1.4 TOOLS FOR THE STUDY OF CONFIGURATIONS

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There are many different ways to relate configurations of points and lines to other mathematical objects; these are often useful in investigating configurations. In Section 1.3 we encountered one example of such a tool — the underlying set-configuration and its *configuration table*. A related concept, which is helpful in the context of more general combinatorial structures as well, is the *incidence matrix* of the configuration. Specifically for configurations, incidence matrices seem to have been introduced by Levi [L3], and we shall usually call them "Levi incidence matrices". Levi makes the rows of a matrix correspond to the points of the configuration, and the columns to the lines (or vice versa). An incidence between a point and a line is indicated by a marking of the corresponding element of the matrix. This can be done by assigning to such matrix elements the value 1, and to the others 0, or by some other specification. Levi's own preference is to use an array of small squares, and an X marked in each square that represents an incidence. As an illustration, consider the set-configuration we shall encounter in Section 2.2, and denote there by $(10_3)_4$. A configuration table is shown in Table 1.4.1, and a Levi incidence matrix in Figure 1.4.1(a). By a suitable permutation of the rows and columns of that matrix, we obtain the Levi incidence matrix in Figure 1.4.1(b). This latter form demonstrates one of the uses of incidence matrices: It shows at a glance that the configuration in question is selfdual, since the matrix is symmetric with respect to the main diagonal.

a	b	c	d	e	f	g	h	i	j
1	1	1	8	2	3	2	3	4	5
2	4	6	9	4	6	5	7	6	7
3	5	7	0	8	8	9	9	0	0

Table 1.4.1. A configuration table for a configuration discussed in Section 2.2, and denoted there by $(10_3)_4$. The notation is adapted from Schroeter [S8]. Two versions of the Levi incidence matrix of this configuration are shown in Figure 1.4.1, and its configuration graph in Figure 1.4.2. A realization with pseudolines appears in Figure 1.2.2.

	a	b	c	d	e	f	g	h	i	j
1	**X**	**X**	**X**							
2	**X**				**X**		**X**			
3	**X**					**X**		**X**		
4		**X**			**X**				**X**	
5		**X**					**X**			**X**
6			**X**			**X**			**X**	
7			**X**					**X**		**X**
8				**X**	**X**	**X**				
9				**X**			**X**	**X**		
0				**X**					**X**	**X**

(a)

	c	b	i	j	e	f	a	h	g	d
3						**X**	**X**	**X**		
2					**X**		**X**		**X**	
8					**X**	**X**				**X**
9								**X**	**X**	**X**
4		**X**	**X**		**X**					
6	**X**		**X**			**X**				
1	**X**	**X**					**X**			
7	**X**			**X**				**X**		
5		**X**		**X**					**X**	
0			**X**	**X**						**X**

(b)

Figure 1.4.1. Two versions of the Levi incidence matrix of the configuration table in Table 1.4.1. In (a) the matrix is formed in the obvious way, while the rearranged form in (b) illustrates the selfduality of the configuration. Part (b) is adapted from Laufer [L1].

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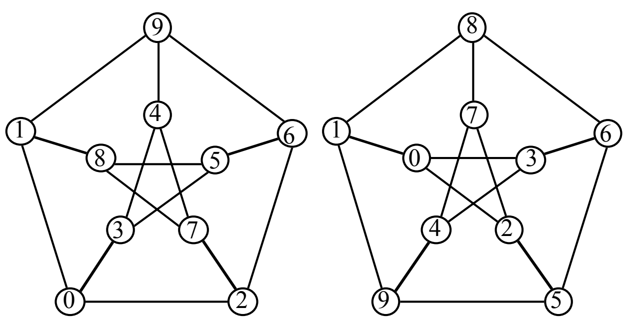
Several graphs have been attached to configurations; it is probably simplest to describe them in terms of geometric configurations, although the concepts apply to topological and combinatorial configurations (with appropriate changes in wording), and even to more general combinatorial systems.

The earliest of these graphs was proposed by K. Menger in a course on projective geometry at Notre Dame University in 1945. It seems that he never published on the topic; the first publication discussing it is a paper (doctoral thesis under Menger's supervision) of M. P. van Straten [V3] in 1949. The name "Menger graph" appears to have been introduced by Coxeter in [C5] and [C6]; it has been used in other works as well, for example in van Maldeghem [V2]. It will also appear later in this book, in Section 5.1. The **Menger graph** M(C) of a configuration C is the graph with vertices corresponding to those of C; an edge of M(C) connects two of its vertices if and only if the corresponding points of the configuration are collinear on a line of the configuration. There seem to have been only few uses of this kind of graph. One mention of the Menger graph of a configuration is in the paper by Di Paola and Gropp [D6], in connection with their definition of the "configuration graph" of a configuration. It will also appear later in this book, in Section 5.1. For a configuration C, the **configuration graph** is the graph with the same vertices as C, and with an edge connecting two vertices of the graph if and only if the corresponding points are not on any line of C; Gropp [G32] calls it the "Martinetti graph", Mendelsohn *et al.* [M5] call it the **deficiency graph**. We shall use the latter term. Obviously, in graph theoretic terminology the deficiency graph of a configuration is the complement of its Menger graph. As we shall see in Section 1.7 and in Chapter 2, constructs related to deficiency graphs have been used in some of the earliest papers on configurations, under the name "Restfigur" ("remainder figure") in enumerations of configurations (n_3) for small values of n. We shall enlarge on these remainder figures below. An interesting recent application is also given in Section 1.7.

The major shortcoming of both the Menger and the deficiency graphs is that they do not uniquely determine the configuration. This was noted already in [M5] and [D6]. The simplest example of distinct configurations having the same configuration graphs is that of the two (10_3) configurations which we shall denote (in Section 2.2) by $(10_3)_1$ and $(10_3)_4$; this is illustrated in Figure 1.4.2. In [M5, p. 96] a family of such pairs is indicated.

Of far greater importance is the third graph associated with a configuration, its "Levi graph". This concept was introduced by Levi in [L4], and the name was first used by Coxeter in [C6]; see also van Maldeghem [V2]. The **Levi graph** L(C) of a configuration C is the bipartite graph, with "black" points corresponding to the points of C and



(a) (b)

Figure 1.4.2. The configuration graph of the configuration $(10_3)_4$ specified by the configuration table given in Table 1.4.1 and by the Levi incidence matrices in Figure 1.4.1 is shown in (a). In (b) is the isomorphic configuration graph of the Desargues configuration, denoted by $(10_3)_1$ in Section 2.2, and using the label shown there.

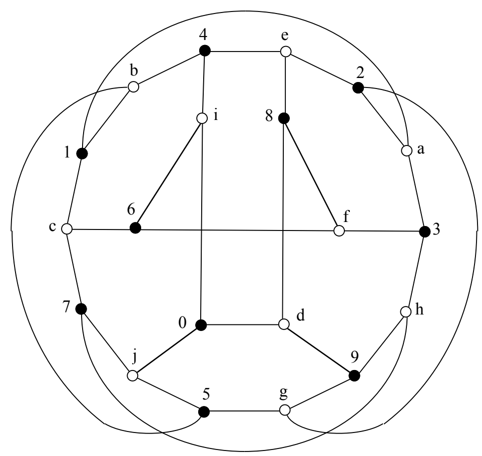


Figure 1.4.3. The Levi graph L(C) of the topological configuration C = $(10_3)_4$. The color-reversing mirror symmetry of the graph shows that the configuration $(10_3)_4$ is self-dual, under the correspondence implied by the symmetry.

"white" points to the lines of C; two points of the Levi graph determine an edge if and only if one represents a point and the other a line incident with that point. As an illustration we show in Figure 1.4.3 the Levi graph L(C) of the (combinatorial and topological) configuration $(10_3)_4$.

The importance of Levi graphs derives from their one-to-one correspondence with combinatorial configurations. More specifically, we have the following widely used result:

A bipartite graph G is the Levi graph L(C) of a [q,k]-configuration C if and only if:

- All black vertices are q-valent, all white vertices are k-valent;
- G has *girth* at least 6, that is, all circuits in G have length at least 6. In particular, G has no loops or digons.

The correspondence between G = L(C) and C is one-to-one, up to isomorphism in each class of objects.

Various graph-theoretic concepts can be transferred to configurations by the use of Levi graphs. For example, we say that a configuration is **connected** or **c-connected** for some integer c if and only if its Levi graph is connected resp. c-connected. Translated into the terminology of configurations, connectedness means that any two elements can be included in a multilateral path; a configuration C is c-connected if on selecting any c–1 elements of C, any pair of the remaining elements can be connected by a multilateral path that does not use any of the selected elements.

A not entirely trivial result, easily provable using Levi graphs, is that any k-connected configuration of is 2-connected. We shall see this in Section 5.1, together with a discussion of some related questions.

As mentioned earlier, a concept related to configuration graphs are the "remainder figures". We shall put it in a more general setting, and we define:

Definition 1.4.1. Given a configuration C and a point P of C, the **complementary graph of** C **at** P consists of the points of C that are not collinear with P on any line of C, and the segments connecting pairs of these points. The **complementary graph complex** of C is the family consisting of complementary graphs of C at all its points P. It is usually more convenient to take, instead of the family, the union of the members of the family.

1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	1
4	5	6	7	8	1	2	3

Table 1.4.2. The configuration table of a set configuration (8_3).

For example, for the configuration (8_3) given by the configuration table in Table 1.4.2, the complementary graph at each point is a singleton vertex, and the complementary graph complex consists of eight isolated vertices. Analogously, the complementary graph complex of the combinatorial configuration (14_4) shown in Table 1.4.3 consists of 14 isolated vertices. For each of the combinatorial configurations (9_3) (see Figure 1.1.5) and (15_4) (see section 3.1) the complementary graph complex consists of one or more circuits that cover all the vertices of these configurations; more about these cases in Sections 2.2 and 3.1. It should be noted that in general, the isomorphism of the complementary graph complexes of two configurations does not imply the isomorphism of the configurations; this is illustrated in Section 2.3.

A	A	A	A	B	B	B	C	C	C	D	D	D	E
B	F	G	H	G	H	E	H	E	F	E	F	G	F
C	L	N	P	L	M	P	L	M	N	L	Q	M	G
D	M	R	Q	Q	N	R	R	Q	P	N	R	P	H

Table 1.4.3. A configuration table (adapted from [M8]) of a combinatorial configuration (14_4).

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Each given geometric configuration C determines an **arrangement** A(C) of the Euclidean or projective plane. By this is meant the 2-complex consisting of the intersection points of the lines of C (**vertices** of A(C)), of the open segments (**edges** of A(C)) of each line constituting the complement of the points of the line, and the 2-dimensional open convex polygons (**cells** of A(C)) that constitute the connected components of the complement of the union of the lines. For example, in Figure 1.4.4 we show another drawing of the (10_3) configuration C from Figure 1.3.3, in which we made visible all the intersection points of its lines; the points that are not configuration points are shown by hollow circles. It is easy to count that, in the Euclidean plane, A(C) has 25 points, 70 edges (twenty unbounded), and 46 cells (20 unbounded). If considered in the projective plane (that is, the extended Euclidean plane), then it has 60 edges and 36 cells. In either case, one can apply the appropriate Euler relation, or other results that have been established for arrangements to deduce properties of configurations. The concept of arrangement associated with a configuration can be applied to topological configurations as well. This will be useful in Sections 2.1 and 3.2.

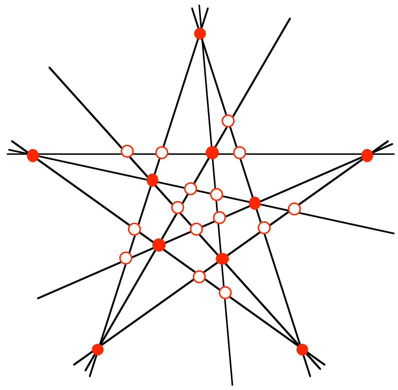

Figure 1.4.4. The arrangement associated with the (10_3) configuration from Figure 1.3.3.

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Other tools have been found very useful in some questions concerning configurations are of a more algebraic character. They have been explained and applied in several widely quoted and studied works, in particular those by Bokowski and Sturmfels [B25] and Bokowski [B21]. We shall not have any occasion to use them here, so we advise the reader interested in seeing how this approach works to consult with the literature.

Another topic we are not going into in this book are programs for computers and computer graphics. Many of the enumeration results (especially those of a purely combinatorial character) have been obtained by using computers. We shall mention a number of such cases, but do not find it appropriate to enter into details beyond giving credit and references to the original works.

Much of the work presented in this book could not have been done at all without the use of widely available software. My main tools were the various consecutive versions of Mathematica®, Geometer's Sketchpad®, and ClarisDraw®, as implemented on several generations of Apple® computers. Other computer-related programs and graphics were generated by various collaborators in the joint papers we shall mention in due course, and I sometimes adapted them to the formats used here.

Exercises and problems 1.4.

1. Find a "configuration" table for the prefiguration in Figure 1.3.3, and a Levi incidence matrix for it. Find a version of the incidence matrix that exhibits the selfduality of the prefiguration.

2. Consider the graph determined by the vertices and (finite) segments in Figure 1.4.4. Does it admit a Hamiltonian circuit?

3. For each mark of the set-configuration (14_4) in Table 1.4.3 find its complementary graph. Does the complementary graph complex determine any configuration?