1.6 REDUCED LEVI GRAPHS

We introduced Levi graphs of configurations in Section 1.4. Now, with the symmetry concepts available, we can modify Levi graphs in such a way that for symmetric geometric configurations the information appears in a much more condensed form. We call these graphs "reduced Levi graphs", and describe them separately for cyclic and for dihedral symmetry groups[1].

The **reduced Levi graph** R(C) of a geometric (or topological, or combinatorial) configuration C with a **cyclic** group c_r and of orbit type $[h_1, h_2]$ is a bipartite graph, that consists of h_1 black vertices and h_2 white ones, corresponding to the **orbits** of points and of lines of C. An edge connects two vertices (of different colors, naturally) if and only if points of the corresponding orbit are incident with the lines of the corresponding line orbit. For the (12_3) configuration C in Figure 1.6.1 this first step leads to the graph in Figure 1.6.2(a).

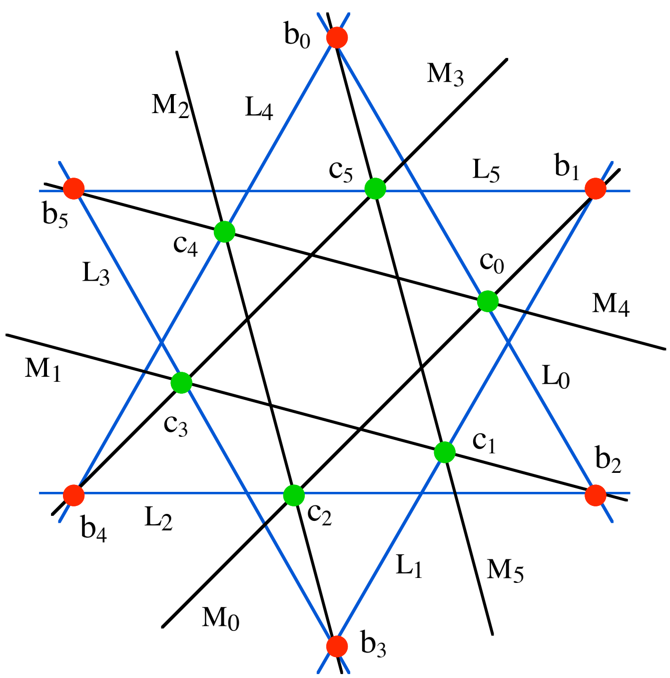


Figure 1.6.1. An astral (12_3) configuration C, with cyclic symmetry group c_6 and with labels and colors convenient in the construction of its reduced Levi graph R(C).

[1] The term "reduced Levi graph" has been used with a different meaning by R. Artzy in [A2]. We shall discuss this in Section 2.11.

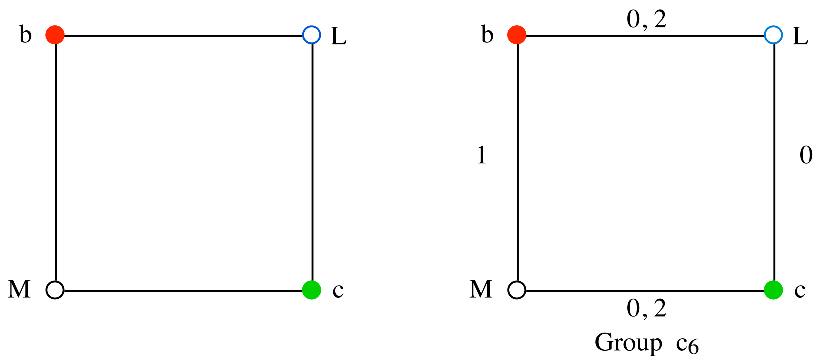


Figure 1.6.2. The formation of the reduced Levi graph R(C) of the configuration C in Figure 1.6.1. The first step is shown at left, the complete graph R(C) at right. All subscripts are understood mod 6.

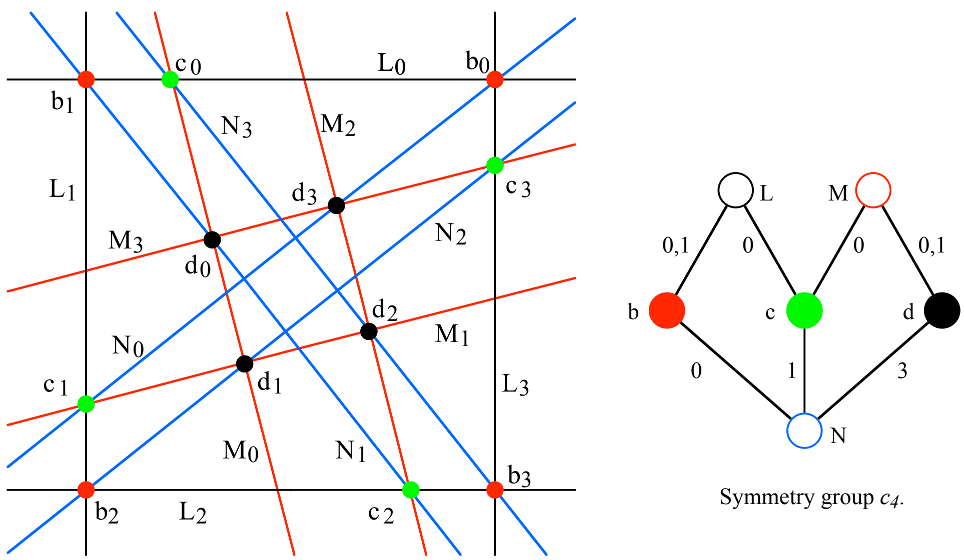
The reduced Levi graph R(C) relies on the labeling of the points and vertices in a way that corresponds to the action of (a generator of) the symmetry group. With such a labeling, an edge of the R(C) carries as labels the differences between the labels of the points that led to the particular edge. Thus in the case illustrated in Figures 1.6.1 and 1.6.2, each line L_j is incident with points b_j and b_{j+2} and c_j, leading to the labels shown on the edges connecting L to b and c in the reduced Levi graph. Similarly for the lines M_j. An indication of the relevant symmetry group completes the reduced Levi graph.

As we shall see in Section 2.8, changing only the symmetry group in this graph helps specify a whole infinite family of graphs of astral configurations.

A different example is presented in Figure 1.6.3. It is a slightly more symmetric version of a configuration shown in Figure 2 of Daublebski [D2] and specified combinatorially in his list as #88. It is interesting to note that (as stated by Daublebski) the *automorphisms* of the underlying set-configuration act transitively on the points; in fact the automorphisms are transitive on the lines, and even on the flags, see Exercise 1.6.6 below. This difference between the symmetry group of a geometric configuration and its underlying set-configuration is a frequent phenomenon; we noted it in connection with the configuration in Figure 1.5.4 and its Levi graph in Figure 1.5.5.

It is quite obvious that the reduced Levi graph R(C) of a configuration C can be used to find the configuration table or its equivalent Levi graph L(C) (as originally defined in Section 1.4). It is also clear that the construction of a reduced Levi graph can be carried out for combinatorial configurations that have a cyclic symmetry group.

A slight modification of the construction outlined above is necessary in case some lines M_j of a configurations C are mapped onto themselves by a c_2 subgroup of the cyclic group of symmetries of C. Then the order of the cyclic group c_k must be even, k = 2t, and for each j we have $M_j = M_{j+t}$. In the reduced Levi graph R(C) we indicate this fact by writing $M^{\sim}$. An illustration is given in Figure 1.6.4, in the case of a configuration $(4_3, 6_2)$ with cyclic symmetry group c_4. Similar to this is the modification required if the configuration includes points-at-infinity; such points are mapped onto themselves by a 180° rotation, that is, by a c_2 symmetry. This is illustrated by the configuration and its reduced Levi diagram in Figure 1.6.5.



Symmetry group c_4.

Figure 1.6.3. The (12_3) configuration listed as #88 in the enumeration of Daublebski [D2], and its reduced Levi graph.

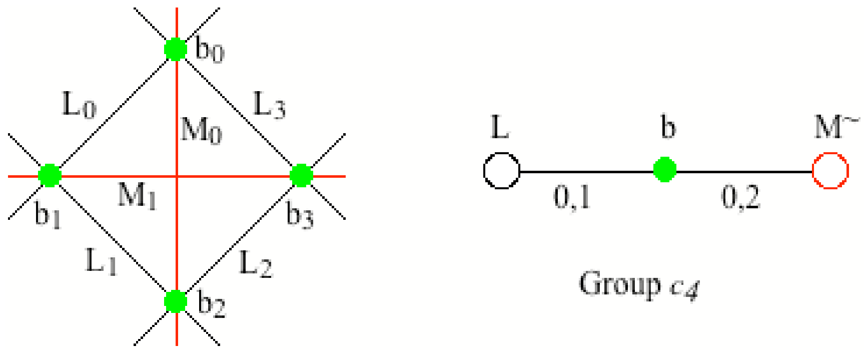
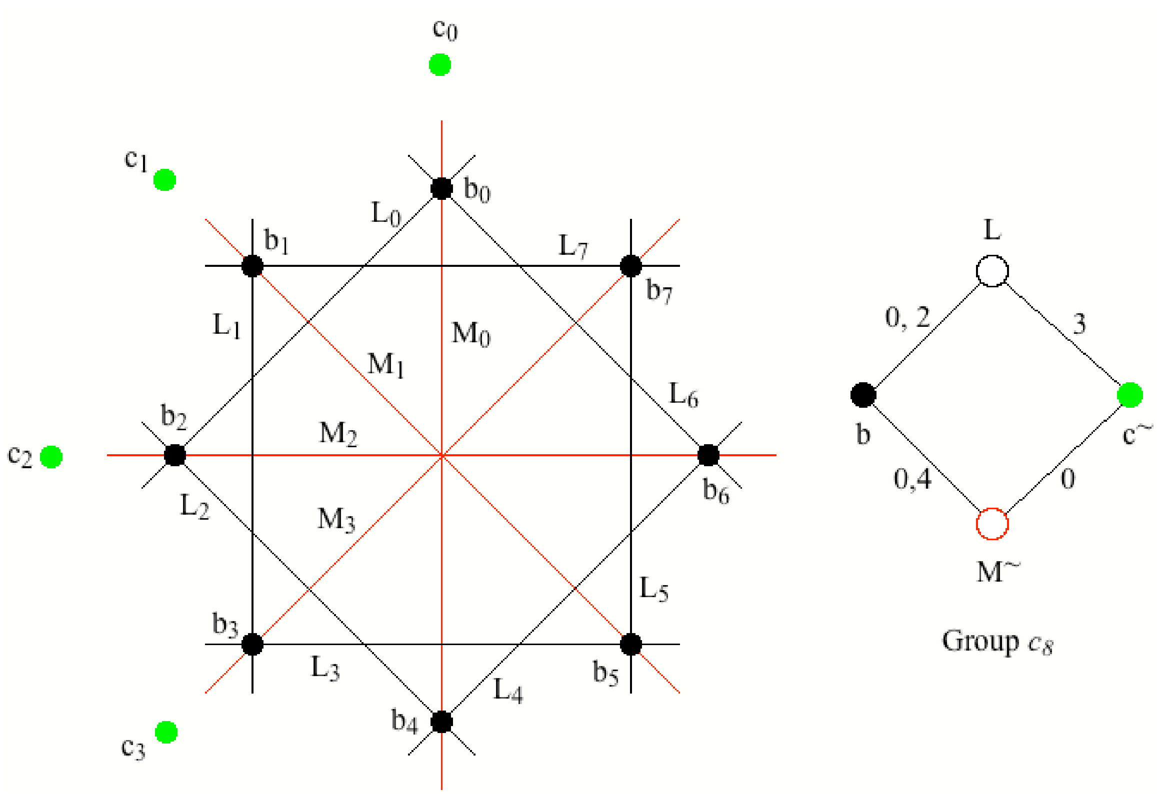


Figure 1.6.4. An example illustrating the formation of the reduced Levi diagram in the presence of lines that are mapped onto themselves by a subgroup c_2 of the symmetry group. The cyclic symmetry group of the (3,2)-configuration shown at left is c_4.



(a) (b)

Figure 1.6.5. (a) A (12_3) configuration astral in the extended Euclidean plane, with cyclic symmetry group c_8; the points at infinity (indicated by the detached dots) and the lines M_j are individually invariant under a c_2 subgroup. (b) The reduced Levi diagram of this configuration.

A concept closely related to reduced Levi graphs of configurations with cyclic symmetry group was introduced by T. Pisanski under the name "voltage graphs", and first published in [B20]. The description presented above differs from the one used in [B20] in that our starting point is a given symmetric (geometric or topological) configuration, while Pisanski starts with a combinatorial configuration that admits a non-trivial cyclic automorphism. For general graphs the concept of "voltage" was introduced by Gross [G33], and presented in detail in several chapters of [G34] and [G35].

For configurations with dihedral symmetry the construction of reduced Levi graphs is slightly more complicated — but in some cases it allows for a commensurately more compressed encoding of the configuration. Naturally, a configuration with dihedral symmetry group d_j can also be considered under the action of the cyclic symmetry group c_j. However, this disregard of reflections as symmetry operations leads in many cases to an increase in the *number* of orbits (and decrease in the *size* of some orbits). For example, if the configuration in Figure 1.5.1(b) is considered under the symmetry group c_5, all orbits have size 5 and the orbit type is [5, 5].

The main steps to consider in the construction of a reduced Levi graph of a configuration with dihedral symmetry are:

- One reflection and its images under the cyclic part of the group are selected and kept throughout. In each orbit a representative pair of elements or a singleton is chosen. The labels of mirror-image chosen pairs carry + or – superscripts, the superscript is ± if the chosen element is mapped onto itself by the mirror. In the latter case it is often convenient to drop the superscript entirely.

- Each black point of the graph corresponds either to a pair of configuration points related by the appropriate reflection, or to a point on a mirror; white points (empty circles) correspond analogously to lines — pairs related by reflection in the mirror, or single lines invariant under the reflection chosen. Rotations carry the labels to all points and lines of the configuration.

- The edges of the graph carry the information regarding which signed labels of lines lead to which signed labels of points, as well as the actual difference in label subscripts. The positive or negative superscripts are indicated by p and n, respectively.

Details of the construction and labeling are best understood by examples. In Figure 1.6.6 we show a labeled configuration (12_3) with symmetry group d_3. Its reduced Levi diagram is shown in Figure 1.6.7. More complicated examples are shown in Figure 1.6.8 to 1.6.10. In cases where the configuration is part of a family with varying numbers of points, it is sometimes convenient to use negative integers to indicate the difference in the labels of points and lines. This is illustrated by the examples in Figure 1.6.10, that belong to families we shall discuss in Section 3.X.

As in the case of cyclic symmetry groups, for configurations with dihedral symmetry group that contain lines or points that are mapped onto themselves by halfturns a few special conventions are needed for the labeling.

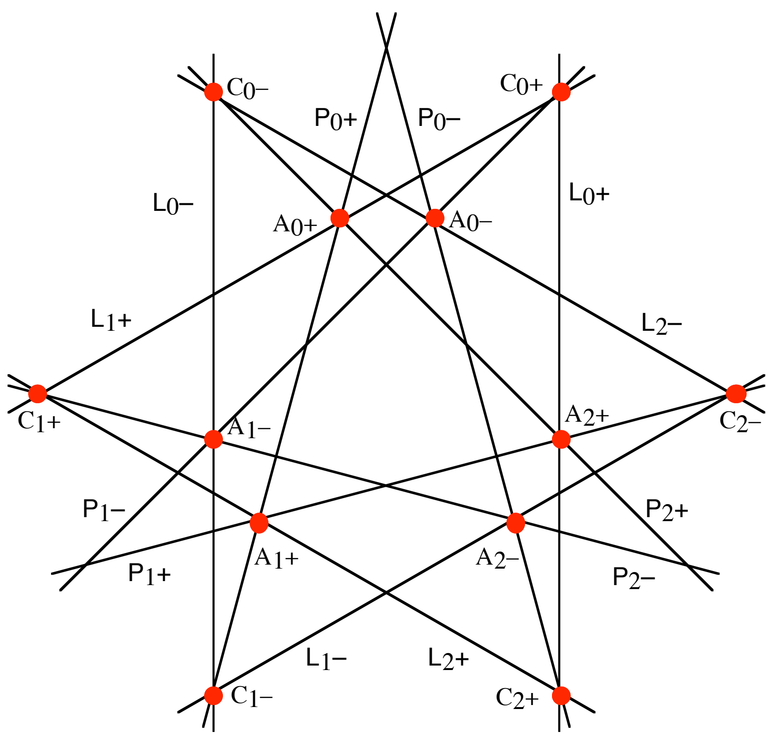


Figure 1.6.6. An astral configuration (12_3) with symmetry group d_3.

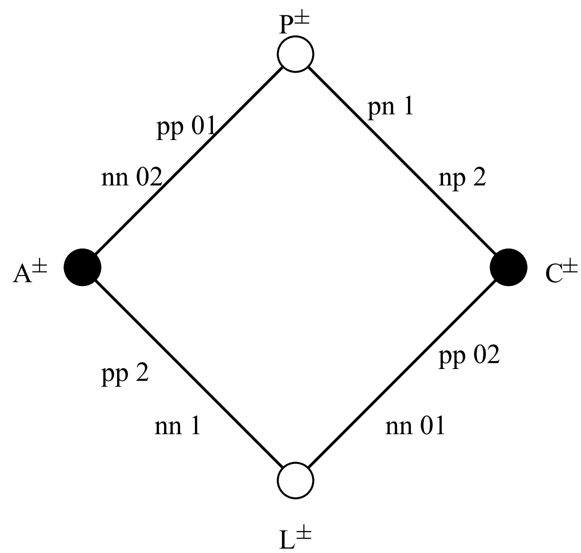


Figure 1.6.7. The reduced Levi graph of the configuration in Figure 1.6.6.

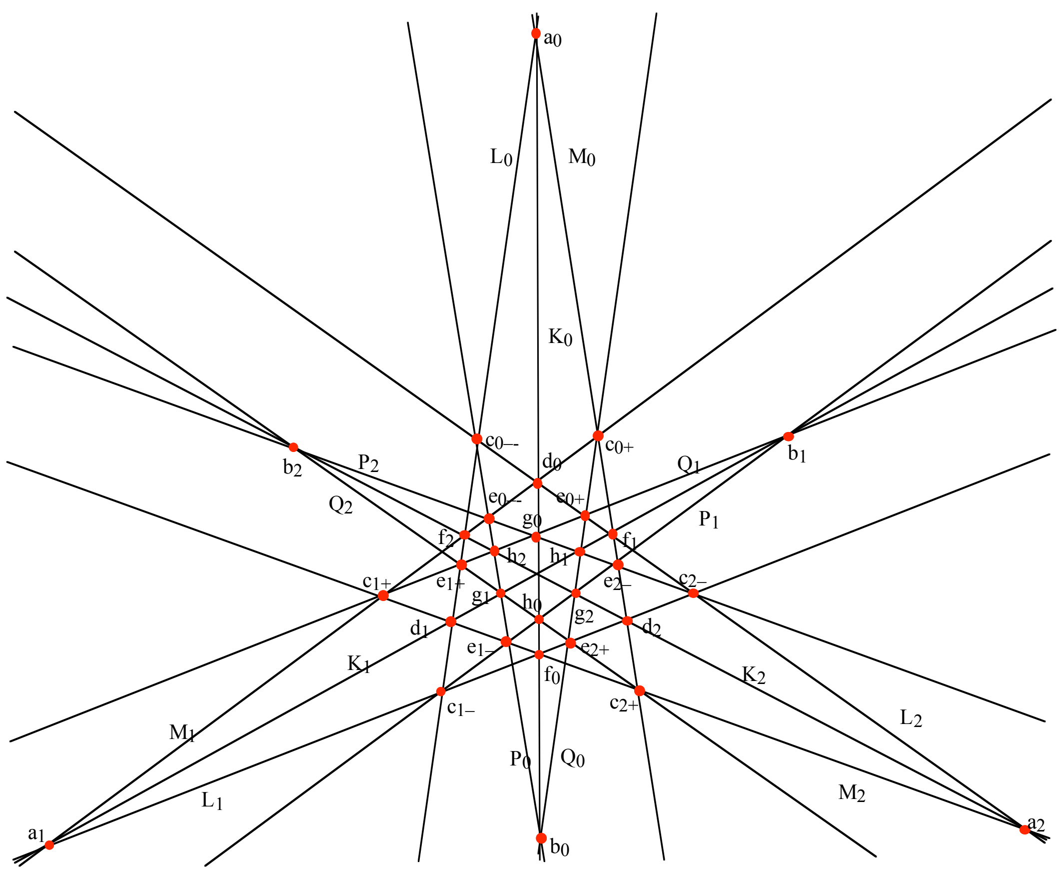


Figure 1.6.8. A $(30_3, 15_6)$ configuration with symmetry group d_3.

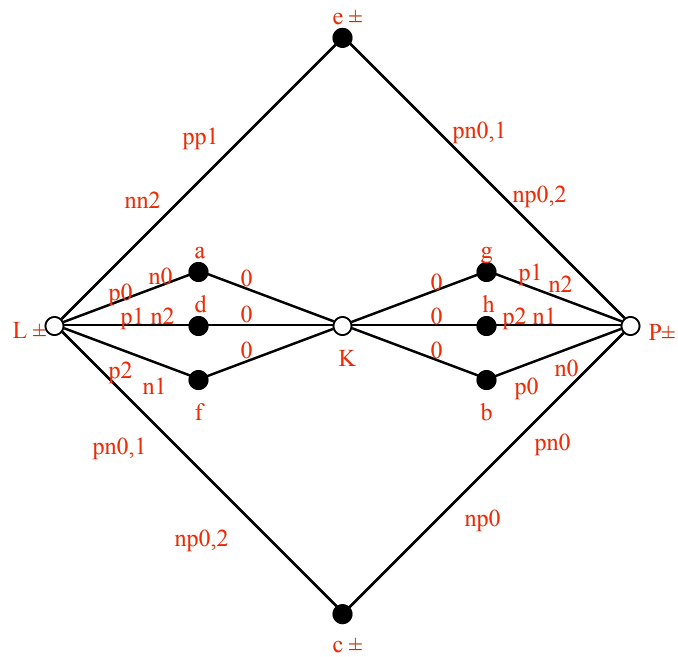
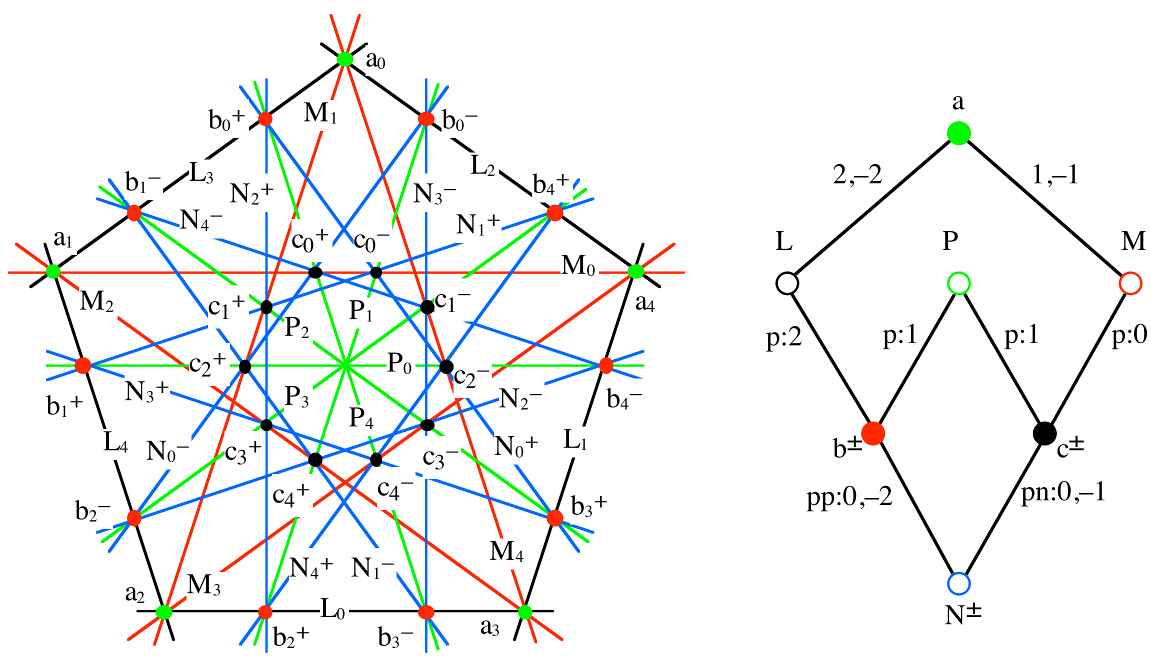
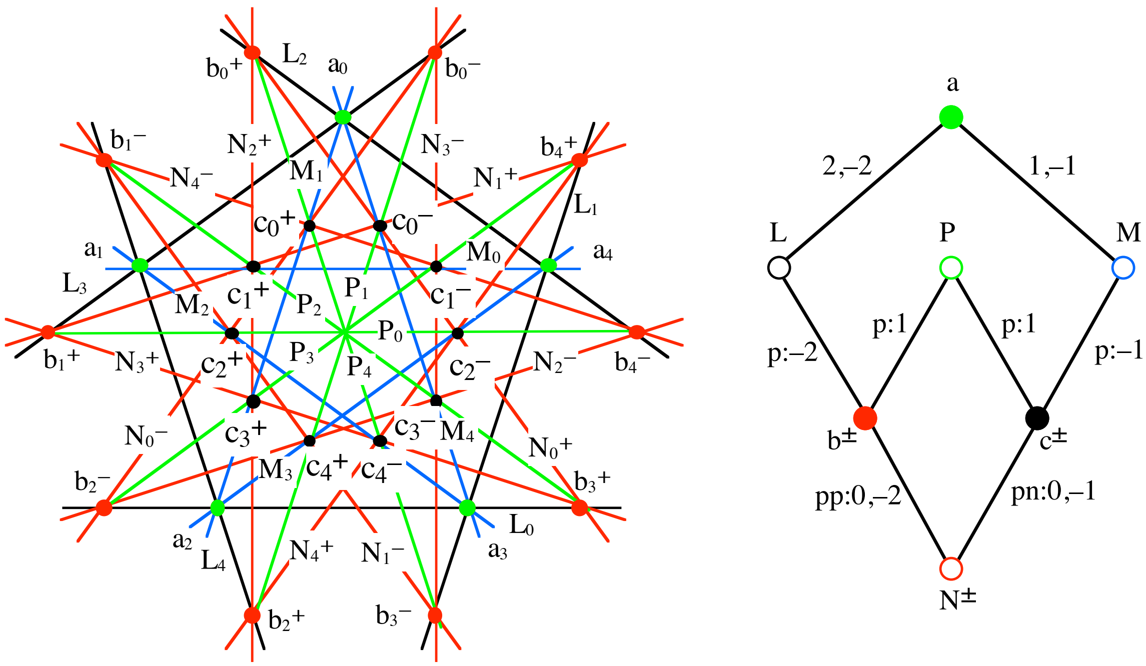


Figure 1.6.9. The reduced Levi graph of the configuration in Figure 1.6.8.



(a)



(b)

Figure 1.6.10. On the left, two configurations (25_4), each of orbit type [3, 3] and with symmetry group d_5. The second one is the same as the configuration in Figure 1.5.1(b). On the right — the reduced Levi graphs of these configurations. All subscripts are mod 5. The configuration in (a) is due to J. Bokowski, see Section XXX.

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Exercises and problems 1.6.

1. Find the reduced Levi graph of the (14_3) configuration in Figure 1.5.2a, and compare it with the reduced Levi graph of the (12_3) configuration shown in Figure 1.6.1

2. Find labels for the configuration in Figure 1.5.2b that yield the reduced Levi graph shown in Figure 1.6.11.

3. Show that the (10_3) configuration in Figure 1.6.12 is isomorphic with the Desargues configuration (10_3) shown in Figure 1.1.1.

4. Find a reduced Levi graph for the (10_3) configuration in Figure 1.6.12.

5. Find the reduced Levi graphs of the configurations in Figure 1.3.8.

6. (i) Use the labels in Figure 1.6.3 to label the Levi graph in Figure 1.6.12.

(ii) Use the Levi graph in Figure 1.6.12 to show that all points of the (12_3) configuration in Figure 1.6.3 form one orbit under its group of automorphisms.

(iii) Use the Levi graph in Figure 1.6.12 to show that the (12_3) configuration in Figure 1.6.3 is selfdual.

(iv) Use the Levi graph in Figure 1.6.12 to show that all flags of the (12_3) configuration in Figure 1.6.3 form one orbit under its group of automorphisms.

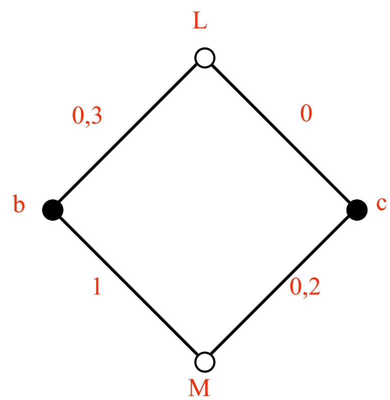

Figure 1.6.11. A reduced Levi graph used in Exercise 2.

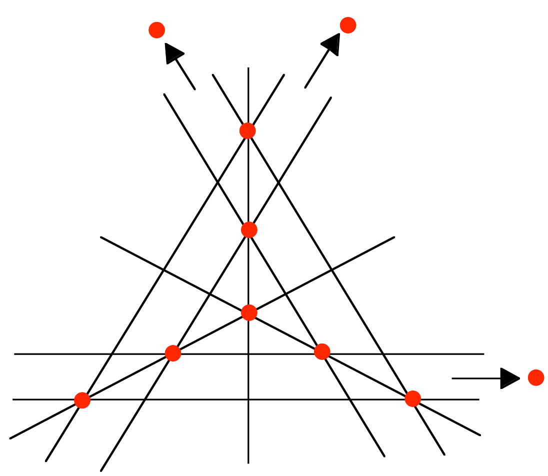

Figure 1.6.12. A configuration used in Exercises 3 and 4.

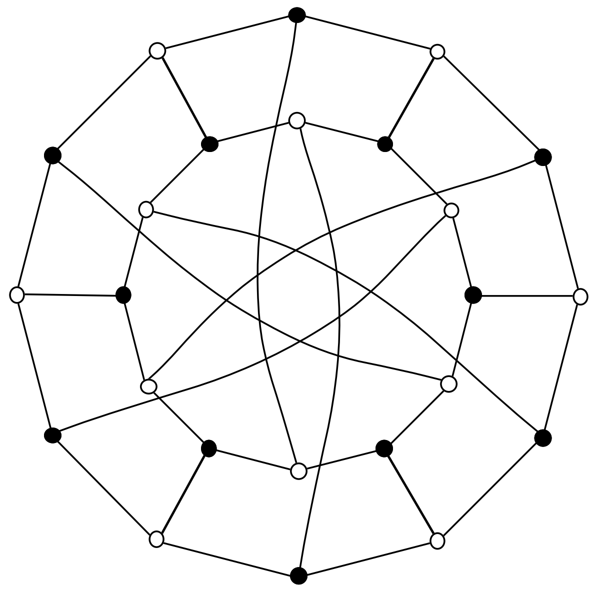

Figure 1.6.13. The Levi graph of the (12_3) configuration in figure 1.6.3.