1.5 SYMMETRY

By a **symmetry** of an object we generally mean a mapping of the object onto itself that preserves some relevant features of the object. For configurations, by **symmetry** we shall understand that the incidence relations are preserved, but will also impose other requirements that will depend on the kind of configuration considered and on other aspects of the discussion.

More specifically, for combinatorial configurations a **symmetry** is just an incidence preserving one-to-one mapping (permutation) of the elements of the configuration onto themselves. We find it convenient to distinguish between **automorphisms**, that is symmetries that map marks to marks and blocks to blocks, and **dualities**, that map marks to blocks and vice versa. By an **automorphism** of a geometric or topological configuration we shall understand an automorphism of the underlying combinatorial configuration.

For topological configurations a **symmetry** is a *homeomorphism* of the plane onto itself that maps the configuration onto itself. However, in different contexts, this definition should be understood in one of three ways, depending on the **plane** we are considering. This can be either the Euclidean plane E^2, or the extended Euclidean plane E^{2+}, or the projective plane P^2. Although the projective plane is homeomorphic with the extended Euclidean plane, when considering symmetries of E^{2+} we require that the line-at-infinity be mapped onto itself. It follows that the symmetries of a topological configuration in E^{2+} can be a proper subset of the symmetries of such a configuration in P^2.

Analogously, **symmetries** of geometric configurations in E^2 are *isometries* of the plane that map the configuration onto itself. For geometric configurations in E^{2+} we need an isometry of E^2 that maps the finite part of the configuration onto itself and permutes the points-at-infinity.

For both topological and geometric configurations it is sometimes useful to include dualities among their symmetries. In particular, for geometric configurations a special type of duality is called **polarity** or **reciprocation**, since it arises by the polarity (also called reciprocation by some) in a circle.

As is obvious, in each of the interpretations of the term "symmetry", all symmetries of a configuration form a group, **the symmetry group** of the configuration (in the appropriate sense). Quite often it is convenient to consider only a subgroup of the symmetry group of a configuration. In such a case we shall say that the group in question is **a group of symmetries** of the configuration.

We shall soon see examples of these various interpretations of "symmetry". But first we should discuss two aspects of symmetries of geometric configurations (that apply in some cases to the other kinds of configurations as well) that lead to classifications of the appropriate configurations.

Our first concern is the collection of orbits of the configuration under *the* symmetry group of the configuration. If a configuration has h_1 orbits of points and h_2 orbits of lines we shall occasionally say that it is of **orbit type [h_1, h_2]** or **[h_1, h_2]-orbital**. We note that no *geometric* or *topological* (n_k) configuration with $k \geq 3$ can have a single orbit of points, or a single orbit of lines; in contrast, there are many such *combinatorial* configurations of this type, and even of [1,1] orbit type. More generally, if a geometric [q,k]-configuration (that is, a (p_q, n_k) configuration) is [h_1, h_2]-orbital, then $h_1 \geq [(k+1)/2]$ and $h_2 \geq [(q+1)/2]$. This is a consequence of the fact that no isometric symmetry can map the middle one of three collinear points onto one of the other points of the triplet, and analogously for lines. In case equality holds in both inequalities we shall say that the configuration is **astral**. Most interesting seem to be [h_1, h_2]-orbital configurations with $h = h_1 = h_2$; in that case we shall simplify the language by calling the configuration **h-orbital** or, more often, **h-astral**. If the values of h_1, h_2 or h are not relevany or not known, we shall speak of **multiastral** configurations. (More on this terminology at the end of the present section.)

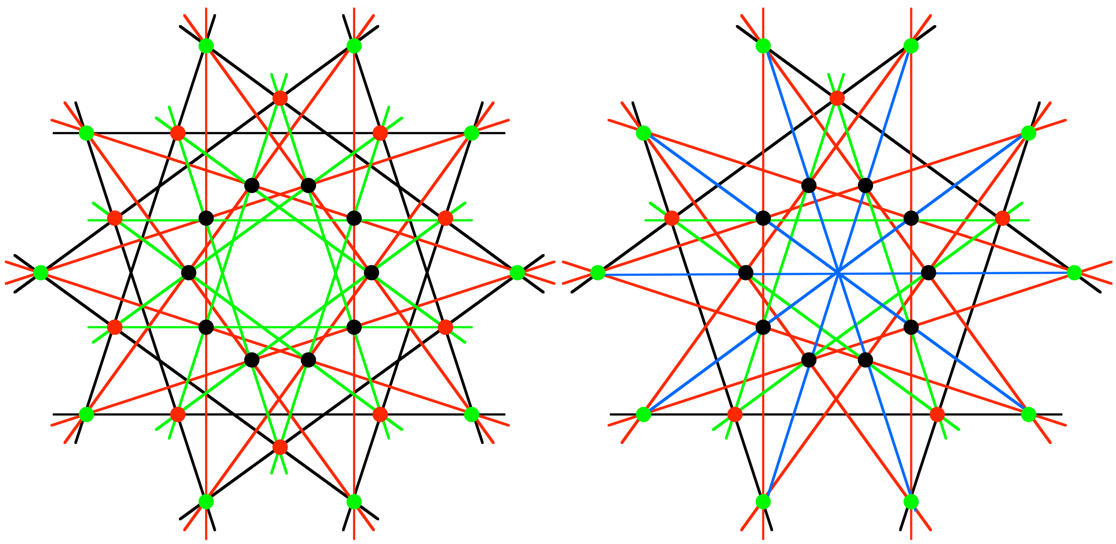
In many situations we shall be dealing with configurations in which all orbits have the same number of elements; however, some cases in which this condition is not fulfilled do have interesting features, and lead to various questions. In any case, this is not a requirement included in the definition.

The other aspect of symmetry considerations for a geometric configuration is the determination of its **symmetry group**. From the well-known classification of isometries of the Euclidean plane it follows that the symmetry group of a geometric or topological configuration is either a **cyclic group** c_r or a **dihedral group** d_r, where r is a positive integer. The group c_r consists of rotations about a **center** through integer multiples of

2p/r, the zero multiple being the identity. The group d_r consists of the same rotations as its subgroup c_r, together with r **mirrors**, that is lines of reflective isometry.

For example, the configuration in Figure 1.5.1(a) has symmetry group d_{10}, the one in (b) has symmetry group d_5. Other illustrations are given in Figure 1.5.2 and 1.5.3.

Although configurations with non-trivial symmetry group occurred in the literature from time to time, it is the recent — last twenty years or so — systematic concern with very symmetric configurations that led to the revival of interest in the whole topic of configurations. We shall investigate symmetric configurations of various kinds in several of the following sections.



(a) (b)

Figure 1.5.1. Geometric configurations (30_4) of orbit type [3,3] (that is, 3-astral) and (25_4) of orbit type [3,4]; the orbits are color-coded. The configuration in (a) has symmetry group d_{10} and all (point and line) orbits of size 10. The configuration in (b) has symmetry group d_5, two orbits of size 10 and one of size 5 for points, and one orbit of size 10 and three of size 5 for lines.

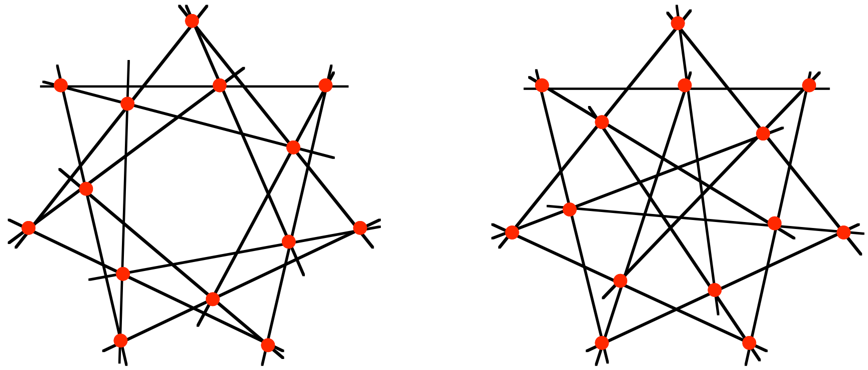


Figure 1.5.2. Two geometric configurations (14_3) of the orbit type [2,2], and with symmetry group c_7. Both are astral.

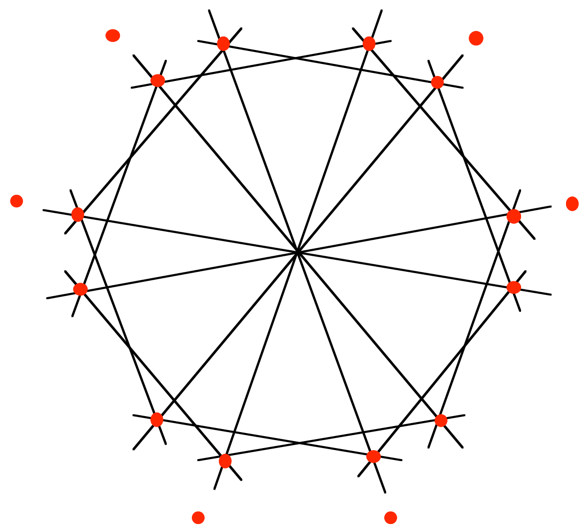


Figure 1.5.3. A geometric configuration (18_3) of the orbit type [2,2] with symmetry group d_6 in the extended Euclidean plane. This configuration cannot be represented with high symmetry in the Euclidean plane, but it is astral in the extended Euclidean plane E^{2+}.

As an example of the use of the different notions of symmetry we reproduce as Figure 1.5.4 once more Figure 1.3.3, and show its Levi graph in Figure 1.5.5. Although the symmetry group of this configuration is c_5, the symmetries of its Levi graph show that the automorphism group of this configuration is c_{10}, and the group of automorphisms and selfdualities is d_{10}.

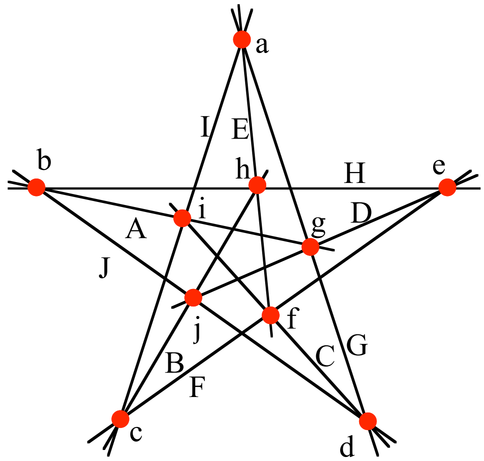


Figure 1.5.4. This (10_3) configuration has symmetry group c_5, and orbit type [2,2]; hence it is astral.

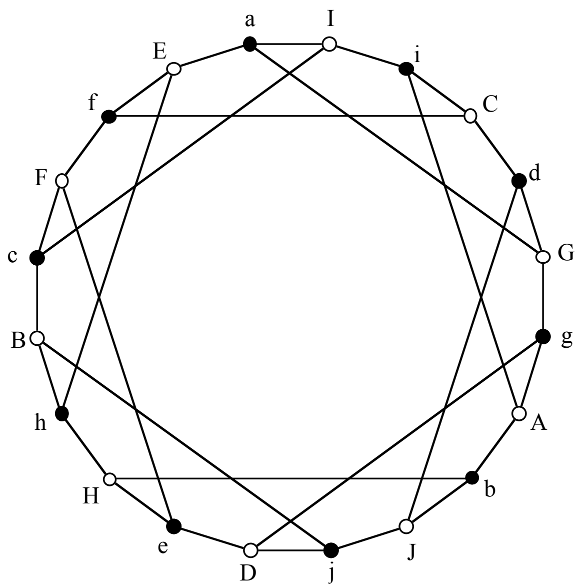


Figure 1.5.5. The Levi graph of the (10_3) configuration shown in Figure 1.5.4.

One urgent note of caution.

As is the case in many rapidly developing fields, the terminology of configurations with varying degrees of symmetry is still unsettled. One could almost claim that each author introduces separate concepts, and often even changes them from paper to paper. This has certainly been the case with the present writer — naturally, on each occasion there was some good reason for the terms introduced and used.

The astral, h-astral and $[h_1, h_2]$-astral terminology we shall use in this book is a development of the various similar concepts introduced in [G39], [G40], and [G46]. It should be stressed that although astral configurations are visually attractive and theoretically most easily investigated, for many kinds of configurations they are not the smallest possible.

In [B20] M. Boben and T. Pisanski introduced a related terminology, dealing with *polycyclic* configurations in the Euclidean plane. They call a configuration C **k-cyclic** provided there exists an automorphism α of order k of the underlying abstract configuration such that all orbits of points and lines of C under α have the same size k (that is, number of elements in each is k). In this terminology the configuration in Figure 1.5.1(a) is 10-cyclic (with three orbits of points, and three orbits of lines), while the configuration in Figure 1.5.1(b) is 5-cyclic (with five orbits each of points and lines). The two configurations in Figure 1.5.2 are 7-cyclic.

Another related concept is that of **celestial configurations**, considered by L. Berman in [B7]. We shall discuss it in Chapter 3.

Exercises and problems 1.5.

1. Decide whether the two (14_3) configurations in Figure 1.5.2 are isomorphic.
2. Find the symmetry group of each of the configurations (12_3) in Figure 1.3.8.
3. Consider the different realizations of the Pappus configuration $(9_3)_1$ in Figure 1.5.6. Maps between the different realizations establish automorphisms of the underlying combinatorial configuration. Find permutation representations for each of these mappings. Which of them correspond to geometric (isometric) symmetries?
4. Show that all points of the combinatorial configuration underlying the Pappus configuration $(9_3)_1$ form a single orbit (under automorphisms). What about the lines? What about the other two configurations (9_3) (see Figure 1.1.6)?
5. Show that there exist combinatorial configurations such that all the points are in one orbit (under automorphisms) but the lines belong to more than one orbit.

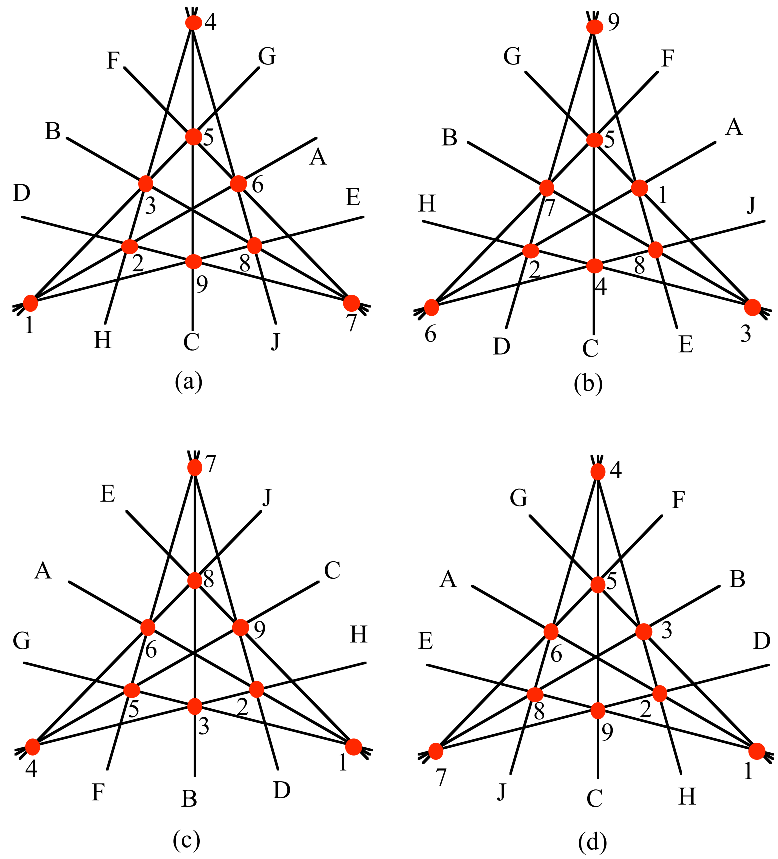


Figure 1.5.6. Four realizations of the Pappus configuration $(9_3)_1$.