GRASSMANN ANGLES OF CONVEX POLYTOPES

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Dedicated to Hugo Hadwiger on his 60th anniversary, December 23, 1968

1. Introduction

The aim of the present note is to generalize to quantities we call Grassmann angles the results of Perles–Shephard [6] and Shephard [9] concerning angles and deficiencies of convex polytopes.

We shall define, for each d-polytope ($=d$-dimensional convex polytope) P^d and for each m, $1 \leqslant m \leqslant d-1$, a $(d-m)$-dimensional vector $\gamma^m(P^d) = (\gamma_0^m(P^d), \ldots, \gamma_{d-1-m}^m(P^d))$ related to the shape of P^d at its different faces. It will turn out that the *Grassmann angle sums* $\gamma_j^m(P^d)$ satisfy equations similar to those of Euler and Dehn–Sommerville, and that they satisfy certain inequalities. For $m=1$, $m=2$, or $m=d-1$, the Grassmann angles are related to the usually considered angles, deficiencies, or exterior angles, and our results therefore specialize to known facts concerning those entities.

As a preliminary stage we investigate the corresponding Grassmann angles for convex polyhedral cones; the results obtained contain as special case the theorem of Sommerville [10] relating (for even d) the volume of a spherical d-polytope to its angles, as well as a result of Fáry [2].

The main tool used, which was similarly applied in special cases already in Fáry [2], Perles–Shephard [6], and Shephard [9], is the invariant measure on the Grassmann manifold of all m-flats through the origin of the Euclidean d-space E^d. The measure-theoretic approach permits to restrict the consideration to m-flats which are in "general position" with respect to the polytope or cone considered, thus eliminating the necessity to take into account various "singular situations".

We start by investigating the Grassmann angles of convex cones (Section 2); in Section 3 we consider the Grassmann angles of polytopes, while the concluding Section 4 is devoted to some additional remarks and problems.

We shall freely use the standard results on convex polytopes; facts for which no references are given may be found in [4]; an account of the older results on angle-sums and their history is also given in [4].

(1) Research supported in part by Office of Naval Research contract N00014–67–A–0103–0003.

2. Grassmann angles of cones

Throughout the paper we shall denote by $G^{m,d}$ (where $1 \leqslant m \leqslant d-1$) the Grassmann manifold consisting of all the m-flats ($=m$-dimensional subspaces) through the origin O of the Euclidean d-space E^d. It is well known (see, for example, Petkantschin [8], Hadwiger [5], Bourbaki [1, p. 118] that there exists a (unique) invariant measure $\mu=\mu_{m,d}$ on $G^{m,d}$ such that $\mu(G^{m,d})=1$.

We shall denote by C^k any k-cone ($=k$-dimensional polyhedral convex cone) in E^d such that O is an apex of C^k. As is well known, the set of all apices of C^k forms a linear subspace of E^d, which is a face of C^k and is contained in every proper face of C^k. If this subspace is j-dimensional we shall sometimes denote the cone C^k by $(C^{k,j})$. Throughout the paper, we shall assume that $j<k$.

For a given k-cone C^k in E^d we shall denote by $G_0^{m,d}=G_0^{m,d}(C^k)$ the subset of $G^{m,d}$ consisting of all the m-flats which are in *general position* with respect to C^k. More precisely, this means that $M^m \in G_0^{m,d}$ if and only if for each i-face C^i of C^k

$$\dim\,(C^i \cap M^m)=\max\,\{0,\, i+m-d\}.$$

Simple considerations of dimensions show that $\mu(G_0^{m,d})=1$.

For each C^k in E^d, it is convenient to use the complementary characteristic functions $\varkappa$ and χ, defined for $M^m \in G_0^{m,d}$ by

$$\varkappa(M^m, C^k)=1-\chi(M^m, C^k)=\begin{cases}1 & \text{if} \quad M^m \cap C^k=\{0\}\\ 0 & \text{if} \quad M^m \cap C^k \neq \{0\}.\end{cases}$$

The *Grassmann angles* $\gamma^{m,d}(C^k)$ are defined by

$$\gamma^{m,d}(C^k)=\int_{G^{m,d}} \varkappa(M^m, C^k)\,d\mu=\int_{G_0^{m,d}} \varkappa(M^m, C^k)\,d\mu.$$

At times it is more convenient to use the complementary angles defined by

$$\beta^{m,d}(C^k)=1-\gamma^{m,d}(C^k)=\int_{G^{m,d}} \chi(M^m, C^k)\,d\mu.$$

In order to illustrate the significance of the Grassmann angles in some special cases, let $C^d=C^{d,j}$ be a d-cone in E^d, with face of apices C^j. The *angle* $\varphi(C^j, C^d)$ spanned by C^d at C^j is defined as the fraction of the unit sphere S^{d-1} belonging to C^d (see [4, p. 297]), while for $j=0$ the *external angle* $\psi(C^0, C^d)$ of C^d at its apex C^0 is defined as the fraction of S^{d-1} covered by outward normals to hyperplanes supporting C^d at C^0 (see [4, p. 308]). It is immediate that

$$\beta^{1,d}(C^{d,j}) = 1 - \gamma^{1,d}(C^{d,j}) = 2\varphi(C^j, C^d)$$
$$\gamma^{d-1,d}(C^{d,0}) = 1 - \beta^{d-1,d}(C^{d,0}) = 2\psi(C^0, C^d).$$

We note first some rather obvious facts concerning the Grassmann angles.

(2.1) *For all* m, k, d, $0 \leqslant \gamma^{m,d}(C^k) \leqslant 1$.

If $m+k \leqslant d$ then $\varkappa(M^m, C^k) = 0$ implies $M^m \notin G_0^{m,d}$; hence

(2.2) *If* $m+k \leqslant d$ *then* $\gamma^{m,d}(C^k) = 1$.

If $m+j \geqslant d$ then $\varkappa(M^m, C^{k,j}) = 1$ implies $M^m \notin G_0^{m,d}$; hence

(2.3) *If* $m+j \geqslant d$ *then* $\gamma^{m,d}(C^k) = 0$.

It follows that the only interesting cases are those for which $m+k \geqslant d+1$ and $m+j \leqslant d-1$, that is
$$d+1-k \leqslant m \leqslant d-1-j.$$

From now on we shall assume that this inequality is satisfied; then, in particular,
$$j+2 \leqslant k.$$

With these assumptions, (2.1) may be strengthened to

(2.4) *If* $1 \leqslant m \leqslant d-j-1$ *then* $0 < \gamma^{m,d}(C^{d,j}) < 1$.

Proof. The assumptions imply that each of the sets
$$\{M^m \in G_0^{m,d} \mid \varkappa(M^m, C^{d,j}) = 0\} \quad \text{and} \quad \{M^m \in G_0^{m,d} \mid \varkappa(M^m, C^{d,j})) = 1\}$$
is non-empty and open in $G_0^{m,d}$, hence the inequalities are obvious.

Our next aim is to indicate how the computation of all the Grassmann angles may be reduced to the computation of angles of the type $\gamma^{m,d}(C^{d,0})$. We shall first establish

(2.5) *Let* $d+1-k \leqslant d-1-j$; *then* $\gamma^{m,d}(C^{k,j}) = \gamma^{m+k-d,k}(C^{k,j})$.

Proof. Let $E^k = \operatorname{aff} C^k$ be the subspace of E^d spanned by C^k. If $M^m \in G_0^{m,d}$ then $M^{m+k-d} = M^m \cap E^k$ belongs to $G_0^{m+k-d,k}$; moreover, whenever $M^m \in G_0^{m,d}$ and $M^{m+k-d,k} \in G_0^{m+k-d,k}$ are in this relation then $\varkappa(M^m, C^{k,j}) = \varkappa(M^{m+k-d}, C^{k,j})$. By the invariance of the measures involved it follows that
$$\gamma^{m,d}(C^{k,j}) = \int_{G_0^{m,d}} \varkappa(M^m, C^{k,j})\, d\mu = \int_{G_0^{m+k-d,k}} \varkappa(M^{m+k-d}, C^{k,j})\, d\mu = \gamma^{m+k-d,k}(C^{k,j}),$$
and the proof of (2.5) is completed.

To formulate the next step in the reduction procedure, let $(L^p)^\perp$ denote the $(d-p)$-flat through O orthogonal to the p-flat L^p in E^d. For a d-cone $C^d = C^{d,j}$ with face of apices C^j let $\bar{C}^{d-j} = C^d \cap (C^j)^\perp$; then $\bar{C}^{d-j}$ is a pointed $(d-j)$-cone, and we have

(2.6) *Let* $1 \leqslant m \leqslant d-j-1$; *then* $\gamma^{m,d}(C^{d,j}) = \gamma^{m,d-j}(\bar{C}^{d-j})$.

We could establish (2.6) by a proof similar to that of (2.5). Instead, we shall prove the more interesting result (2.7), from which (2.6) may be deduced by applying (2.7) to $C^{d,j}$ and then once more to the polar cone of $C^{d,j}$, considered as a subset of E^{d-j}.

Let $C^{d,j}$ be a d-cone in E^d, and let $(C^{d,j})^* = \tilde{C}^{d-j,0}$ be the pointed $(d-j)$-cone polar to $C^{d,j}$ in E^d; that is, $\tilde{C}^{d-j,0}$ is the set of all points of E^d lying on outward normals to hyperplanes supporting $C^{d,j}$ at 0. With this notation we have

(2.7) *Let* $1 \leqslant m \leqslant d-j-1$; *then* $\gamma^{m,d}(C^{d,j}) + \gamma^{d-m,d}(\tilde{C}^{d-j,0}) = 1$.

Proof. The correspondence between $M^m \in G^{m,d}$ and $M^{d-m} = (M^m)^\perp \in G^{d-m,d}$ is well known to be an isometry between $G^{m,d}$ and $G^{d-m,d}$. Also, it is easily verified that for $M^m \in G_0^{m,d}(C^{d,j})$ and $(M^m)^\perp = M^{d-m} \in G_0^{d-m,d}(\tilde{C}^{d-j,0})$ one and only one of the relations $M^m \cap C^{d,j} = \{0\}$ and $M^{d-j} \cap \tilde{C}^{d-j,0} = \{0\}$ holds. Therefore, for each such M^m we have

$$\varkappa(M^m, C^{d,j}) + \varkappa((M^m)^\perp, \tilde{C}^{d-j,0}) = 1.$$

The proof is completed by integrating this relation over $G_0^{m,d}$, using the isometry between $G^{m,d}$ and $G^{d-m,d}$ mentioned above.

For the formulation of the subsequent results it is convenient to introduce the following notation. Let $C^{d,j}$ be a d-cone in E^d, and let $j \leqslant k \leqslant d$ and $1 \leqslant m \leqslant d-j-1$; we define

$$\beta_k^m(C^{d,j}) = \sum_{C^{k,j}} \beta^{m,k}(C^{k,j}),$$

the summation being over all k-faces $C^{k,j}$ of $C^{d,j}$.

We shall next establish

THEOREM 2.8. *For* $1 \leqslant m \leqslant d-j-1$,

$$\sum_{i=d-m+1}^{d} (-1)^i \beta_i^{m+i-d}(C^{d,j}) = (-1)^{m+d-1} \beta_d^m(C^{d,j}).$$

Proof. Let $M^m \in G_0^{m,d}$ satisfy $\chi(M^m, C^{d,j}) = 1$; then $D^m = M^m \cap C^d$ is a pointed m-cone in M^m. For $1 \leqslant i \leqslant m$, each i-face of D^m is the intersection of M^m with a $(d-m+i)$-face C^{d-m+i} of $C^{d,j}$, such that $\chi(M^m, C^{d-m+i}) = 1$; moreover, M^m meets (in points different from 0) only those $(d-m+i)$-faces of $C^{d,j}$ for which this relation holds. Therefore, for each such M^m and for $1 \leqslant i \leqslant m$, we have

$$f_i(D^m) = f_i(M^m \cap C^d) = \sum_{C^{d-m+i}} \chi(M^m, C^{d-m+i}),$$

the summation being over all the $(d-m+i)$-faces C^{d-m+i} of $C^{d,j}$. Since the numbers $f_i(D^m)$ of i-faces of D^m satisfy Euler's equation

$$\sum_{i=1}^{m} (-1)^{i+1} f_i(D^m) = 1,$$

we have for all such M^m

$$1 = \sum_{i=1}^{m} (-1)^{i+1} f_i(D^m) = \sum_{i=1}^{m} (-1)^{i+1} \sum_{C^{d-m+i}} \chi(M^m, C^{d-m+i}).$$

On the other hand, if $M^m \in G_0^{m,d}$ satisfies $\chi(M^m, C^{d,j}) = 0$, then $\chi(M^m, C^k) = 0$ for each k-face C^k of $C^{d,j}$, $j < k \leqslant d$. Hence, for *each* $M^m \in G_0^{m,d}$ we have

$$\chi(M^m, C^{d,j}) = \sum_{i=1}^{m} (-1)^{i+1} \sum_{C^{d-m+i}} \chi(M^m, C^{d-m+1}).$$

An integration over $G_0^{m,d}$ and an application of (2.5) complete the proof of Theorem 2.8.

In case $m=1$ Theorem 2.8 is obviously trivial. However, in case $m=2$ Theorem 2.8 reduces to the non-trivial assertion $\beta_{d-1}^1(C^{d,j}) = 2\,\beta_d^2(C^{d,j})$ which is equivalent to

$$\gamma^{2,d}(C^{d,j}) = \delta(C^j, C^d);$$

the last expression denotes the *deficiency* of the cone C^d at its face of apices C^j (see Shephard [9]).

A pointed d-cone $C^{d,0}$ is called *simple* provided it is the cone spanned from the origin O by a simple $(d-1)$-polytope P^{d-1} such that $O \notin \operatorname{aff} P^{d-1}$. More generally, a d-cone $C^{d,j}$ with face of apices C^j is called *simple* provided $C^{d,j} \cap (C^j)^\perp$ is a (pointed) simple $(d-j)$-cone. Using the same method as in the proof of Theorem 2.8, but applying the Dehn–Sommerville equations instead of Euler's to the simple cones D^m, we obtain

THEOREM 2.9. *For $1 \leqslant k \leqslant m \leqslant d-j-1$ and for each simple d-cone $C^{d,j}$ we have*

$$\beta_{d-m+k}^k(C^{d,j}) = \sum_{i=1}^{k} (-1)^{i+1} \binom{m-i}{m-k} \beta_{d-m+i}^i(C^{d,j}).$$

Additional results on the Grassmann angles of cones are discussed on pp. 301 and 302.

3. Grassmann angles of polytopes

Let K^d be a d-dimensional polyhedral set in E^d, let K^j be a j-face of K^d, $0 \leqslant j < d$, and let $z \in \operatorname{relint} K^j$. We define the d-cone $C^{d,j}$ by

$$C^{d,j} = \operatorname{cone}_0 (-z + K^d);$$

it clearly has a j-dimensional face of apices $C^j = \operatorname{aff}(-z + K^j)$. We shall extend to the pair (K^d, K^j) the functions $\varkappa, \chi, \gamma^{m,d}$, etc. by setting, for $M^m \in G^{m,d}$,

$$\varkappa(M^m, K^d, K^j) = \varkappa(M^m, C^{d,j}),$$

etc.

The subset $G_0^{m,d} = G_0^{m,d}(K^d)$ of $G^{m,d}$ is defined as the intersection of all the (finitely many) sets $G_0^{m,d}(C^{d,j})$, for K^j ranging over all the proper faces of K^d; note that $\mu(G_0^{m,d}) = 1$.

We find it convenient to define $\varkappa(M^m, K^k, K^j)$ and $\gamma^{m,d}(K^k, K^j)$ even if K^j is not a face of K^k; in this case we define both expressions to be 0.

Let K^d be a d-polyhedral set in E^d, and let K^k be any face of K^d. For each $M^m \in G_0^{m,d}$, we denote by $K^k(M^m)$ the polyhedral set obtained by projecting K^k orthogonally into the $(d-m)$-flat $(M^m)^\perp$. If $k \leqslant d-m$ then $K^k(M^m)$ is k-dimensional because $M^m \in G_0^{m,d}$ implies that the mapping of K^k onto $K^k(M^m)$ is one-to-one.

An easy application of the separation theorem for convex sets shows that (for $0 \leqslant k \leqslant d-m-1$) $K^k(M^m)$ is a proper face (hence a proper k-face) of $K^d(M^m)$ if and only if $\varkappa(M^m, K^d, K^k) = 1$. Therefore we have

(3.1) *For* $0 \leqslant k \leqslant d-m-1$

$$f_k(K^d(M^m)) = \sum_i \varkappa(M^m, K^d, K_i^k),$$

the summation being over all the k-faces K_i^k of K^d.

In order to simplify the notation, we define

$$\gamma_k^m(K^d) = \sum_i \gamma^{m,d}(K^d, K_i^k),$$

the summation being extended over all the k-faces K_i^k of K^d. (Note that this definition is essentially different from the definition of β_k^m in Section 2.)

The main lemma may now be stated as follows:

(3.2) *If the f-vector of $K^d(M^m)$ satisfies an equation*

$$\sum_{k=0}^{d-m-1} \lambda_k f_k = \lambda \tag{*}$$

for each $M^m \in G_0^{m,d}$, then

$$\sum_{k=0}^{d-m-1} \lambda_k \gamma_k^m(K^d) = \lambda.$$

Proof. Multiplying the equation (*) by λ_k, adding for $k=0, \ldots, d-m-1$, and integrating over $G_0^{m,d}$, we have

$$\lambda = \int_{G_0^{m,d}} \lambda d\mu = \int_{G_0^{m,d}} \sum_{k=1}^{d-m-1} \lambda_k f_k(K^d(M^m))\, d\mu = \int_{G_0^{m,d}} \sum_{k=0}^{d-m-1} \sum_i \lambda_k \varkappa(M^m, K^d, K_i^k)\, d\mu$$

$$= \sum_{k=0}^{d-m-1} \sum_i \lambda_k \gamma^{m,d}(K^d, K_i^k) = \sum_{k=0}^{d-m-1} \lambda_k \gamma_k^m(K^d).$$

In particular, if K^d is a d-polytope, we may use Euler's equation for the $(d-m)$-polytope $K^d(M^m)$ to obtain

THEOREM 3.3. *For each d-polytope K^d and for $1 \leqslant m \leqslant d-1$,*

$$\sum_{j=0}^{d-m-1} (-1)^j \gamma_j^m(K^d) = 1 - (-1)^{d-m}.$$

Using the Dehn–Sommerville equations we similarly have

THEOREM 3.4. *For each $(d-m-1)$-simplicial d-polytope K^d and for $0 \leqslant k \leqslant d-m-1$,*

$$\sum_{j=k}^{d-m-1} (-1)^j \binom{j+1}{k+1} \gamma_j^m(K^d) = (-1)^{d-1} \gamma_k^m(K^d).$$

Analogous results hold for $(d-m-1)$-cubical polytopes.

A very similar approach is applicable in the case of d-cones. For reasons of simplicity we shall illustrate the modification needed to derive the analogue of Theorem (3.3) for a pointed d-cone K^d with apex K^0. For a given m, $1 \leqslant m \leqslant d-1$, and for each $M^m \in G_0^{m,d}$ such that $\varkappa(M^m, K^d, K^0) = 1$ we have, as in (3.1),

$$f_k(K^d(M^m)) = \sum_i \varkappa(M^m, K^d, K_i^k)$$

and the Euler equation

$$\sum_{k=1}^{d-m-1} (-1)^{k+1} f_k(K^d(M^m)) = 1 + (-1)^{d-m}$$

for the pointed $(d-m)$-cone $K^d(M^m)$. However, if $\varkappa(M^m, K^d, K^0) = 0$ then $K^d(M^m) = E^{d-m}$, and $f_k(K^d(M^m)) = 0$ for each k. Combining the two cases we see that for each $M^m \in G_0^{m,d}$

$$\sum_{k=1}^{d-m-1} (-1)^{k+1} f_k(K^d(M^m)) = (1 + (-1)^{d-m}) \varkappa(M^m, K^d, K^0).$$

An integration over $G_0^{m,d}$ yields therefore

THEOREM 3.5. *For each pointed d-cone K^d and for $1 \leqslant m \leqslant d-1$,*

$$\sum_{k=1}^{d-m-1} (-1)^{k+1} \gamma_k^m(K^d) = (1+(-1)^{d-m}) \gamma_0^m(K^d).$$

Similar results may be derived for d-cones K^d with a j-dimensional face of apices, $j>0$; relations analogous to Theorem 3.4 hold for appropriately defined simplicial or cubical cones. Relations of these types may also be obtained for arbitrary polyhedral sets, at the expense of involving their lineality and characteristic cone ([4, Sections 2.5 and 8.5]).

For $m=1$ Theorem 3.5 specializes to a result equivalent to Sommerville's theorem on angle-sums of spherical polytopes (Sommerville [10], [11, p. 159], Perles–Shephard [6]), which is a generalization of the fact that the area of a spherical polygon is proportional to its "excess". Similarly, for $m=2$ Theorem 3.5 specializes to Theorem 3.7 of Shephard [9].

Let now K^d be again a d-polytope, let $K_1^{d-m}, \ldots, K_n^{d-m}$ be representatives of all the combinatorial types occurring among the $(d-m)$-polytopes $K^d(M^m)$ for $M^m \in G_0^{m,d}$, and let $B_i \subset G_0^{m,d}$ be the set of all those $M^m \in G_0^{m,d}$ for which $K^d(M^m)$ is combinatorially equivalent to K_i^{d-m}. Then each B_i is an open set in $G_0^{m,d}$, hence $\mu(B_i)>0$; obviously we also have $\sum_{i=1}^n \mu(B_i)=1$. Therefore, integrating over $G_0^{m,d}$ the equation of Lemma (3.1), and denoting by $\gamma^m(K^d)$ the $(d-m)$-vector

$$\gamma^m(K^d) = (\gamma_0^m(K^d), \gamma_1^m(K^d), \ldots, \gamma_{d-m-1}^m(K^d)),$$

we have

THEOREM 3.6. *For each d-polytope K^d and for $1 \leqslant m \leqslant d-1$,*

$$\gamma^m(K^d) \in \operatorname{relint} \operatorname{conv} \{f(K_i^{d-m}) \mid i=1, \ldots, n\}.$$

4. Remarks

(a) The two approaches (section and projection) used above being dual to each other, it is not surprising that the results obtained by them (such as (2.8) and (3.5)) are also duals of each other; the link between them is Lemma (2.7).

(b) For $m=1$ and $m=2$, our results essentially coincide with those of Perles–Shephard [6] and Shephard [9]. The same papers contain also various consequences of these results. For very ingenious applications of their results to some combinatorial problems concerning higher-dimensional polytopes see Perles–Shephard [7].

(c) Lemma (2.5) may be seen to imply (in case $d=3$, $m=2$, $k=2$, $j=0$) the result of Lemma 1 of Fáry [2]. For applications of Fáry's lemma to polyhedral cones in E^3 see Frostman [3].

(d) For a d-polytope K^d let $\mathcal{K}^d$ denote the set of all polytopes affinely equivalent to K^d, and let

$$\gamma^m(\mathcal{K}^d) = \{\gamma^m(K) \mid K \in \mathcal{K}^d\}.$$

For $m=1$, Theorem 3.6 may be strengthened (Perles–Shephard [6]) to the assertion

$$\operatorname{conv} \gamma^1(\mathcal{K}^d) = \operatorname{relint} \operatorname{conv} \{f(K_i^{d-1}) \mid i=1, \ldots, n\}.$$

It may be conjectured that

(i) For $m=1$ and each d-polytope K^d,

$$\gamma^1(\mathcal{K}^d) = \operatorname{relint} \operatorname{conv} \{f(K_i^{d-1}) \mid i=1, \ldots, n\};$$

(ii) For each m with $2 \leqslant m \leqslant d-2$ and for each d-polytope K^d,

$$\dim \operatorname{aff} \gamma^m(\mathcal{K}^d) = \dim \operatorname{aff} \{f(K_i^{d-1}) \mid i=1, \ldots, n\};$$

(iii) For each m with $2 \leqslant m \leqslant d-2$ there exists a d-polytope K^d such that

$$\operatorname{conv} \gamma^m(\mathcal{K}^d) \neq \operatorname{relint} \operatorname{conv} \{f(K_i^{d-1}) \mid i=1, \ldots, n\}.$$

(e) Let $\mathcal{D}^{d,m}$ denote the family of all d-polytopes K^d with the property that all $K^d(M^m)$, for $M^m \in G_0^{m,d}(K^d)$, are of the same combinatorial type. As easily seen (Perles–Shephard [6], Shephard [8] for $m=1, 2$) Theorem 3.6 implies that for polytopes $K \in \mathcal{D}^{d,m}$ the vectors $\gamma^m(K)$ are affine invariants of K. It is not hard to see that zonotopes (i.e., vector-sums of segments), as well as direct sums of (segments and) regular polygons belong to $\mathcal{D}^{d,m}$ for each m. However, $\mathcal{D}^{d,m}$ has not been characterized even in the simplest non-trivial case ($d=3$, $m=1$).

(f) In connection with applications of Theorem 3.6 (Perles–Shephard [7]) the following conjecture is of some interest. For each d-polytope K there exists a K' combinatorially equivalent to K and having the property: For every K'' combinatorially equivalent to K, and for every $M'' \in G_0^{m,d}(K'')$, there exists an $M' \in G_0^{m,d}(K')$ such that $K''(M'')$ is combinatorially equivalent to $K'(M')$. This conjecture is also open even in the seemingly trivial case $d=3$, $m=1$.

(g) It would be interesting to investigate whether the various Grassmann angles (and their sums) satisfy (non-linear?) inequalities which are not trivial consequences of (2.4), (3.6), and the various equations found above.

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Added in proof (October 13, 1968): The following recent papers deal with topics related to Grassmann angles:

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Received May 6, 1968