	.....	n	1	2	3

Table 2.1.1. A configuration table for the cyclic combinatorial configuration $\mathscr{C}_3(n)$ for $n \geq 7$. It also shows that for $n \leq 6$ this would not be a configuration, since some pairs of points would belong to two different lines.

This completes the proof of Theorem 2.1.1.

In the exercises at the end of the section we shall enlarge upon the configurations $\mathscr{C}_3(n)$, and other cyclic configurations.

The configuration $\mathscr{C}_3(7)$ is known as the **Fano configuration**; it was described by Gino Fano in 1891 [F1, p. 111] in connection with his axiomatic studies of projective geometries. In fact, it was found earlier (in 1888) by Schönflies [S2], who dismissed it by saying that "a configuration 7_3 with all points distinct does not exist", as well as by Schroeter [S6], also in 1888. Schönflies' assertion makes a limited sort of sense when one realizes that he was thinking of geometric configurations — albeit in the complex projective plane! Schroeter was the first to stress the distinction between combinatorial and geometric configurations, and between geometric configurations in the real plane as distinct from the ones in the complex projective plane.

The configuration $\mathscr{C}_3(8)$ is known as the Möbius-Kantor configuration. In the prehistory of configurations it was described by Möbius in 1828 [M20], who proved that it cannot be realized geometrically in the real Euclidean plane. The configuration was later described by Kantor in 1881 [K3], although not as a combinatorial configuration but as a configuration *geometric in the complex plane*. Reye noted in 1882 [R2] that $\mathscr{C}_3(8)$ does not exist as a geometric configuration in the real plane. In the same paper [K3] the three configurations (9_3) are described by Kantor for the first time, as geometric configurations in the real plane.

To prove Theorems 2.1.2 and 2.1.3 it is sufficient to show that *geometric* configurations (n_3) exist for each $n \geq 9$, and that *topological* configurations (n_3) do not exist for $n = 7, 8$.

To establish the latter, we shall first prove a lemma.

Lemma 2.1.1. Let C be a family of pseudolines in the real projective plane, such that no point is incident with all members of C. Then there is a point that is contained in precisely two of the pseudolines in C.

Such a point is customarily called an *ordinary point* of the family C.

Proof of the lemma. If all intersection points of pseudolines in C are ordinary, there is nothing to prove. Otherwise, there exist "triangles" (that is, regions of the plane) bounded by three pseudolines and such that, at one of the vertices of this "triangle", one of the pseudolines L incident with that vertex V enters the interior of the "triangle". Indeed, start with any non-ordinary point of C, and three arbitrarily chosen pseudolines through V. Then any pseudoline L* not through V will determine a triangle with the required properties. See Figure 2.1.1. Call such a triangle a *good* triangle. Among the (possibly many) good triangles of C find the (or one) that has a minimal area. Then the pseudoline L that enters the triangle at V must meet L* at some point P on the boundary of the "triangle". Now P has to be an ordinary point of C, since any pseudoline through P different from L and L* would belong to a "good" triangle with smaller area.

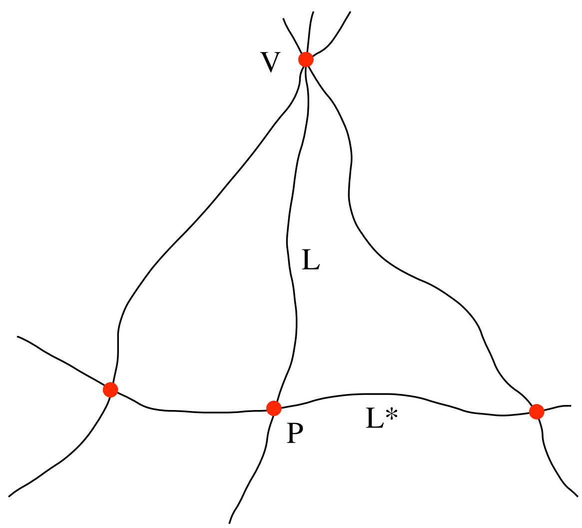

Figure 2.1.1. Any vertex incident with three or more pseudolines of a family C as in Lemma 2.1.1 can serve as one of the vertices of a "good" triangle.

Resuming now the proof of the non-realizability by pseudolines of any combinatorial configuration (7_3), we note that if a realization were possible, then the seven points of such a configuration would account for 21 pairwise incidences of points and pseudolines. On the other hand, seven pseudolines can have at most 7(7–1)/2 = 21 pairwise intersections, that is, at most 21 pairwise incidences with points of the configuration. Thus all intersection points of the pseudolines would be at points of the configuration, hence triple points, and there would be no ordinary points – contradicting Lemma 2.1.1. It follows that there is no realization of combinatorial configurations (7_3) by pseudolines. This completes the proof of this part of our assertion.

Concerning the case of topological configurations (8_3), let us assume we have a realization C of such a configuration by pseudolines. We begin by selecting one of them, say L. It is met by six other pseudolines in the three vertices of C that are on L. Hence there is one pseudoline L* of C that meets L in an ordinary point, which is not a vertex of C. Choosing the line-at-infinity to pass through that point, we can represent L, L* and the vertices of C incident with them as shown in Figure 2.1.2.

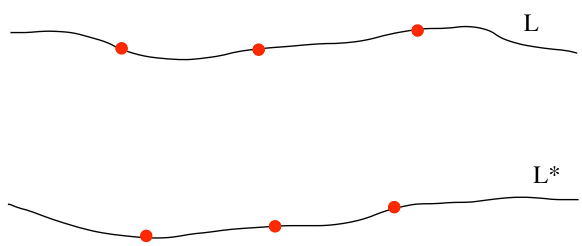

Figure 2.1.2. Two of the pseudolines and the six vertices discussed in the proof.

The remaining six pseudolines must all pass through the three vertices on L and through the three vertices of L*, as well as through the two remaining vertices of C. Let V be one of these two vertices; without loss of generality we can assume that it is in the "strip" between L and L*; then, since at the point of intersection the pseudolines must cross each other, the three pseudolines pass through V as schematically shown in Figure 2.1.3.

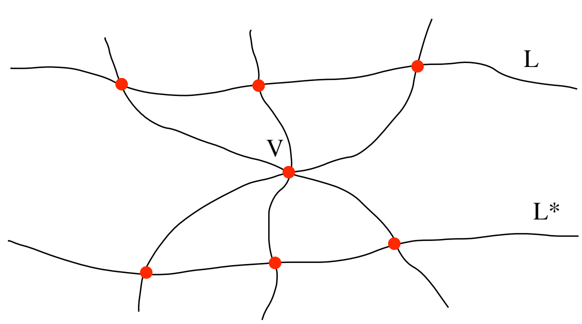

Figure 2.1.3. The arrangement of the pseudolines incident with the vertex V.

Now the last three pseudolines, that are incident with the last vertex of C, must be connected either by the scheme represented in Figure 2.1.4 by the dotted connections, or by the dashed ones. Since in either case two of them meet in a point (that therefore must be the last vertex) that is inaccessible to the third, it follows that the realization of (8_3) by pseudolines is impossible.

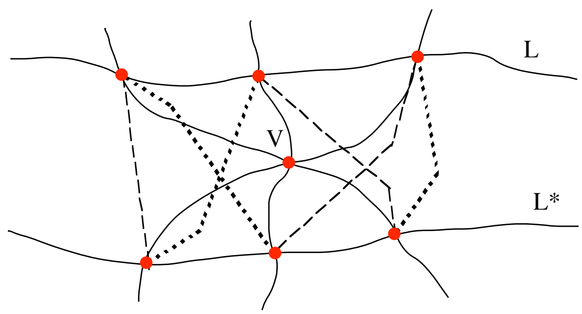

Figure 2.1.4. The three dotted connections or the three dashed ones are schematic representations of the relative positions of the three pseudolines that should be incident with the last vertex of C.

The proof of the weaker result that *geometric* configurations (7_3) and (8_3) do not exist is much older and simpler. For example, Schroeter [S6] argues that, in the notation used in Table 2.2.2, if a geometric realization of the configuration (7_3) were possible, the points 2, 3, 4, 5 would generate a complete quadrangle (in the sense of projective geometry), with diagonal points 1, 6, 7. But these points are collinear in (7_3) while diagonal points of a complete triangle cannot be collinear unless the starting points are collinear; hence there is no geometric configuration. A different proof of the impossibility of geometric realization of the (7_3) configuration appears in Bokowski-Sturmfels [B25, p. 39]; it relies on the method of “final polynomials”. Essentially the same proof is used by Levi [L3, p. 95]. It shows that (7_3) can be “realized” only in projective planes with characteristic 2. Sidorov's statement in [S15] that (7_3) is realizable in the complex plane is plain wrong.

The method of "final polynomials" is used by Bokowski–Sturmfels [B25 p. 35] to show that (8_3) cannot be geometrically realized in the *real* plane, although it can be realized in the *complex* plane. That result itself belongs to the “prehistory” of configurations; it appears (in somewhat different formulation) in Moebius [M20, p. 445], as described by Schroeter [S6, p. 239]. An explicit calculation of feasible coordinates of points of a realization of (8_3) by Moebius [M20] as well as by Levi [L3, p. 99], shows that such coordinatization is possible only using complex numbers.

The only publication I am aware of in which the non-existence of *topological* configurations (7_3) and (8_3) has been considered (and proved) is Levi’s book [L3, pp. 95, 100]. However, his proofs are quite laborious, and part of the argumentation in case of (8_3) is left to the reader to complete. Instead of our Lemma 2.1.1 Levi relies on a lemma (Satz 21, [L3, p. 85]) that we may formulate as follows: No topological configuration (n_k) with $k \geq 3$ contains as vertices all points determined by its pseudolines. This is clearly a weaker version of the Lemma 2.1.1. On the other hand, Kelly and Rottenberg prove in [K7] the stronger result that every family of n pseudolines, not all incident with one point, must determine at least $3n/7$ ordinary points. That result is a generalization of the well-known result of Kelly and Moser [K6] for families of straight lines. This topic has had an interesting history, and is still subject of widespread interest. It is not possible to enlarge upon it here; the interested reader should consult [B28], where Section 7.2 presents details and gives a large number of references.

In order to complete the proof of Theorem 2.1.3 (hence also of Theorem 2.1.2) we shall describe the construction of a suitable geometric configuration (n_3).

We shall show that for $n \geq 9$, the cyclic combinatorial $\mathcal{C}_3(n)$ configuration of Table 2.1.1 can be realized as a geometric configuration of points and lines (see Figure 2.1.5). We begin by placing the first triplet on the x-axis in a coordinate system, with point 2 at the origin and point 4 at $x = 2$; we shall specify the location of the point 1 shortly. We draw a line through 2 with positive slope, and place on it 3 near to 2, so that the line 3,4 has negative slope small in absolute value, and place 5 on the same line sufficiently far to the right so that 4,5 has positive slope. On line 3,4 we locate 6 so that its x–coordinate is larger than the x–coordinate of 5, then on 4,5 we locate 7 so that its x–coordinate is larger than that of 6, and so on up to and including the line through n–5 and n–4 on which we locate n–2 so that its x–coordinate is larger than that of n–3. Clearly, all these steps can be carried out. Now, the choice of location for vertex 1 determines the only possible position of vertex n–1 (since it is on the already determined lines 1,n–2 and n–4,n–3), as well as the position of vertex n (which must be on the by now determined lines 2, n–1 and n–3,n–2). The only remaining question is whether the last triplet 1,3,n is collinear — and this depends on the choice of 1 (see Figure 2.1.5). It is easy to check that if 1 is chosen to be on the x–axis between points 2 and 4 and near to 2 (see part (a) of Figure 2.1.5), then the halfplane determined by the line 1,n and containing the positive x-axis contains the point 3 in its interior. On the other hand, if 1 is chosen between 2 and 4 but near to 4 (see part (b)), then 3 in not in the interior of that halfplane determined by 1,n that contains the positive x–axis. Due to the continuity of all construction steps, it follows that there must exist a position of vertex 1 for which the line 1,n passes through the point 3 — thus yielding the desired geometric realization of the combinatorial configuration in question. ¨

It should be noted that the construction fails unless $n - 5 > 3$; this provides an explanation why this construction requires $n \geq 9$.

This proof is quite analogous to the first published proof by Schroeter [S6]. The main difference is that in the last part, instead of the continuity argument used in our proof, Schroeter gives a purely geometric proof which utilizes properties of sets of points on cubic curves. On this topic he published a book in the same year [S7]. The advantage of Schroeter's proof over ours is that it shows that the cyclic configuration of Table 2.1.1 can be geometrically realized, for every $n > 9$, by a *linear construction* – that is, with just a straightedge. This implies that all these configurations can also be geometrically realized in the rational plane. In their combinatorial guise these configurations were studied, more or less simultaneously, by Schoenflies [S2] and Martinetti [M2].

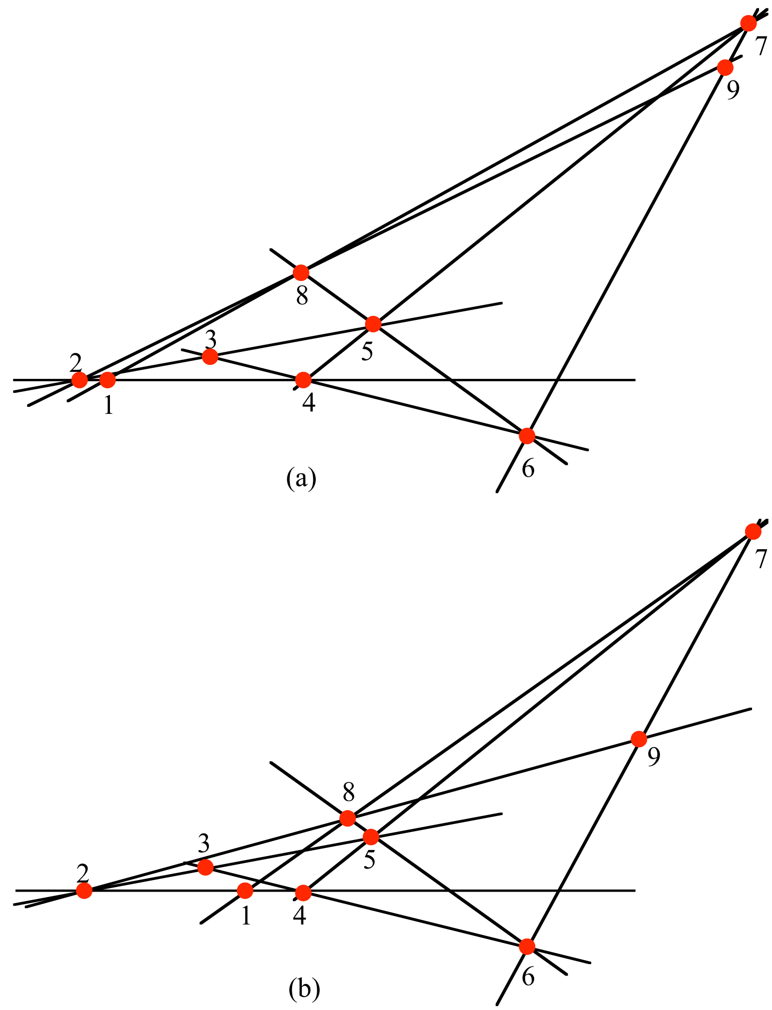

Figure 2.1.5. An illustration (for n = 9) of the construction of a geometric realization of the cyclic combinatorial configuration $\mathscr{C}_3(n)$ from Table 2.1.1.

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A few remarks seem appropriate at this time.

Early papers on configurations often considered configurations in the complex plane. Obviously, all geometric configurations as considered here (that is, in the real Euclidean plane) can also be considered as being in the complex plane. The interesting point is that in the complex plane Theorems 2.1.2 and 2.1.3 require modification: There exists a geometric configuration (n_3) in the complex plane if and only if $n \geq 8$. The fact that a *configuration* (8_3) exists in the complex plane was first announced by Kantor [K3], although in "prehistoric" formulation it goes back at least to Moebius [M20, p. 445].

Like many other writers on configurations in the last quarter of the nineteenth century, Kantor [K3] did not make explicit what kind of configurations (n_3) he is investigating. This is particularly amusing in connection with the configuration (8_3), which he describes as two quadrangles, each inscribed to and circumscribed about the other. Only later does he make an off-hand remark that at most four of the eight vertices of the configurations are real !!!

The cyclic configuration $\mathcal{C}_3(7)$ is the only configuration (7_3); this will be proved explicitly in Section 2.2. The configuration (7_3) does not appear in the paper by Kantor [K3] in which he considers configurations (n_3) for $n \le 9$. Although he relies on some combinatorial arguments, the combinatorial configuration (7_3) was probably invisible to him since it cannot be realized in the complex plane; it seems that he was considering only configurations that have realizations in the complex plane, although he is not explicit about that. On the other hand, (7_3) appears in many other publications and guises – – for example, as a Steiner triple system on 7 elements, as the projective plane of order 2, and several others.

Levi [L3] established Theorem 2.1.1 by considering generalizations of the cyclic configuration $\mathcal{C}_3(n)$ in Table 2.1.1. The same idea appeared earlier, most explicitly in Schönflies [S2]. A generalization of this is the *cyclic configuration* $\mathcal{C}_3(n,m)$, which consists of triples $\{j, j+1, j+m\}$, for $1 \le j \le n$, all entries taken mod n. Such configurations were studied (with slightly different notation) by Levi [L3, p. 91]. Levi proved that $\mathcal{C}_3(n,m)$ is a combinatorial configuration whenever $3 \le m < n/2$. He does not discuss their geometric realizability, and mentions no earlier works on any cyclic configurations.

There are familiar diagrams intended to illustrate the Fano (7_3) and Möbius-Kantor (8_3) configurations, shown in Figure 2.1.6. They are not topological configurations, since they involve one "line" that is a circle. If one of the incidences is not insisted upon, then this line can be "opened up" and a geometric realization of the resulting sub-figuration is obtained.

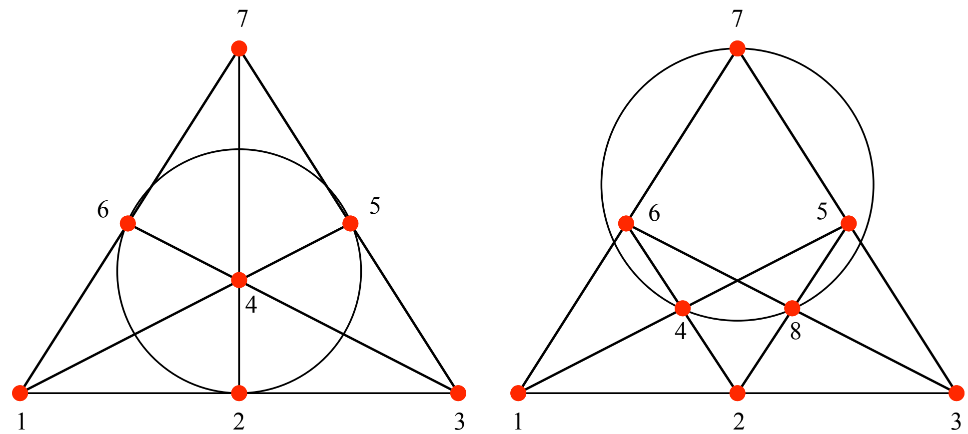

Figure 2.1.6. Diagrams often used to illustrate the combinatorial Fano (7_3) and Moebius-Kantor (8_3) configurations. The labeling shown is the "greedy" one: it uses a new mark only if unavoidable.

	L_1	L_2	L_3	L_4	L_5	L_6	L_7
Q_1	**X**		**X**				**X**
Q_2		**X**				**X**	**X**
Q_3	**X**				**X**	**X**	
Q_4				**X**	**X**		**X**
Q_5			**X**	**X**		**X**	
Q_6		**X**	**X**		**X**		
Q_7	**X**	**X**		**X**			

Figure 2.1.7. A Levi incidence matrix for the Fano (7_3) configuration, which shows that it is cyclic and selfdual.

Exercises and problems.

1. Find the isomorphism between the labeling of the points of the configurations (7_3) and (8_3) in Figure 2.1.6 and the cyclic configuration $\mathscr{C}_3(7)$ and $\mathscr{C}_3(8)$.

2. Use the illustration of the Moebius-Kantor (8_3) configuration given in Figure 2.1.6 to find a Levi incidence matrix for it. Can you use it to show that the configuration is selfdual?

3. A *general cyclic configuration* $\mathscr{C}_3(n,a,b)$ consists of triples $\{j, a+j, b+j\}$, for given a, b with $0 < a < b < n$ and for $1 \le j \le n$, all entries taken mod n. The configuration $\mathscr{C}_3(n)$ in Table 2.1.1 is, in this notation, $\mathscr{C}_3(n,1,3)$. Determine for which n, a, b is $\mathscr{C}_3(n,a,b)$ a combinatorial configuration. As a first step, investigate configurations $\mathscr{C}_3(n,1,m)$. (This was done as early as 1895, by G. Brunel in [B30].)

4. Is $\mathscr{C}_3(n,1,4)$ geometrically realizable for some n? Generalize.