1.2 AN INFORMAL HISTORY OF CONFIGURATIONS

Somewhat parallel to the history of Western civilization, the development of the theory of configurations can be assigned to distinct periods. However, in the case of configurations it is possible to assign precise dates to each of these periods.

We begin with the **prehistory**. By this I mean the relevant results developed by various mathematicians prior to the year 1876, at which time the concept of configurations was first formulated by T. Reye [R1]; see also [R2], where the notation n_k was introduced[1]. The results in question were formulated as theorems pertaining to certain sets of points and lines; in retrospect we can see that the results can be interpreted as configurations, or as implying the existence of certain configurations. Typical for this "prehistoric" results is the theorem of Pappus (see, for example [C7, p. 67], [C12, p. 231], which is usually formulated as follows (see Figure 1.2.1):

> If alternate vertices of a hexagon [2,6,8,3,5,9] lie on two lines, then the three intersections 1, 4, 7 of opposite sides of the hexagon are collinear.

It should be noted that this formulation requires additional explanation and modifications in some special cases, but this is not really relevant to the present discussion.

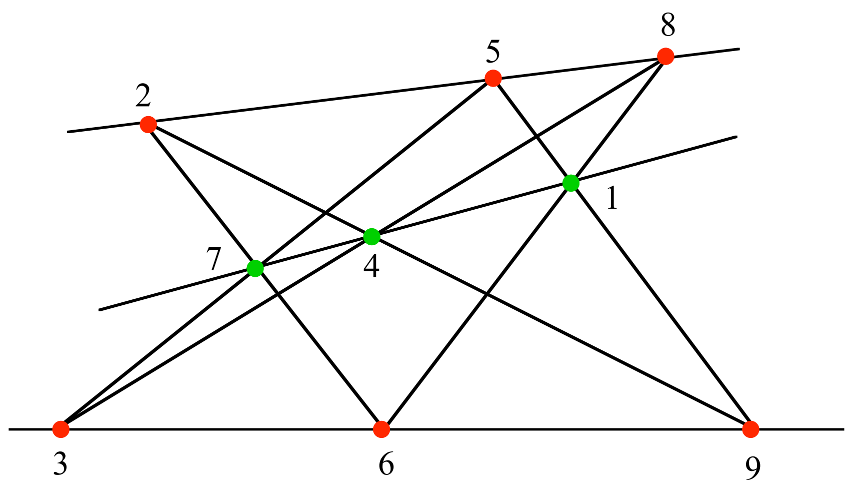

Figure 1.2.1. An illustration of the theorem of Pappus. Solid red dots are the two triplets of collinear vertices of a hexagon, solid green dots are the collinear intersection points of the three pairs of opposite sides of the hexagon.

---

[1] This notation, or the slight modification (n_k), are in general use. Since in many cases n is either irrelevant or not known, we shall also use "k-configuration" in such instances.

In our interpretation, Pappus' theorem amounts to the assertion that the incidences indicated in Figures 1.1.4 and 1.1.5 are correct. We used there the same notation as in Figure 1.2.1.

Other results from the prehistoric period can be found in works of Desargues, Steiner, Moebius, Cayley, and others. We shall mention them in due course.

*   *   *   *   *

Next comes the **classical period**, which runs from 1876 to 1910. It starts with the publication of Reye's book [R1], and ends with the publication of the survey of configurations by Ernst Steinitz [S19] in the Encyklopädie der mathematischen Wissenschaften. The period covers the formulation of the configuration concept as well as a number of basic results, in particular those related to questions (A) and (B) in Section 1.1. This includes works by such mathematicians as Reye, Kantor, Martinetti, Schroeter, Schönflies, Brunel, Burnside, Daublebsky, Steinitz and others. For example, Kantor [K3], [K4] gives answers to question (B) for n = 8, 9, 10. Unfortunately, his results are not correct as claimed. One of his errors concerns the enumeration of the combinatorial types of configurations (10_3). Kantor claims that there are precisely ten different types of such configurations, and presents diagrams purporting to illustrate these types. In Figure 1.2.2 we reproduce Kantor's drawing of one of these configurations. However, in contrast to the situation with the Pappus configuration explained above, a configuration of this type **cannot** be drawn with **straight** lines. The reviews [R6] of [K4] by C. Rodenberg and [S10] of [K3] and [K4] by H. Schubert contain several incorrect assertions. The impossibility was first proved by Schroeter [S8], and by other methods more recently by Carver [C2], Laufer [L1], Bokowski-Sturmfels [B25], Glynn [G2], and Sternfeld et al. [S22]. Claims by Dolgachev [D8] that this result is due to Kantor [K3] (sic) and that "a modern proof" is given in [B15] are both incorrect. We shall return to this topic in Section 2.2.

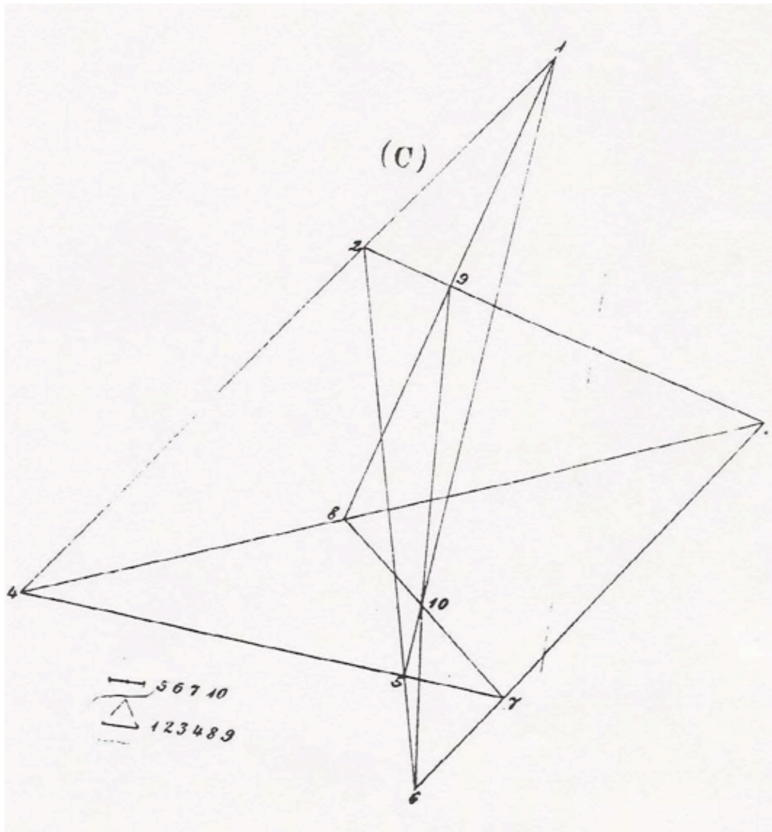

Figure 1.2.2. A diagram from Kantor [K4] which was supposed to illustrate a configuration (10_3), but which cannot be drawn by straight lines (as we shall see in Section 2.2).

While Kantor's error was discovered shortly after its publication, other deficiencies of his paper did not receive attention till very recently (see, for example, Section 2.3). The same late discovery of errors occurred with certain works of Martinetti and Steinitz, which were considered as basic for the theory of configurations throughout the twentieth century; however, the "Ernst Steinitz" article in the *MacTutor History of Mathematics* [O1] ignores all of Steinitz' work on configurations! We shall discuss Martinetti's and Steinitz's claims, and their corrected versions, in Chapter 2.

It is not clear how the various errors arose, and it is even more mysterious why they remained hidden for close to a century. A possible answer to the latter question is that for a long time apparently nobody cared enough about the topic to immerse oneself in the long and murky expositions of the original papers.

Other investigations during this period dealt with specific types of configurations, such as "mutually inscribed and circumscribed polygons". As relating to k-configurations, with the single exception of a paper by Brunel [B31], the considerations were limited to k = 3. We shall consider these results, as well as some more recent one on these topics, in many of the following pages.

* * * * *

The classical period, which was characterized more by enthusiasm for configurations than by solid mathematical achievement, was followed by long lasting **dark ages** of configuration theory. While configurations did not disappear completely from the mathematical horizon, in the period from 1910 till 1990 there were few significant publications on this topic. It may well have been no fault of configurations that the creative attention of mathematicians was directed to other fields. Rather it is the excitement caused by spectacular development in other mathematical disciplines that attracted most researchers; it led to general neglect of the intuitively accessible parts of geometry. The interest shifted to more technically intricate fields, leaving configurations dead in the water.

A few exceptions from this gloomy picture deserve to be mentioned. The first is the only book-length serious publication on configurations, by Levi [L3]. Unfortunately for the theory of configurations, the book had practically no influence on later works on configurations. Possible explanations for this lack of effect may be the very restricted types of questions considered (fifty pages are devoted to the consideration of the lines associated with the theorem of Pascal – a topic that was very popular in the "prehistory", but has practically no relevance to more modern investigations of configurations), together with the dry and pedestrian tone of exposition, the pedantic discussion of topics not connected to configurations (such as the long chapter on regular polyhedra), and the almost complete absence of references to previous work on configurations. Naturally, the fact that the center of gravity of mathematicians' interests shifted to other fields must also be considered as relevant to the book's lack of influence. We shall have occasion to mention Levi's book several times in connection with some of the original results contained in it – but this is meager pickings for a book of more than 300 pages.

Three years after the publication of Levi's work an extremely well received book was published by Hilbert and Cohn-Vossen [H4] (see, for example, [F2]). Twenty years later an English translation was published, but unfortunately the editors of Mathematical Reviews did not feel it deserves more than a listing; a second German edition in 1996 did not get a even a listing. This is a grievous mistake, since many later workers (the present author included) became interested in configurations by reading the account in [H4]. This presentation of the basics of configuration theory is contained in just a part of one chapter of [H4], but presents an attractive approach to the topic. It has been often mentioned as a justification for studying configurations, by quoting the following sentence from [H4, English translation p. 95]:

> "... there was a time when the study of configurations was considered the most important branch of geometry."

I would like to conjecture that this is the greatest exaggeration of the truth that can be found in any of Hilbert's writings. While it is a fact that — as mentioned above — in the "classical period" of the history of configurations there were quite a few people interested in the topic, configurations were never a central topic of mathematical (or geometric) research.

Even so, the influence of these two books can on occasion still be discerned today. For example, the recent work [P2] by K. Petelczyc mentions only these two sources for its information about configurations — ignoring all earlier and later publications.

Another relevant publication is the book by H. L. Dorwart [D9].

Other points of light during the "dark ages" of the configurations were several papers by Coxeter. Two of his early contribution to the topic are [C5] and [C6]. The latter – reproduced in [C11] – introduced several new ideas and popularized some older ones; we shall mention it frequently in various section of this book. Coxeter's other contributions to configurations are his papers [C8], [C9] and [C10], in which he presents detailed studies of certain specific configurations.

Some other papers on configurations that were published during the "dark ages" will find mention in appropriate places. They did not amount to much, and some of them were wrong.

* * * * *

After the long "dark ages" of the theory of configurations came a **renaissance** that continues to this day. It amounted to posing questions different from the ones considered previously, as well as to application of new tools and methods. For example, the investigation of Euclidean symmetries of a configuration was found to lead to more meaningful presentations of various configurations, and also to lead to the construction of previously unimagined ones. This is evident in the illustrations of the 4-configurations in Figure 1.1.2, and will become a leading topic in several later sections. Another difference concerns the more careful attention given to questions of graphic presentation of configurations, the use of new computational methods and computer graphics in the study of configurations, and – last but not least – to precise definitions, formulation, and proofs of results.

Rather immodestly, I believe that I played a significant role in the "renaissance" of the theory of configurations. Influenced by the chapter on configurations in the Hilbert and Cohn-Vossen's book [H4], in the mid-1980's I started studying configurations and presenting some of the results and problems in various seminars at the University of Washington, and in some courses I was giving; for example, [G36], which is mentioned in [B25]. The first formal publications resulting from these actions were the publication of the Grünbaum and Rigby paper [G50] in 1990, and the Sturmfels and White paper

[S24] in the same year. The former contains the first published images of any (n_4) configurations, while the latter answers (affirmatively for $n \leq 12$) the question I posed earlier (in [G36]) whether every configuration (n_3) can be realized in the *rational* plane. These and other publications that followed led soon to a revival of interest in configurations, with significant advances in many directions. These will be discussed in Chapters 2 and 3, and seem to justify the use of the term "renaissance".

Readers interested in the history of configurations and the individuals involved in its creation may wish to consult the various papers of H. Gropp ([G10], [G15], [G26], [G28], [G27], [G29], [G30]). However, it should be borne in mind that Gropp's attitude towards geometric configurations may be inferred from his statement in [G30]: "From the point of view of pure combinatorics the problems of realizing and drawing configurations may be artificial."

* * * * *

History does not stop today. There is no doubt that the theory of configurations, and related topics, will continue to be studied and to evolve. It is not possible to predict what direction the future investigations will take, but it is possible to hope that deeper connections will be established with algebraic geometry — a rather natural home to configurations — and to topology.

Exercises and problems 1.2

1. Define what you understand by say that "two configurations are essentially the same". Use your definition to decide whether the configurations in Figure 1.2.3 fit your definition of "sameness".

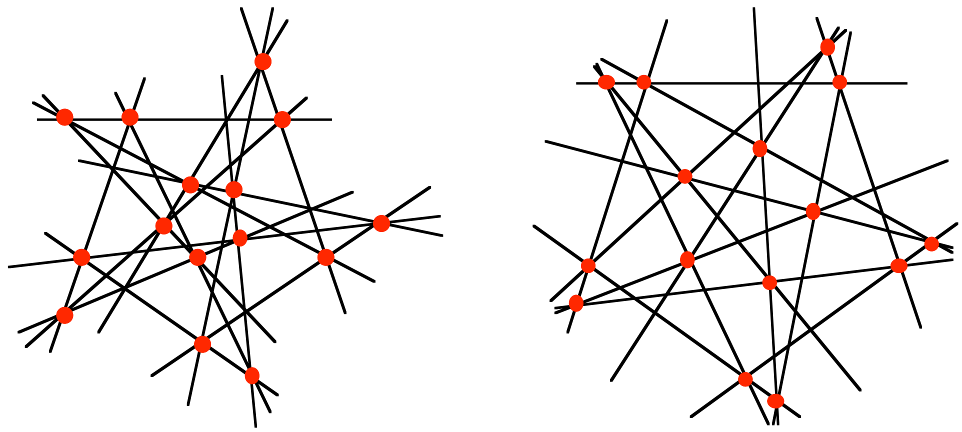

Figure 1.2.3. Two configurations (15_3). Are they "essentially the same" according to your definition?

2. Can you find a configuration that is "essentially the same" according to your definition as the ones in Figure 1.2.3 but has 3-fold rotational symmetry?

3. Find (in a book, or Google, or ...?) a formulation of the Desargues' theorem, that leads to the Desargues configuration in Figure 1.1.1.