CHAPTER 1. BEGINNINGS

1.1 INTRODUCTION.

The word "configuration" has many meanings in both colloquial and technical use. In the present work, however, it will be used in one meaning only, although with several nuances which will be explained soon. By a *k-configuration*, specifically an (n_k) configuration, we shall always mean a set of n *points* and n *lines*, such that every point lies on precisely k of these lines, and every line contains precisely k of the points. The variants of the meaning will concern the interpretation of "point" and "line", with additional distinctions regarding the space in which the points and lines are taken. However, in this Introduction it is simplest to interpret the words at just their most naïve meaning – – points and lines in the Euclidean plane. It is probably surprising that even with this simple interpretation there is sufficient material to consider writing a book about, and that there are many problems that are easily stated but are still unsolved.

For a quick orientation (see Figure 1.1.1), here are three examples of well-known

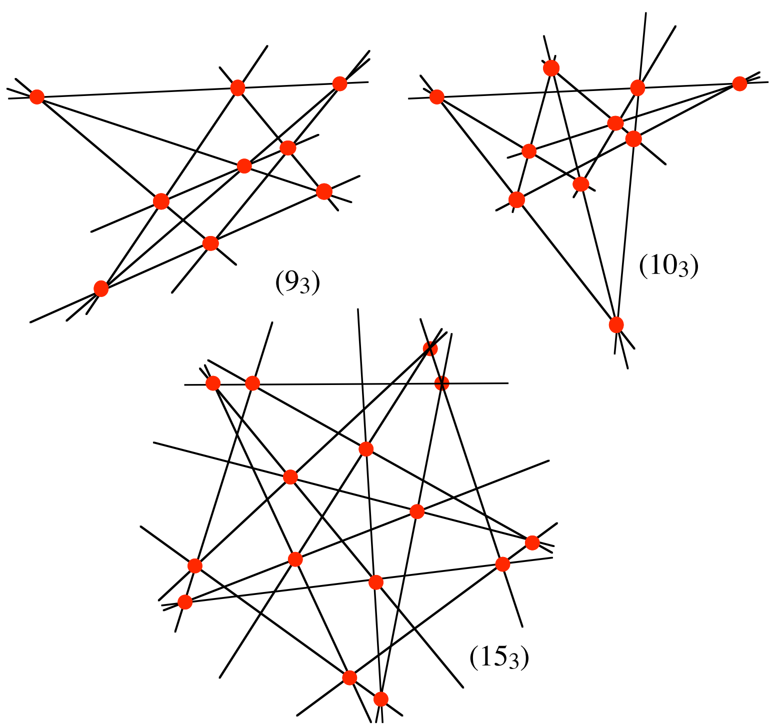

Figure 1.1.1. The 3-configurations of Pappus (9_3), Desargues (10_3), and Cremona-Richmond (15_3).

configurations (n_3), about which much has been written and which are known by the names of specific mathematicians. Each will appear several times in our discussions. Much less known are 4-configurations; three examples are shown in Figure 1.1.2.

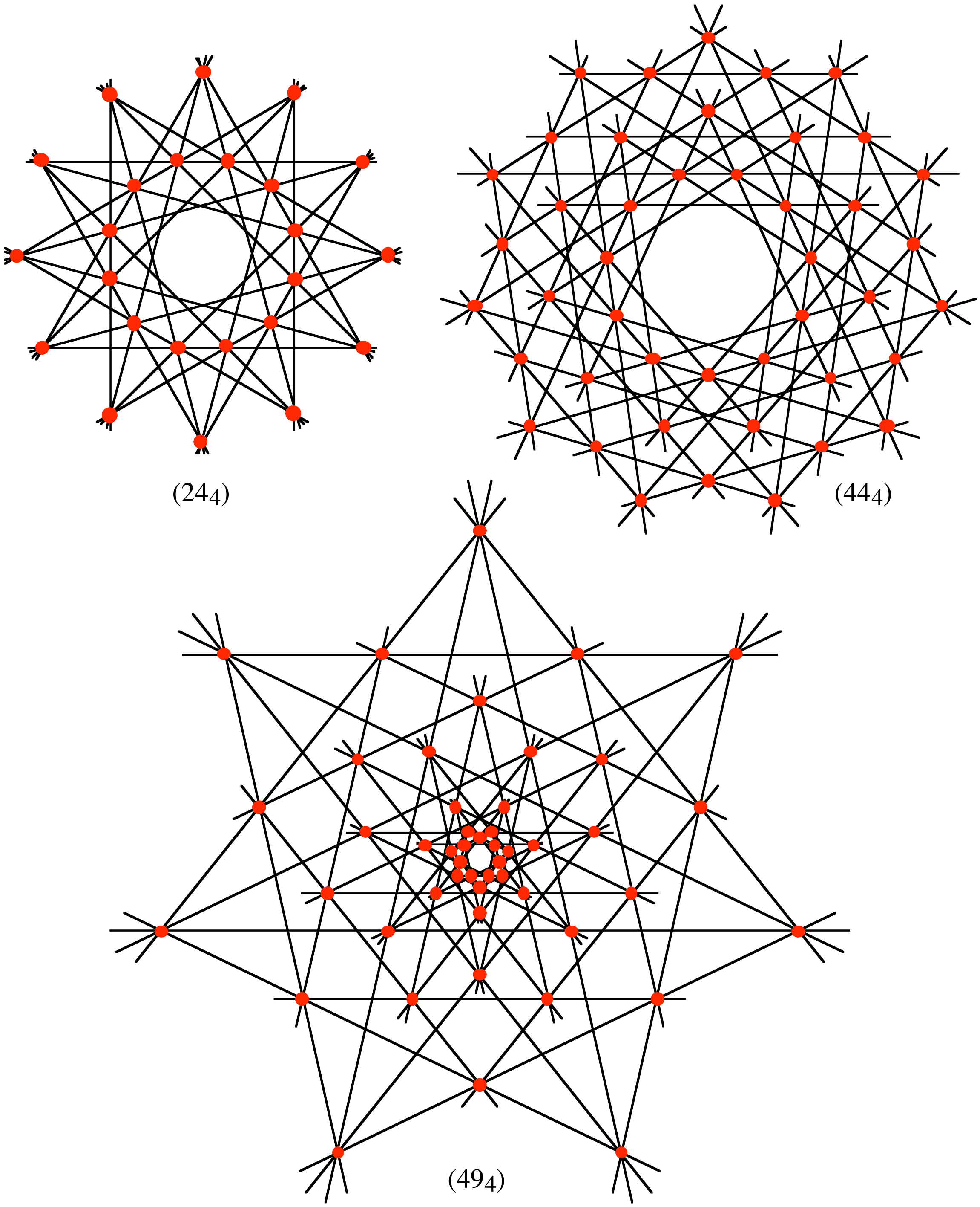

Figure 1.1.2. Three examples of 4-configurations.

A configuration (50_5) is illustrated in Figure 1.1.3.

It is both clear and natural that, with increasing k, the images of k-configurations become more complicated. In fact, the smallest n for which a configuration (n_6) is known to exist has a value of n = 110. (This topic will be discussed in detail in Chapter 4.) One concern that can be answered easily is whether for an arbitrary integer k there exists a configuration (n_k). Indeed, taking in the k-dimensional Euclidean space a "box" consisting of $n = k^k$ points of the integer lattice, together with the $n = k^k$ lines through them that are parallel to the coordinate axes, we see that there exists a configuration (n_k) with points and lines in the k-dimensional space. But then an appropriate projection onto a suitable plane yields the required configuration in the plane. We shall have repeated use of this configuration, hence we give it a special symbol LC(k). In [P5], T. Pisanski calls these the "generalized Gray configurations". The drawback of this construction is, obviously, that already for k = 7 this yields $n = 7^7 = 823{,}543$ — a rather unwieldy number. One may expect that with some ingenuity this number can be reduced, just as the corresponding $6^6 = 46{,}656$ has been reduced to 130. A different construction of some k-configurations with arbitrary k was proposed by Kantor [K2].

Some very specific question that can be asked for any k, but which we shall here illustrate for k = 3 only, are:

(A) For which n do configurations (n_3) exist?

(B) For each n such that configurations (n_3) exist, determine all distinct ones.

(C) How can given geometric configurations be represented symbolically? Given a type of symbolical representation, how can one decide whether it corresponds to a geometric configuration, and if it does, how can one draw it?

As we shall see in Section 2.1, question (A) has a simple answer: Configurations (n_3) exist if and only if $n \geq 9$. However, the question becomes much harder for configurations (n_k) with $k > 3$; in that form it was first posed by Reye [R2] in 1882, and is listed as Problem 12 in Section 7.2 of [B28]. We shall consider these cases in Chapters 3 and 4.

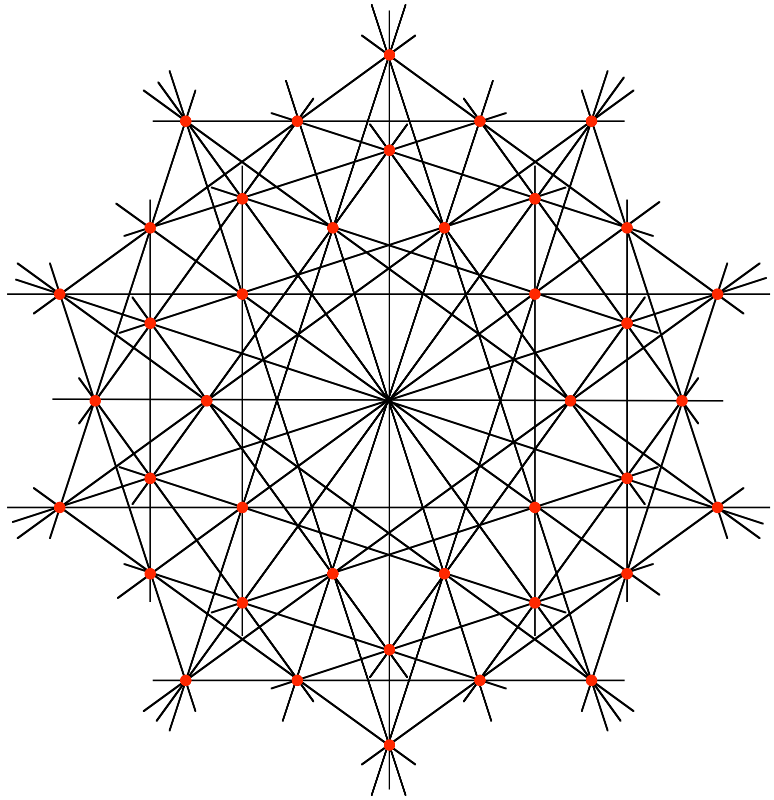

Figure 1.1.3. A configuration (50_5) in the projective (extended Euclidean) plane. There are ten points at infinity, in the direction of the sets of five parallel lines in the diagram. The smallest (n_5) configurations known are two (48_5) shown in Figure 4.1.4.

In contrast, to answer question (B) we first have to decide under what circumstances are two configurations considered to be the same, that is, not to be distinguished from each other for the purposes of the intended classification. As it turns out, in analogy to many other geometric topics, there are several sensible ways of classification, each leading to its own answer to question (B).

As an illustration of these differences we consider the case of configurations (9_3), which include the Pappus configuration from Figure 1.1.1. The three configurations in Figure 1.1.4 are obtained from each other by a simple affine transformation. They are considered the same under the so-called *projective* equivalence, which assigns to the same class configurations obtained as affine (or, more generally, projective) images of each other.

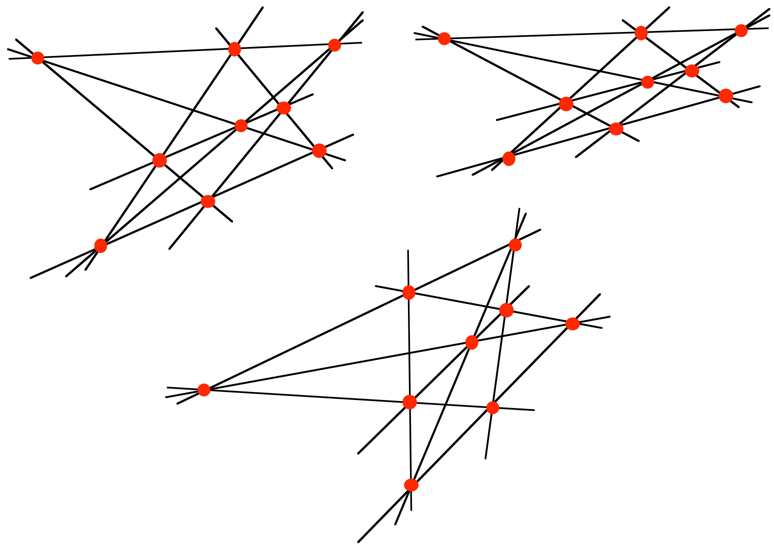

Figure 1.1.4. Three affinely equivalent configurations (9_3).

The three configurations in Figure 1.1.5 are not projectively equivalent, but have the same *incidences*.

Here and until further notice, incidences are defined between points and lines, and an incidence means that the point lies on the line or, equivalently, that the line passes through the point. Two configurations have the same incidences provided their points and lines can be given such labels that a point and a line are incident in one of them if and only if they are incident in the other. Since affine (and projective) transformations preserve lines and incidences, it is obvious that classification by incidences is coarser that the projective classification. The labels attached to the configurations show that they have the same incidences.

Concerning (C) we shall see that the available resources are rather modest. Some of the approaches will be discussed in the appropriate sections of the book. However, there is practically nothing relevant to configuration (n_k) with $k \geq 5$.

In Figure 1.1.6 are shown three configurations (9_3) that do not have the same incidences. While it may appear that proving their difference may be a staggering task, we shall see in Section 2.2 that — using appropriate tools — it can be accomplished in a few seconds. In fact, with just slightly greater effort, it can be shown that in this classification there are precisely three distinct configurations; one of each equivalence class is shown in Figure 1.1.6.

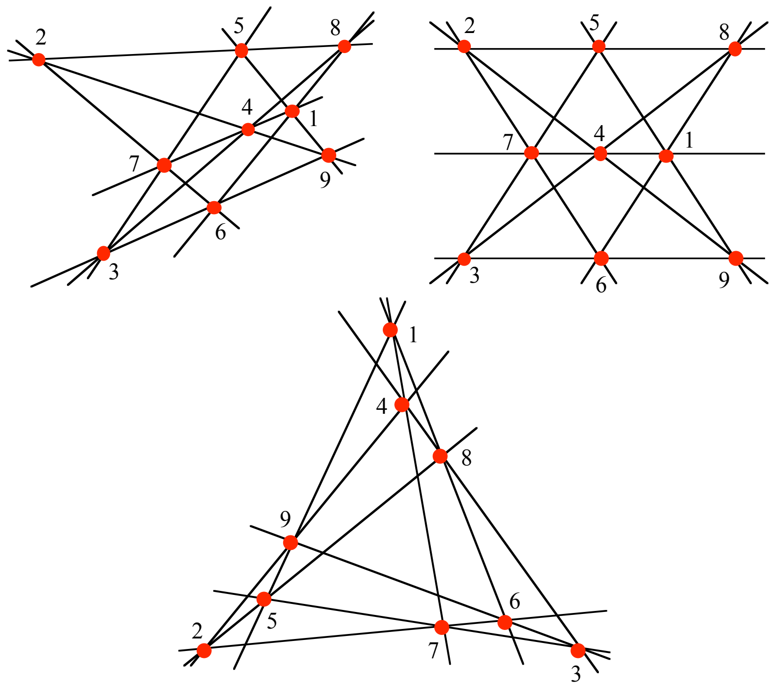

Figure 1.1.5. Three configurations (9_3) that have the same incidences but are not projectively equivalent.

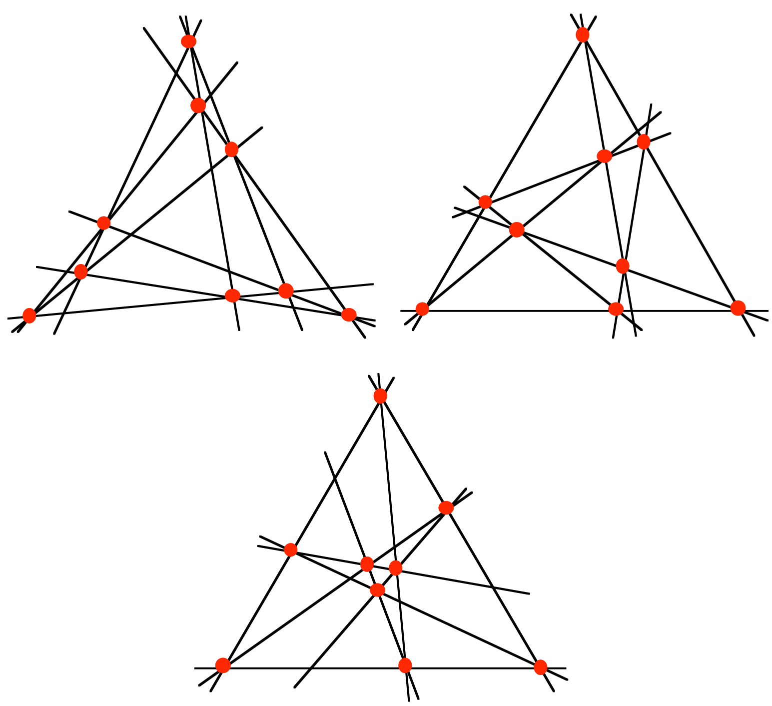

Figure 1.1.6. Three configurations (9_3) that have different incidences.

To simplify the expressions used in the sequel, we shall say that two configurations are *isomorphic* (or *combinatorially equivalent*, or of the *same combinatorial type*) if and only if they have the same incidences. In this terminology, all configurations in Figures 1.1.4, 1.1.5 and 1.1.6(a) have the same combinatorial type, different from the types in Figure 1.1.6(b) and (c).

In the next section we shall give an informal historical survey of the theory of configurations. In Section 1.3 we shall give formal definitions of the various concepts that are used in the book. Later sections of this chapter will present a selection of tools that have been found useful in the study of configurations. Chapter 2 will be devoted to a detailed study of 3-configurations, and Chapter 3 will deal with 4-configurations. Chapter 4 will present information about k-configurations for $k \geq 5$, as well as some other kinds of configurations. Chapter 5 will discuss known results on various properties of configurations (among them connectivity, Hamiltonicity, movability). In many sections we shall also pay attention to *combinatorial configurations* and on *configurations of pseudolines*. (The italicized concepts will be explained in the following sections.)

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A number of other directions of investigation start with families of lines and/or points in the plane, but have distinct aims from the study of configurations. The closest one to configuration has only recently been named, although some of its results go back close to two centuries. In an attempt to find a common framework for the various results that are more-or-less well known the term "aggregate of lines (or points)" has been proposed. The topics covered in this discipline deal – like configurations – with incidences of lines and points, but without the assumption of equal numbers for all lines and all points. The most famous among them are known as "orchard problems" and "Sylvester's problem". The former typically asks to locate a certain given number of points so that a maximal possible number of lines are incident with precisely 3 (or some other chosen number) of these points. For references see [B28, Chapter 7], [B27], [C13, Section F12], and [I1].

Sylvester's problem is to show that if a family of n lines is such that they are not in a pencil (that is, all incident with a single point), then there is an *ordinary point* -- a point incident with precisely two of the lines. In fact, the number of points is always greater than one, and a longstanding conjecture is that there are at least [n/2] such points. For more details about this problem and related ones see [B27], [B28, Chapter 7], or [C13, Section F12].

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A guiding principle of this book is the conviction that mathematics should be both interesting and attractive, and that it should give pleasure while working on it or reading about it. That is why the pace of the presentation is rather leisurely, eschewing acronyms and *ad hoc* abbreviations. It is also the reason for the inclusion of many diagrams – even in situations in which a formal argument could have been supplied. It is my hope that the reader will find this approach inviting, and the appeal to geometric intuition useful and stimulating.

Exercises and problems 1.1.

In the following sections we shall present lots of exercises that deal with configurations. As a warm-up, here are some questions that are only marginally relevant to configurations, but have much in common with the spirit that permeates the study of configurations.

1. The *orchard problem* is: For sets of p points in the plane, find t(p), the maximal possible number of lines containing precisely three of these points; see [B33]. It is known that t(12) = 19. Can you find a set of 12 points with this property? Concerning t(13) it is known only that $22 \leq t(13) \leq 24$. Can you improve on this?
2. Particular case of the generalized *Sylvester problem*. Let s(p) denote the minimal possible number of points incident with precisely two of a set of p lines, not all concurrent and no two parallel. It is known only that $7 \leq s(15) \leq 9$. What is the correct value?
3. Given n lines in the plane, no three concurrent and no two parallel. Show that among the bounded regions they determine in the plane there are at least n-2 triangular regions. This is known as *Roberts' theorem* (see [G36, p. 398]).