1.7 DERIVED FIGURES AND OTHER TOOLS.

Testing whether a **given** mapping between the elements of two configurations (p_q, n_k) is an isomorphism is straightforward, though tedious and somewhat time-consuming; its complexity is polynomial in p (and n) for given q and k. However, deciding whether **there exists** an isomorphism is much harder if only brute force is used. That is why, since the beginning of the study of configurations in the nineteenth century, variants of the idea of **derived figures** have been found useful in finding isomorphisms between configurations or establishing that they are not isomorphic. The method is, in essence, a sort of "preprocessing" and is particularly timesaving if many configurations are to be considered simultaneously, or if they are known to have only few transitivity or symmetry classes. It is also very helpful if one aims at determining the automorphism group of a configuration.

The idea underlying "derived figures" is to associate with each point (or line) of a given configuration C a small "figure", determined by the point (or line) and the incidences in C. The associated ("**derived**") figure should be easy to determine, and it should be easy to see whether two such figures isomorphic. The vagueness of the above description should not bother you too much: it is not supposed to be an algorithm, just a heuristic approach which has been found convenient, and which can best be explained by examples.

Consider the (10_3) configuration indicated in Figure 1.7.1(a), which we shall call $(10_3)_9$ as in Section 2.2. We would like to determine whether it is isomorphic to the Desargues (10_3) configuration shown in Figure 1.7.1(b). We would also like to determine its automorphism group, the transitivity classes of its elements, and the possible symmetry groups that isomorphic configurations can have.

We start by observing that since in any (10_3) configuration C each point lies on three lines which together contain six other points of C, for each point of C there are three points of C to which it is not connected by any line of C. (For example, the point 2 in Figure 1.7.1(a) is connected by no line of the configuration to any of 5, 7, 8.) One kind of "derived figure" associates to each point of C the set of the three points of C

that are not connected to it, together with any lines of C that contain two of these points or all three of them. (In the literature, this is often called the "**remainder** figure", or "Restfigur" in the German literature.) Clearly, the derived figure could be any of the five schematically indicated in Figure 1.7.2. It is easily checked that for the configuration $(10_3)9$ of Figure 1.7.1(a) the derived figure is of type (iv) at points 0, 1, 4, 7, and is of type (iii) for the other six points. Thus the points of $(10_3)9$ form at least two transitivity classes; since the points of the Desargues configuration form one transitivity class, these two configurations cannot be isomorphic. The same conclusion can be reached without appealing to the automorphisms group of the Desargues configuration: We note that the derived figure at each vertex of the Desargues configuration is of type (v) -- and even a single such derived figure shows the impossibility of an isomorphism to $(10_3)9$.

A closer examination of the derived figures of $(10_3)9$ shows that the points 3 and 8, which lie on lines determined by 0, 1, 4, 7, cannot be in the same transitivity class as 2, 5, 6, 9; hence there are at least three transitivity classes. Each of the sets {0, 1, 4, 7}, {2, 5, 6, 9}, {3, 8} is either an equivalence class of points of $(10_3)9$ or a union of such classes. We shall try to determine which is the case. We start by looking for a permutation of the vertices that maps 0 to 1. Since they determine a line, 1 must be mapped onto 0 (since no line connects 1 to 4 or 7). Hence the permutation we are trying to find has the cycle (0, 1), and also the singleton cycle (8). The points 4 and 7 are either invariant, or else interchanged. In the former case, we would have 9 invariant as well; but then the line through 1 and 9 would be mapped on the line through 0 and 9 -- which is not a line of the configuration. On the other hand, if we assume that the permutation contains the cycle (4, 7) then we find, successively, that it must contain the cycles (3), (2,5) and (6, 9) as well. Hence the only candidate for an automorphism that maps 0 to 1 is the permutation s = (0, 1) (2, 5) (3) (4, 7) (6, 9) (8). A check reveals that s is indeed an automorphism of $(10_3)9$. A similar analysis shows that there is no automorphism that maps 0 to 4, or 2 to 6, or 3 to 8. Hence the decomposition

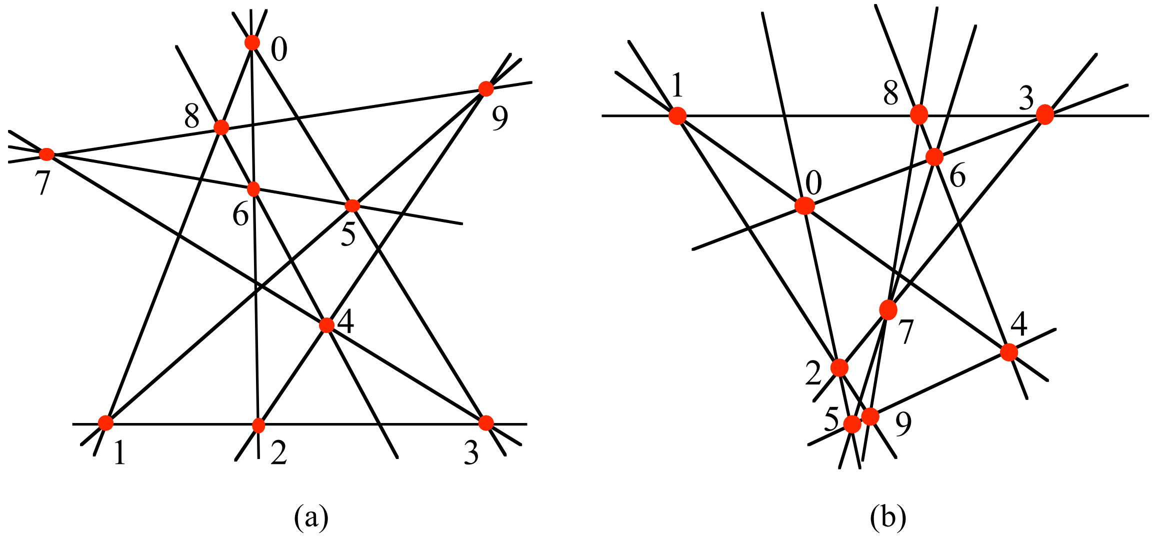


Figure 1.7.1. Two configurations (10_3).

of the vertices of $(10_3)_9$ into transitivity classes is {0, 1} {2, 5} {3} {4, 7} {6, 9} {8}. Moreover, the *automorphism* group consists of two elements, the identity and s. Thus a labeling of the points of $(10_3)_9$, more rational than the one in Figure 1.7.1(a), is as shown in Figure 1.7.3; now s interchanges the starred and double-starred versions of each letter, while keeping those without stars invariant. Concerning the *symmetry* groups of geometric configurations isomorphic to $(10_3)_9$ we can say that they either have the trivial symmetry group c_1 as in Figure 1.7.3, or else d_1 or c_2. However, since two of the points of the configuration remain invariant under s, it cannot have symmetry group c_2. By an elementary but slightly longer argument it can be shown that d_1 is impossible as well; hence no geometric realization has any nontrivial symmetry.

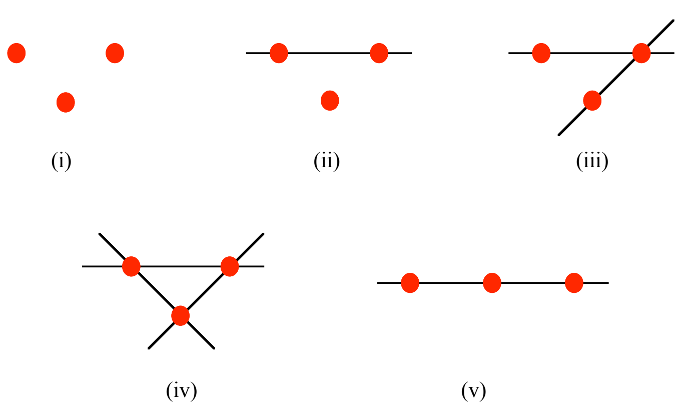


Figure 1.7.2. The possible "derived figures" for points of configurations (10_3).

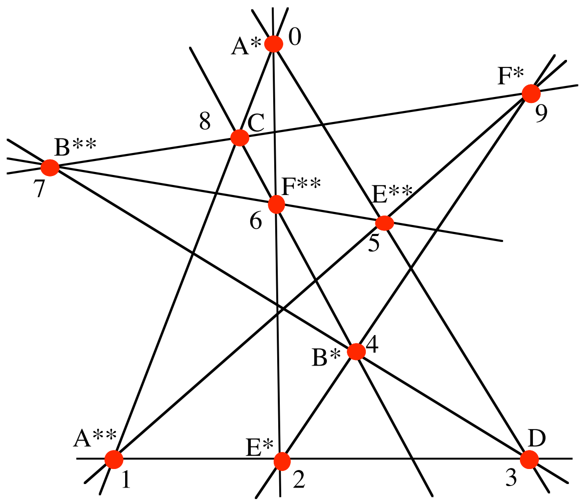


Figure 1.7.3. A revised labeling of the configuration $(10_3)_9$, making visible its automorphism group.

As an illustration of a second variant of the method of derived figures we investigate the four $(24_3, 18_4)$ configurations shown in Figure 1.7.4.

We shall call them C_1, C_2, C_3 and C_4. In this case we shall associate a "derived figure" with every line L of the configuration, by taking the 9 lines of the configuration that do not meet L in a point of the configuration, as well as the points of the configuration incident with these 9 lines. Since the configurations in Figure 1.7.4 have only two symmetry classes of lines, it is easy to determine all the derived figures; the isomorphism types that occur are **schematically** indicated in Figure 1.7.5. (Since we are interested only in isomorphisms, the relative positions of the points and lines are not relevant; only the incidences that occur in the configuration matter.) The results are:

C_1 has 6 derived figures of type (i), and 12 of type (ii), and so does C_4;

C_2 has 6 derived figures of type (iii), and 12 of type (iv);

C_3 has 18 derived figures of type (v); note that these are isomorphic to the ones of type (iii).

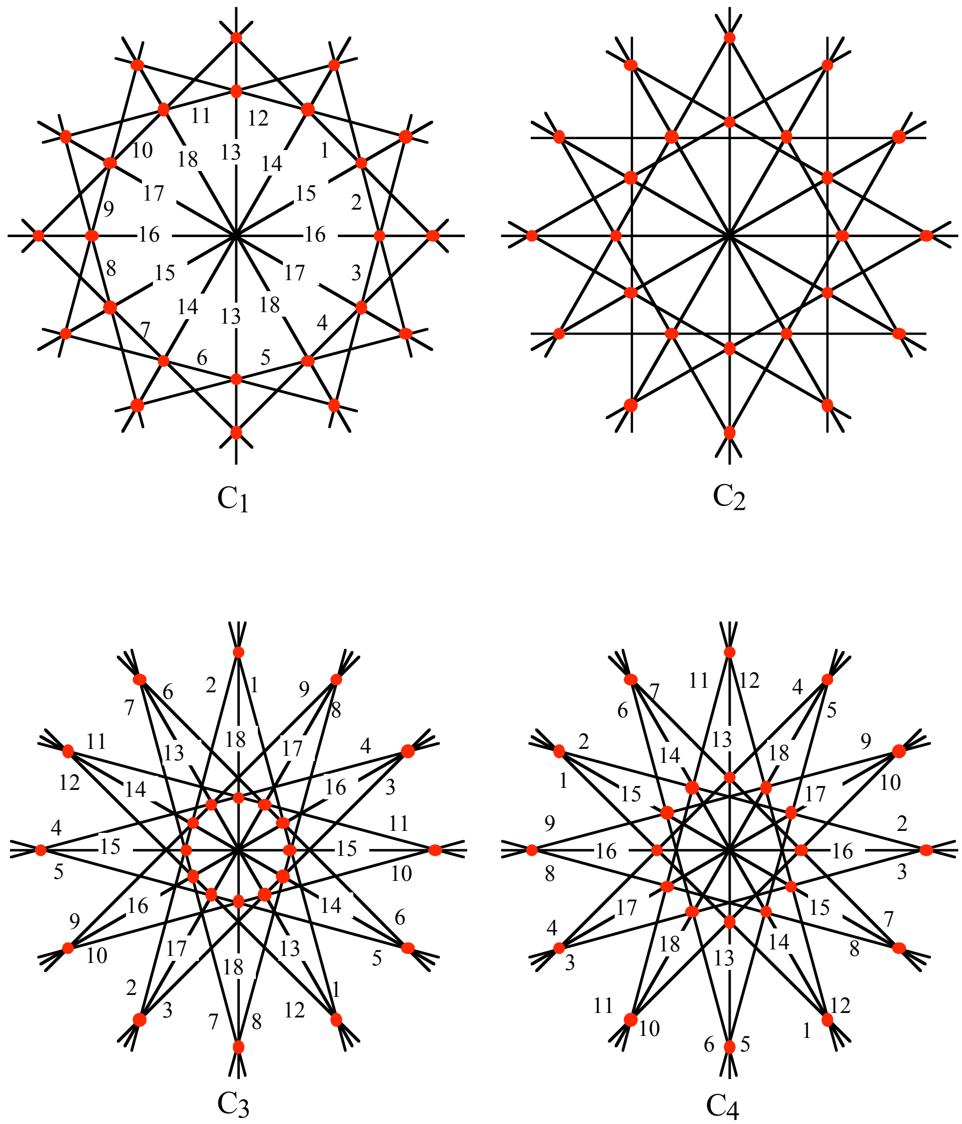


Figure 1.7.4. Four configurations $(24_3, 18_4)$.

It follows that the lines of C_1 form two transitivity classes, and so do the lines of C_2 and those of C_4. As far as the derived figures are concerned, the lines of C_3 could all belong to the same transitivity class. They, in fact, do so, as is shown, for example, by the permutation $t = (1)\ (2, 18)\ (3, 11, 9, 5)\ (4, 14, 10, 17)\ (6, 13)\ (7)\ (8, 15)\ (12, 16)$.

It also follows that C_2 and C_3 are isomorphic neither to each other nor to either of C_1 and C_4. However, as far as the derived figures are concerned, C_1 and C_4 could

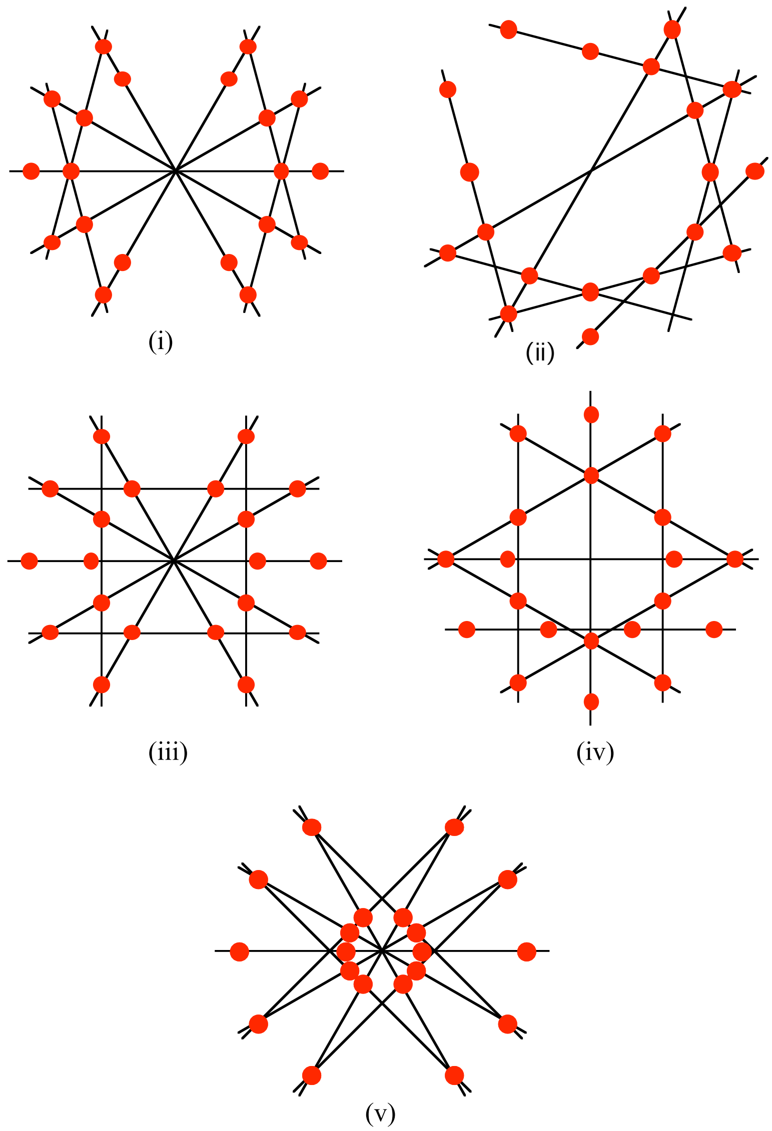


Figure 1.7.5. Derived figures that are possible for the configurations $(24_3, 18_4)$ in Figure 1.7.4.

be isomorphic. The labeling of these configurations in Figure 1.7.4, which was obtained with the help of the derived figures, is easily checked to represent an isomorphism between the two configurations.

As the next example we consider the three (9_3) configurations in Figure 1.1.6, and again find the derived figures for the various points. (We will consider these three configurations again in Section 2.2. There we will use a slightly different method.) It is clear that for all points, in all three configurations, the derived figure consists of two points. In $(9_3)_1$ none of these pairs is incident with a line of the configuration; in $(9_3)_2$ each such pair of points is incident with a line of the configuration, while in $(9_3)_3$ in six of the pairs the two points are incident with a line of the configuration and in the three remaining pairs this is not the case. Therefore no two of the configurations in Figure 1.1.6 are isomorphic. Moreover, the points in $(9_3)_3$ form (at least) two transitivity classes, while it is possible that in each of $(9_3)_1$ and $(9_3)_2$ they are in a single transitivity class. This is indeed the situation, as is easy to verify and as we will see in Section 2.2. (Note that $(9_3)_1$ is the Pappus configuration, which we encountered in Section 1.1.)

So far all the derived figures we considered were of a "local" type, related to features of the configuration that depended on the relations of each individual point with the other elements of the configuration. We shall now consider a "global" derived figure, which was used in several works of H. Gropp.

In Section 1.4 we briefly mentioned the Menger graph of a configuration. Its nodes are all the vertices of the configuration, and edges connect pairs of nodes that correspond to vertices incident with a line of the configuration. As easy to see, this graph is almost always rather unwieldy, and has had very little use. However, in many instances its **complement**, that is, the graph on the same nodes, but with edges connecting precisely those pairs which are not endpoints of an edge in the Menger graph. This graph has been called by Gropp the **configuration graph**; in Section 1.4 following [M5] we called it the **deficiency graph**. The deficiency graphs of the three (9_3) configurations in Figures 1.1.6 and 2.2.1 are shown in Figure 2.2.2 and used in Section 2.2 to distinguish between the three possible configurations (9_3).

The deficiency graph can be used to quickly decide whether the vertices of the $(12_4, 16_3)$ configuration in Figure 1.7.6 form one transitivity class. We consider its deficiency graph shown in Figure 1.7.7. The nodes P, Q, X are not in any 3-circuit, while the nine other nodes are in such circuits. Together with the obvious d_3 symmetry of the geometric realization in Figure 1.7.6, this means that P, Q, X are in one orbit, and that the other vertices form either one orbit, or they are in two orbits. It is easy to verify that the permutation (NMO)(RTV)((SWU)(P)(Q)(X), together with the geometric symmetries, establishes the single transitivity class of the nine points.

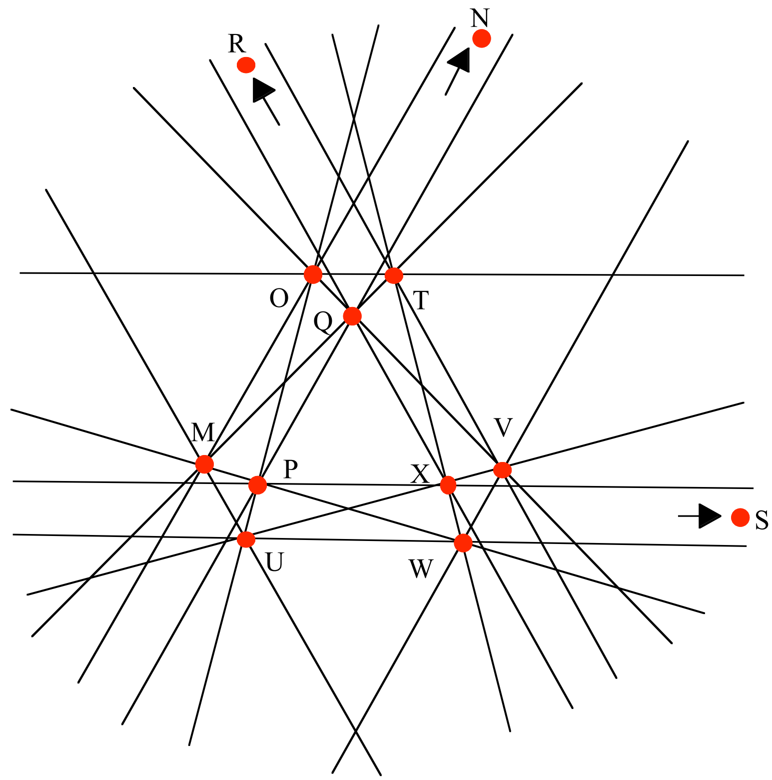


Figure 1.7.6. A $(12_4, 16_3)$ configuration.

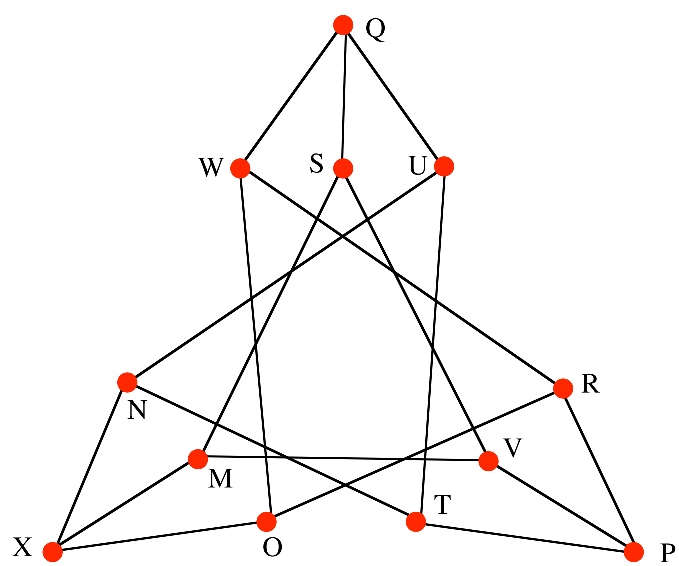


Figure 1.7.7. The deficiency graph of the $(12_4, 16_3)$ configuration in Figure 1.7.6.

Another example of the use of deficiency graphs is given in DiPaola-Gropp [D6], where combinatorial configurations (21_4) are studied. Using various combinatorial techniques they produce 200 non-isomorphic configurations of this kind, 12 of which are selfdual. They are recognized as non-isomorphic by the use of their deficiency graphs — except that one pair of non-isomorphic configurations has the same graph. They do not list the other configurations, but present configuration tables for these two (and their configuration graphs). They also do not make any statements regarding geometric realizability. I was curious whether the selfdual geometric configuration (21_4) from [G50] (shown in Figure 1.7.8) is isomorphic to one of these – and using the given configuration graph of the DiPaola-Gropp paper, it is easy to verify that our configuration is not isomorphic to either of these. In Table 1.7.1 we show a corrected copy of their tables. Since the vertices of our (21_4) form one orbit under automorphisms, in order to show that it is not isomorphic to the DiPaola-Gropp configurations it is enough to compare one of its remainder figures with any one of the latter. The remainder figure of the vertex A in Figure 1.7.8 is shown in Figure 1.7.10(a). The remainder figure of node 1 of the first graph in Figure 1.7.9 is shown in Figure 1.7.10(b); since it has a 5-valent vertex, the geometric configuration is not isomorphic to either of the two combinatorial ones.

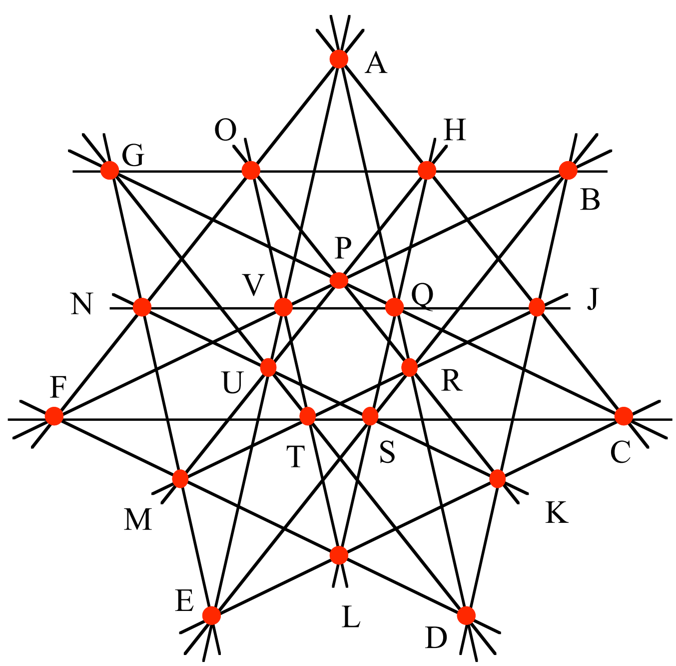
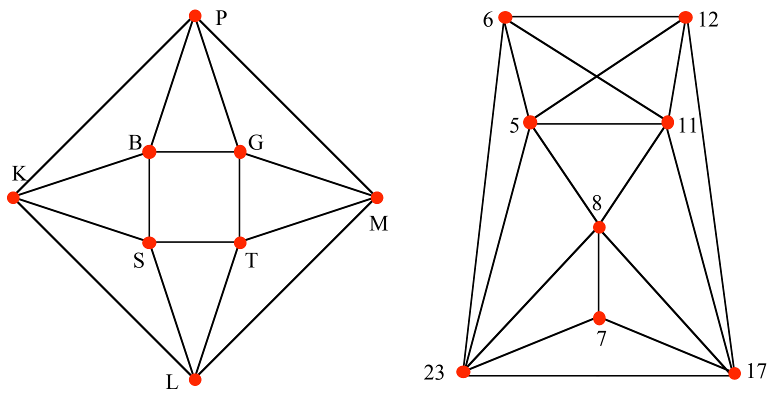


Figure 1.7.8. A geometric configuration (21_4) from [G50].

Configuration 749				Graph (749)			
1	2	3	22	(1,5)	(1,6)	(1,7)	(1,8)
1	4	13	24	(1,11)	(1,12)	(1,17)	(1,23)
1	9	16	18	(2,4)	(2,6)	(2,8)	(2,9)
1	10	14	15	(2,10)	(2,12)	(2,18)	(2,23)
2	5	14	24	(3,4)	(3,5)	(3,7)	(3,9)
2	7	16	17	(3,10)	(3,11)	(3,16)	(3,23)
2	11	13	15	(4,8)	(4,9)	(4,14)	(4,15)
3	6	15	24	(4,16)	(4,18)	(5,7)	(5,9)
3	8	17	18	(5,13)	(5,15)	(4,16)	(5,17)
3	12	13	14	(6,7)	(6,8)	(6,13)	(6,14)
4	5	6	23	(6,17)	(6,18)	(7,11)	(7,12)
4	7	10	22	(7,14)	(7,15)	(8,10)	(8,12)
4	11	12	17	(8,13)	(8,15)	(9,10)	(9,11)
5	8	11	22	(9,13)	(9,14)	(10,13)	(10,17)
5	10	12	18	(10,21)	(10,23)	(11,14)	(11,18)
6	9	12	22	(11,21)	(11,23)	(12,15)	(12,16)
6	10	11	16	(12,21)	(12,23)	(13,16)	(13,17)
7	8	9	24	(13,22)	(14,17)	(14,18)	(14,22)
7	13	18	23	(15,15)	(15,18)	(15,22)	(16,22)
8	14	16	23	(16,24)	(17,22)	(17,23)	(18,22)
9	15	17	23	(18,24)	(22,23)	(22,24)	(23,24)

Configuration 799				Graph (799)			
1	2	3	22	(1,5)	(1,6)	(1,8)	(1,9)
1	4	13	24	(1,10)	(1,12)	(1,16)	(1,23)
1	7	17	18	(2,4)	(2,6)	(2,7)	(2,9)
1	11	14	15	(2,10)	(2,11)	(2,17)	(2,23)
2	5	14	24	(3,4)	(3,5)	(3,7)	(3,8)
2	8	16	18	(3,11)	(3,12)	(3,18)	(3,23)
2	12	13	15	(4,8)	(4,9)	(4,14)	(4,15)
3	6	15	24	(4,17)	(4,18)	(5,7)	(5,9)
3	9	16	17	(5,13)	(5,15)	(5,16)	(5,18)
3	10	13	14	(6,7)	(6,8)	(6,13)	(6,14)
4	5	6	23	(6,16)	(6,17)	(7,11)	(7,12)
4	7	10	22	(7,13)	(7,14)	(8,10)	(8,12)
4	11	12	16	(8,14)	(8,15)	(9,10)	(9,11)
5	8	11	22	(9,13)	(9,15)	(10,15)	(10,16)
5	10	12	17	(10,23)	(10,24)	(11,13)	(11,17)
6	9	12	22	(11,23)	(11,24)	(12,14)	(12,18)
6	10	11	18	(12,23)	(12,24)	(13,16)	(13,18)
7	8	9	24	(13,22)	(14,16)	(14,17)	(14,22)
7	15	16	23	(15,17)	(15,18)	(15,22)	(16,22)
8	13	17	23	(16,24)	(17,22)	(17,24)	(18,22)
9	14	18	23	(18,24)	(22,23)	(22,24)	(23,24)

Table 1.7.1. The configuration tables of two non-isomorphic combinatorial configurations (21_4) and their isomorphic configuration graphs (from [D6]). The isomorphism is established by the permutation mapping Graph (749) onto Graph (799): (1,13,2,14,3,15)(4,8,6,7,5,9)(10,17,11,18,12,16)(22,23). The peculiar names of the marks (19,20,21 not used) are from [D6]. The two red entries are not correct in the original.



(a) (b)

Figure 1.7.9. (a) The remainder figure of vertex A in Figure 1.7.8. All other vertex figures are isomorphic to this. (b) The remainder figure of vertex 1 in the configuration (749) of Table 1.7.1.

Exercises.

1.7.1. For the $(16_3, 12_4)$ configuration of Figure 1.7.4(c) consider the derived figures of lines of each of the two symmetry classes. Find an automorphism that maps one line in one symmetry class to a line in the other class, and use that to show that all the lines belong to a single transitivity class. Show that all points belong to a single transitivity class, and that, in fact, all flags are in one transitivity class.

1.7.2. Use derived figures of lines to show that the two $(20_3, 15_4)$ configurations in Figures 4.3.3 and 4.3.4 are not isomorphic. Show also that in each of these configurations the lines (as well as the points) form two transitivity classes. How many transitivity classes of flags are there?

1.7.3 For the configuration we denote $(10_3)_3$ in Section 2.2, conduct an analysis of its automorphisms and symmetries analogous to the one we did above for $(10_3)_9$.

1.7.4. In continuation of our discussion concerning the configuration $(12_4, 16_3)$ shown in Figure 1.7.6, decide how many orbits of lines are there under the group of automorphisms.

1.7.5 Investigate the orbits, automorphisms and symmetries of the configuration $(12_4, 16_3)$ shown in Figure 4.3.7(b).

1.7.6 Are any of the three configurations in Figure 4.3.7 isomorphic?

1.7.7. Show that the vertices of the (21_4) configuration in Figure 1.7.8 are in a single orbit under automorphisms.

1.7.8 Find the automorphisms of the configuration denoted (749) in Table 1.7.1.