1.3 BASIC CONCEPTS AND DEFINITIONS

In this section we shall clarify the fundamental concepts involved in the study of configurations. We shall start with very general definitions that will enable us to specialize and particularize the concepts as we find appropriate. Considerable care is required in distinguishing the various related concepts; neglect to do so led to many of the problems we mentioned in the historical account in Section 1.2. The reader who finds the plethora of words confusing, should skip the details and return to this section when the text uses terms that need explanation. This author's position in connection with the abundance of terms is that Johann Wolfgang Goethe had it all wrong when (in "Faust", lines 1995-1996) he has Mephistopheles say:

"Denn eben wo Begriffe fehlen,
Da stellt ein Wort zur rechten Zeit sich ein"[1]

On the contrary; words are needed (even if they have to be invented) when concepts appear that need to be distinguished from other concepts. So, here we go.

A *configuration* C is a family of "points" (sometimes called *vertices*) and a family of "lines" such that, for positive integers p, q, n, k each of the p "points" that constitute the family is "incident" with precisely q of the n "lines", while each of these "lines" is "incident" with precisely k of the "points". The use of quotation marks is meant to indicate that these objects can be of any nature whatsoever, as soon as the "incidence" relation satisfies what can be considered the natural conditions:

- It is a symmetric (that is, mutual) relation;
- An "incidence" can involve only a "point" and a "line", never two "points" or two "lines"; and
- Two "points" (or "lines") can be incident with at most one "line" ("point").

Moreover, it is assumed throughout the book that use of the word *configuration* implies that there is a "point" and a "line" that are not "incident" with each other. The totality of "points" and "lines" of a configuration will be called its **elements**.

[1] Freely translated: "Where concepts are missing, a word soon appears."

A configuration C with **parameters** p, q, n, k will in general be denoted by (p_q, n_k). The concept of (p_q, n_k) configurations and the notation for it were introduced by de Vries [D5] in 1888. If the particular values of p and n are not important, we shall say that we have a **[q,k]-configuration**. If q = k we shall simplify the notation by dispensing with the brackets and write **k-configuration**. In the terminology of a parallel universe, what we call a [q,k]-configuration is known as a "geometry of order (q-1, k-1)", see, for example, van Maldeghem [V2]. Note that a [q,k]-configuration is called a "slim geometry" in the terminology of that universe, and a 3-configuration is a "bislim geometry".

By counting the total number of incidences in a configuration (p_q, n_k), it follows that a necessary condition for the existence of a configuration with parameters p, q, n, k is the equation $pq = nk$.

There are additional necessary conditions. Each "point" of (p_q, n_k) is "incident" with q "lines", each of which is "incident" with k–1 other "points". Hence there are at least $p \geq q(k-1) + 1$ "points". A similar argument shows that $n \geq k(q-1) + 1$. To avoid trivialities we shall generally assume that $q \geq 2$ and $k \geq 2$, which implies that $p \geq 3$ and $n \geq 3$. (Exceptions will be signaled explicitly.) Although these necessary conditions are in many cases sufficient for the existence of some configuration (p_q, n_k), we shall see in the following sections that this is not always the case.

Much of the time we are interested in configurations (p_q, n_k) with p = n (and therefore q = k). As already mentioned, it is customary to simplify the notation for such configurations, and designate them as **(n_k) configurations**, or **k-configurations**. The (n_k) notation (or, more precisely, the very similar n_k notation) was introduced by Reye [R2] in 1882; it is convenient in many contexts, but in cases the number of points and lines is not known or is nor relevant, it seems illogical to insert the letter n that has no meaning in such a situation; hence the alternative "k-configuration" notation.

In the literature, k-configurations are often called "symmetric"; but this term is highly unsuitable since the configurations in question may fail to have any symmetry whatsoever, geometric or combinatorial. (The ill-advised use of "symmetric" in this context seems to go back to erroneous understandings in some late 19th century writings on configurations, and possibly to its use in the theory of BIBDs – balanced incomplete block designs; the latter are totally irrelevant to the theory of configurations. More about this in Section 1.5.) The main objection to the use of "symmetric" in this meaning is that, as we shall see throughout the text, configurations exhibiting certain genuine symmetries – combinatorial or geometric -- are very important. With much justification it may be claimed that configurations with *geometric symmetries* have been the motivating factor in the recent great expansion of knowledge about configurations. Hence bestowing the descriptor "symmetric" to configurations that may be totally devoid of symmetries is downright misleading. In the present book we shall say that k-configurations are **balanced**; it is hoped that this term will become the accepted designation for k-configurations. (The use of "balanced" in the BIBD theory is not compromised by this use for configurations, just as the use of "symmetric" for BIBDs raises no problems for configurations.)

We shall be concerned with configurations at three levels of generality. In the most restricted sense, “points” and “lines” are interpreted as being the points and lines in some space in which these concepts are defined, so as to satisfy the first two groups of Hilbert’s axioms. (For these axioms see, for example, Hilbert [H3], Noronha [N1], Sibley [S14], Stahl [S16].) In particular, this interpretation includes the traditional Euclidean plane and higher-dimensional Euclidean spaces, as well as the real projective plane and higher-dimensional projective spaces. Unless specifically stated otherwise, we shall consider configurations at this level of generality to be in the real Euclidean plane, or in the real projective plane, and call such configurations **geometric**. In Appendix A the necessary facts about the Euclidean and projective planes are collected. The usual way of presenting geometric configurations is by **diagrams**, in which “points” are represented by solid dots and “lines” by straight lines. Naturally, it must be true (and possible to verify) that the presumed lines are straight and that the incidences actually occur; some diagrams can be misleading, as illustrated by Figure 1.2.2. More details on this topic will be found in Section 2.2. For general discussions of the topic of drawing configurations see [G21], [G26].

In the most general sense we shall consider **combinatorial** (or **abstract**) configurations; we shall use the term **set-configurations** as well. In this setting "points" are interpreted as any symbols (usually letters or integers), and "lines" are families of such symbols; "incidence" means that "point" is an element of a "line". It follows that combinatorial configurations are special kinds of general incidence structures. Occasionally, in order to simplify and clarify the language, we shall for "points" use the name **marks**, and for "lines" we shall use **blocks**. The main property of geometric configurations that is preserved in the generalization to set-configurations (and that characterizes such configurations) is that two marks are incident with at most one block, and two blocks with at most one mark. The usual way of presenting set-configurations is by **configuration tables**. In such a table the marks of each block are listed in a column that represents the block. If necessary or convenient, a label for the block may be indicated at the head of each column, but in other cases this may not be needed. An example of a set-configuration $(16_3, 12_4)$ is shown in Table 1.3.1 (with block labels); in Table 1.3.3 we dispensed with the block labels. Sometimes a combinatorial configuration (p_q, n_k) is presented simply as a family of n k-tuples formed from p marks, with appropriate restrictions. In [G11] and other papers, H. Gropp faults Hilbert and Cohn-Vossen [H4] and Coxeter [C6] for being interested in structures "realized in real geometry" and not in the "more modern 'schematical' configurations". This seems to the present author to be quite inappropriate, especially since in most of his papers Gropp does not warn the reader that his use of "point" and "line" is meant in purely combinatorial sense.

a	b	c	d	e	f	g	h	i	j	k	l
A	A	A	H	B	B	B	I	C	C	J	D
H	N	G	N	H	N	I	P	O	J	O	K
I	O	L	P	G	M	J	M	M	K	P	L
C	D	E	K	F	E	D	L	F	E	G	F

Table 1.3.1. A configuration table for a set-configuration $(16_3, 12_4)$.

It is immediate from the definitions that every geometric configuration gives rise to a set-configuration: Just label the points of the geometric configuration, and use these labels as marks to construct a configuration table. As can be checked very easily, the set-configuration in Table 1.3.1 corresponds in this way to the $(16_3, 12_4)$ geometric configuration in Figure 1.3.1. However, as we shall illustrate in Section 2.1, the converse is not valid: there are set-configurations that do not correspond to any geometric configuration.

The relationship between the configurations in Table 1.3.1 and Figure 1.3.1 is sometimes formulated by saying that the former **underlies** the latter, and that Figure 1.3.1 is a geometric **realization** of the set-configuration in Table 1.3.1.

A remarkable result goes back to Steinitz [S17], in the classical period of the theory of configurations. In one formulation, it states that every combinatorial k-configuration can be presented in an **orderly** configuration table. A configuration table is orderly if each point (mark) appears in each row of the table once and only once. We shall discuss this result and some of its ramifications in Section 2.5.

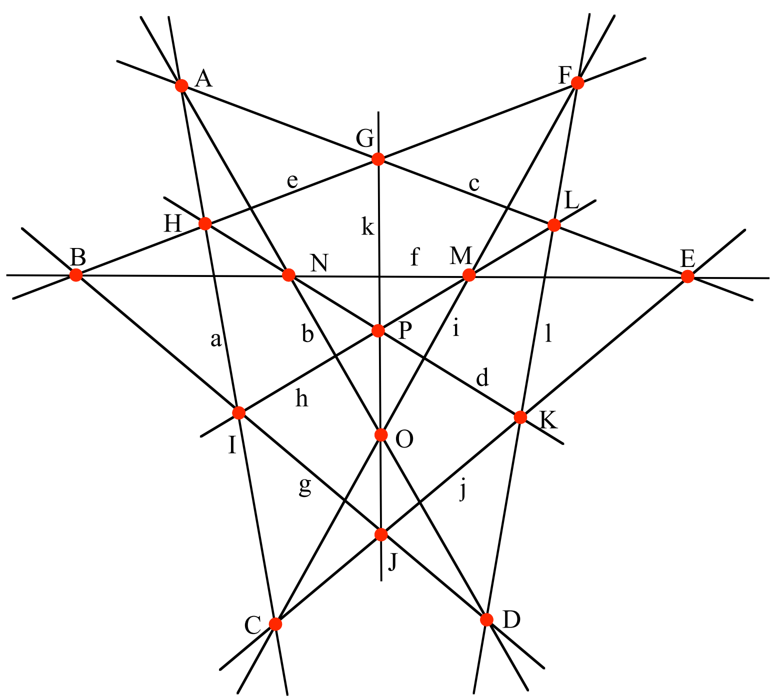

Figure 1.3.1. A geometric configuration $(16_3, 12_4)$, with points labeled in such a way as to yield the configuration table in Table 1.3.1.

If the points and lines of two configurations C' and C" admit labels such that a 1-to-1 correspondence τ of points to points (and lines to lines) preserves incidences, we shall say that C' and C" are **isomorphic** (or **combinatorially equivalent**, or of the same **combinatorial type**); sometime we shall also wish to make explicit the correspondence τ. With appropriate interpretation, this terminology applies to set-configurations as well. For example, the set-configuration in Table 1.3.1 is isomorphic to the geometric configuration in Figure 1.3.1 with τ the identity transformation ι of the labels. If an incidence-preserving correspondence maps points to lines (and vice versa), it is said to be a **duality**, and the configurations are said to be **dual** to each other.[2] A configuration table of the configuration dual to the one in Table 1.3.1 is shown in Table 1.3.2 and in Figure 1.3.2 is a geometric configuration dual to the one in Figure 1.3.1.

Here again a warning seems necessary. Some authors (for example, van Maldeghem [V2]) use the word "realization" in a different meaning. In particular, they allow (geometric) realizations of set-configurations to have additional incidences, not among the ones in the underlying set-configurations. Put differently, they use "realization" for a different concept, which we shall encounter below under the designation "representation". Thus, for us, a geometric realization of a set-configuration is isomorphic to it, while a representation that is not a realization is not isomorphic to it.

Some configurations are isomorphic to configurations dual to them. In such cases we call the configuration **selfdual**. In other words, a configuration C is selfdual if there is an incidence-preserving correspondence τ that maps the points of C onto its lines and vice versa; such correspondence τ is called a **selfduality**, and it is obvious that in this case the inverse τ^{-1} is a selfduality as well. In many cases (but not always) τ^2 is the identity map ι. An example of a selfdual configuration with $\tau^2 = \iota$ is shown in Figure 1.3.3. We shall discuss this topic in much more detail in Sections 2.10 and 5.8.

---

[2] It is unfortunate that Coxeter [C6] uses "dual configurations" to mean any pair consisting of a (p_q,n_k) configuration and a (n_k,p_q) configuration; in this terminology any (n_k) configuration is selfdual. This error has been copied by Evans [E1*].

There is often a need to consider families of points and lines (or marks and blocks) that fail — by a "few" incidences — to be configurations in the sense we use here. We say that such a family is a **prefiguration**. We shall often encounter two types of prefigurations, although other types occur at times.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
a	e	a	b	c	e	c	a	a	g	d	c	f	b	b	d
b	f	i	g	f	i	e	d	g	j	j	h	h	d	i	h
c	g	j	l	j	l	k	e	h	k	l	l	i	f	k	k

Table 1.3.2. A configuration table of the dual of the configuration in Table 1.3.1 and in Figure 1.3.1.

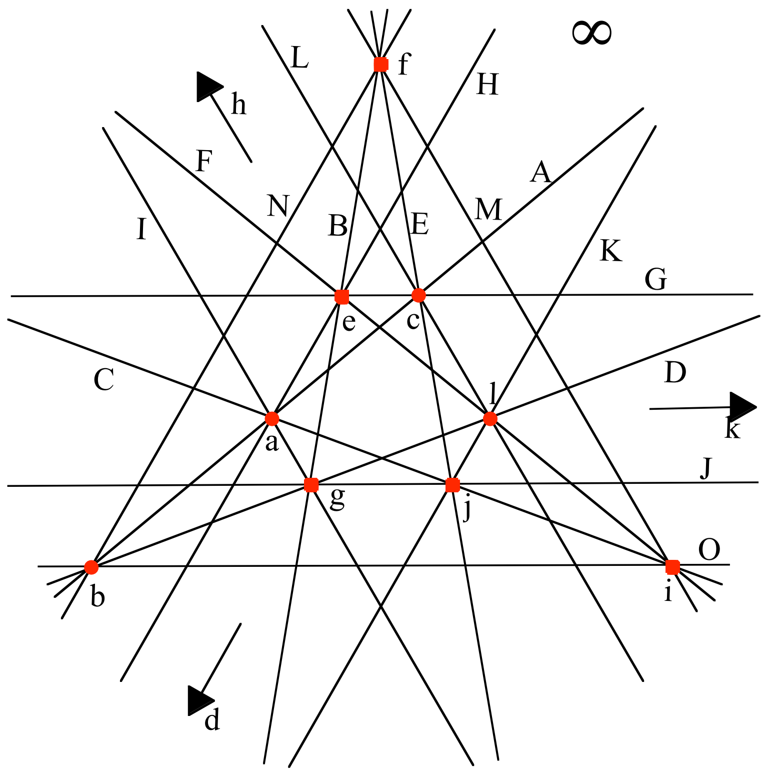

Figure 1.3.2. A geometric configuration that is a realization of the set-configuration in Table 1.3.2. It is dual to the configuration in Figure 1.3.1. The arrows indicate “points-at-infinity”, and the “line-at-infinity”, indicated by the infinity symbol, is also part of the configuration. Explanations of this kind of diagrams in the projective (or extended Euclidean) plane are given in Appendix A.

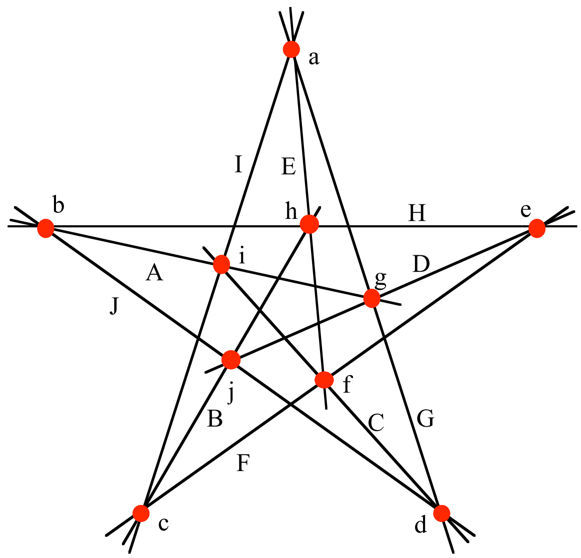

Figure 1.3.3. An example of a selfdual configuration (10_3). The selfduality mapping τ interchanges the upper- and lower-case letters. Hence clearly $\tau^2 = \iota$, the identity.

A **superfiguration** (p_q, n_k) is a family of "points" and "lines" with "incidences" as in the definition of configurations, such that each of the p "points" is "incident" with **at least** q of the n "lines", and each "line" is "incident" with **at least** k of the "points". If the number of incidences exceeding that in a [q,k]-configuration is s, we shall sometime say that it is an **#s-superfiguration**. An example of a remarkable (and quite useful – see Section 2.11) superfiguration is shown in Figure 1.3.4; it is a #2-superfiguration of a 3-configuration, since there is one line incident with four points and one point incident with four lines. (This is taken from [G36]; it also appears in [B25] and [M18], and probably in other places as well.) Superfigurations often arise through "accidental" incidences in configurations that have realizations depending on variable parameters.

In another way of looking at this situation is that sometimes it is necessary or convenient to consider certain superfigurations as *representing* combinatorial *configurations.* As already mentioned, by a **representation** we mean a family of points and lines such that all the combinatorial incidences are satisfied but some points may be on lines with which they are not incident in the combinatorial configuration, or some pairs of distinct points (or lines) of the combinatorial configuration may be represented by single points (or lines). A typical example of such a superfiguration is shown in Figure 1.3.5, which is a *representation* of the set-configuration specified in Table 1.3.3. The point 1 in Figure 1.3.5 lies on the line 089 but is not incident with it according to Table 1.3.3.

1	1	1	2	2	3	3	0	0	0
2	4	6	4	7	5	6	4	5	8
3	5	7	8	9	9	8	6	7	9

Table 1.3.3. A configuration table of a set-configuration (10_3). We shall encounter this configuration again in Section 2.2.

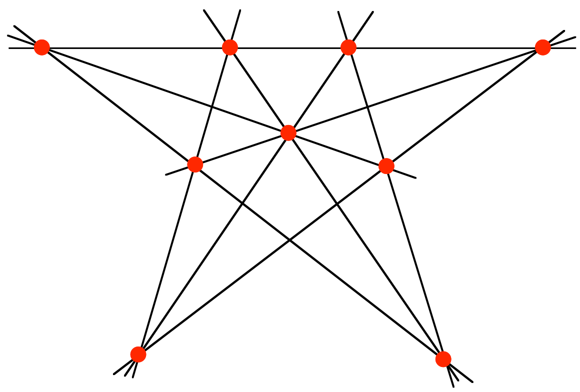

Figure 1.3.4. An example of a geometric #2-superfiguration (9_3), in which one point is incident with four lines, and one line is incident with four points.

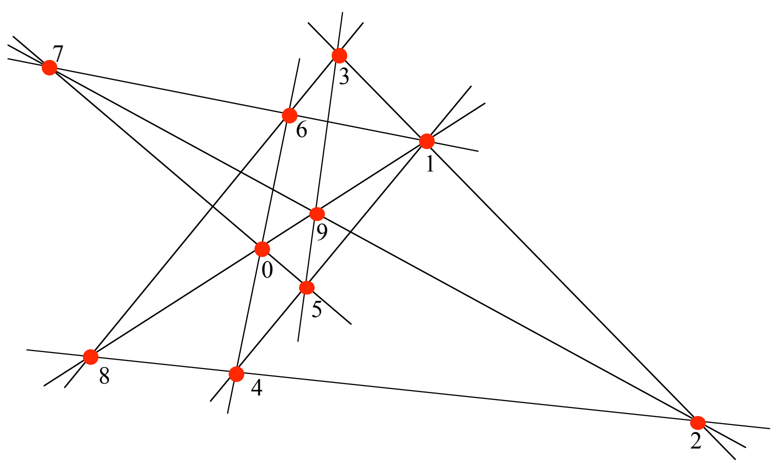

Figure 1.3.5. A #2-superfiguration that is a representation (but not a realization) of the set-configuration specified by the configuration table in Table 1.3.3.

In the same vein we call **subfiguration** (or #**s-subfiguration**) a family that is short by a number of incidences (specifically, s incidences) to be a [q,k]-configuration. An illustration of this concept is provided in Figure 1.3.6. Other examples will become very important in Section 2.5.

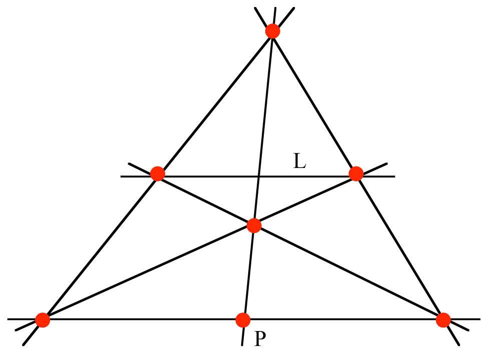

Figure 1.3.6. A subfiguration of a (7_3) configuration. If the line L were incident with the point P, this would be a configuration. However, as we shall see in Section 2.1, there exists no geometric configuration (7_3), although a set-configuration of this type is possible.

* * * * *

Intermediate in generality between set-configurations and geometric configurations are **topological configurations**, also known as **configurations of pseudolines**. A family of simple curves in the (projective or Euclidean) plane is a family of **pseudolines** provided each curve differs from a straight line in at most one segment of the Euclidean (or projective) line, and any two curves have at most one point in common, at which they cross each other. We shall discuss pseudolines in more detail in several of the following sections. The definition implies that any two points are incident with at most one pseudoline.

Configurations of pseudolines are defined in complete analogy to configurations of lines, with pseudolines taking the place of "lines". As an example of a topological configuration we may take Kantor's presumed configuration shown in Figure 1.2.2, which – as we mentioned in Section 1.2 and will prove in Section 2.2 — is not realizable as a configuration of lines. Other examples are shown in Figure 1.3.7. Clearly, any geometric configuration can be understood as a topological configuration, and each of the latter has an underlying set-configuration. It is easily seen that each topological configuration is isomorphic to a configuration in which each pseudoline consists of a finite number of (straight) segments (including rays, considered as segments). For a detailed discussion of the various concepts of pseudolines that appear in the literature see [E4]. Several problems of combinatorial geometry that involve pseudolines are discussed in [G37].

It is clear that if a set-configuration is geometrically *representable* by a superfiguration C, it is also *realizable* by a topological configuration. The pseudolines may be "bent" to avoid offending incidences. However, not every superfiguration can be realized by a topological configuration. For example, the superfiguration in Figure 1.3.4 is not a representation of any set-configuration, and cannot be realized by a topological configuration.

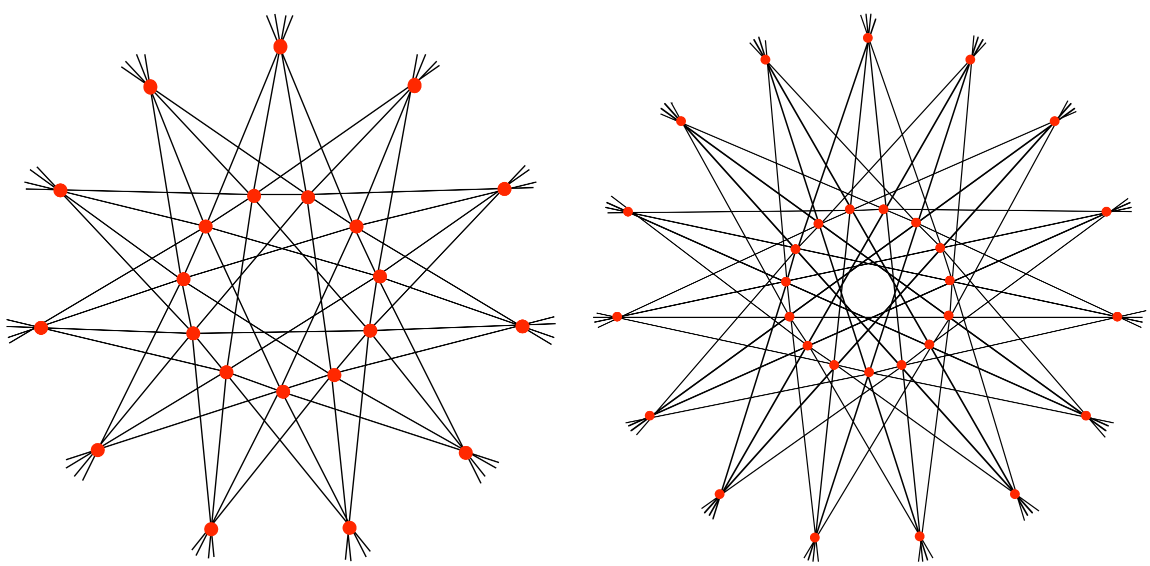

Figure 1.3.7. A (22_4) and a (30_4) configurations of pseudolines. Note how small is the departure from straight lines in these examples. It may be conjectured that these configurations are not isomorphic to any geometric configuration; however, this has not been established.

* * * * *

The terminology of the theory of configurations is very unsettled. Different writers, and "schools" use terms that are often quite unrelated to each other, and sometimes even carry a different meaning while using the same words. An example of the former is the use of terms like *slim* and *bislim geometries* to indicate what we call configurations.

On the other hand, some writers discuss at length "configurations" without bothering to mention that they have **combinatorial** – and *not* geometric – configurations in mind. This was to some extent excusable in the nineteenth century, when the essential difference between the concepts had not yet been recognized. A hundred year later this is exemplary carelessness, or at the least, total disregard for traditional terminology and the work of earlier authors. Still other writers state that configurations are "partial linear spaces with constant and equal point rank and line rank"; despite the use of explicit geometric terms (including "n-gons") the configurations they consider are only combinatorial (see, for example, [P2], [K1]).

The internet also abounds with vague, misleading, or wrong entries. This applies, in particular, to the frequently consulted Wikipedia (see [W3], as modified 9 November 2007) and Mathworld (see [W1], quoted from version dated November 30, 2007). In the latter one find, among other inaccurate assertions, that the (7_3) and (8_3) configurations are realizable with a "point at infinity".

* * * * *

It is easy to see that every realization of a set-configuration by points and lines in any Euclidean or projective space (of any dimension) can lead by suitable projection to a geometric configuration in the Euclidean plane. However, the converse question — can a given geometric configuration be realized in a higher-dimensional space in such a way that it is not contained in a subspace — has a negative answer in some cases. For example, each of the three configurations (9_3) in Figure 1.1.6 is easily seen to be contained in the plane spanned by some three of its points. With this example in view, it is meaningful to define the **dimension** of a configuration as the largest dimension of a space that is spanned by the configuration. We shall discuss this topic in Section 5.6.

In many questions about configurations one is concerned with what in the literature is often called "polygons". However, this is a misnomer since in most cases it is not segments that are relevant as "sides" of the "polygons"; instead, the intention is to deal with the lines of the configuration. We shall call **multilateral** any sequence of points and lines of a configuration that can be written as $P_0, L_0, P_1, L_1, \dots , P_{r-1}, L_{r-1}, P_r = P_0$, with each L_i incident with P_i and P_{i+1} (all subscripts understood mod r). For example, in the prefiguration shown in Figure 1.3.5, the sequence of points 1, 2, 8, 0, 4, 5, 7, 9, 3, 6, 1 (and the lines determined by adjacent pairs) determines a multilateral that involves all points and all lines. A **multilateral path** satisfies the same conditions except the coincidence of the first and last elements. A **Hamiltonian multilateral** passes through all points and uses all lines, each precisely once; hence the example just given is a Hamiltonian multilateral. We shall encounter multilaterals in several sections, and in particular Section 5.2 is devoted to Hamiltonian multilaterals.

Exercises and problem 1.3.

1. Show that the superfiguration in Figure 1.3.4 is selfdual. Find a selfduality map τ such that $\tau^2 = \iota$.

2. Decide whether any of the (12_3) configurations in Figure 1.3.8 are isomorphic. (As a practical matter, to show that two configurations are isomorphic it is sufficient to find an isomorphism. To show that they are not isomorphic, it is often simplest to find a property that is invariant under all isomorphisms but regarding which the two configurations behave differently. Neither is all that simple to actually carry out, even for rather small configurations.)

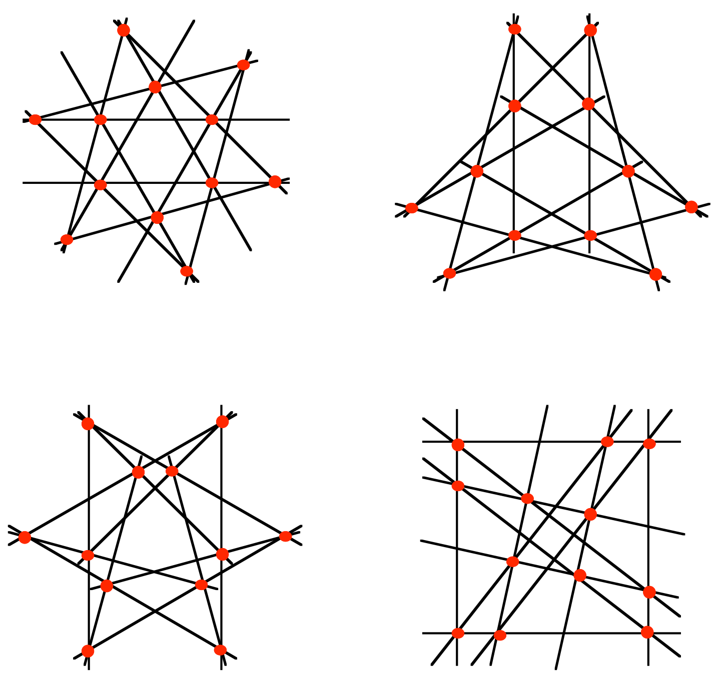

Figure 1.3.8. Four configurations (12_3). Are any isomorphic?

3. What is the smallest n required for the existence of a combinatorial configuration (n_4) ? Can you find a configuration table for it? Can you decide whether it is unique (up to isomorphism) ?

4. Do the topological configurations in Figure 1.3.7 admit Hamiltonian multilaterals? What about the configurations in Figure 1.3.8?

5. Determine the dimension of each of the configurations in Figure 1.3.8, and of the configuration in Figure 1.3.3.