METRICALLY HOMOGENEOUS SETS: CORRIGENDUM

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ABSTRACT

A flaw in Lemma 2 is corrected and other parts of the paper changed accordingly so as to preserve all the theorems.

In reviewing our paper [1] Professor J. F. Rigby has observed that Lemma 2 is not valid as it stands.* The lemma can be repaired by adding the hypothesis $\measuredangle ABC > 120°$, but then additional arguments are required to complete the discussion of Case V in the proof of Theorem 6. In the original text Lemmas 2 and 6 were designed to facilitate the exposition contained in the last two paragraphs of the arguments of that case. We find it advantageous to present modified versions of Lemmas 2 and 6, and to revamp correspondingly the argument in Case V of the proof of Theorem 6.

We note also that the reference to Lemma 2 on page 191, line 31, should be* to Lemma 1, and that in various references throughout the paper "Comparison Lemma" is another term for Lemma 1.

LEMMA 2′. *Let the convex quadrilaterals $ABCD$ and $A'B'C'D'$, each labelled in cyclic order, satisfy*

$$AB = CD = B'C' = x,\ A'B' = C'D' = BC = y,\ AC = BD = A'C' = B'D',$$

$$AD = w,\ A'D' = w', \text{ while } \measuredangle ABC > 120° \textit{ and } y > x;\ \textit{then } w' > w.$$

PROOF. Noting that $\alpha = 180° - \measuredangle ABC < 60°$ we have

$$w = 2x\cos\alpha + y,\quad w' = 2y\cos\alpha + x$$

$$w' - w = 2\cos\alpha(y - x) + (x - y) = (y - x)[2\cos\alpha - 1] > 0.$$

* We are indebted to Professor Rigby for kindly bringing these facts to our attention.

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LEMMA 6′. *If a homogeneous polygon contains consecutive vertices* A, B, C, D, E *with* $AB = CD = a_1$, $BC = DE = a_2$, $AC = BD = a_3$ *and* $AD > a_i$, $i = 1,2,3$, *and if* $\measuredangle ABC \leqq 120°$, *then* AD *is a diameter of the polygon.*

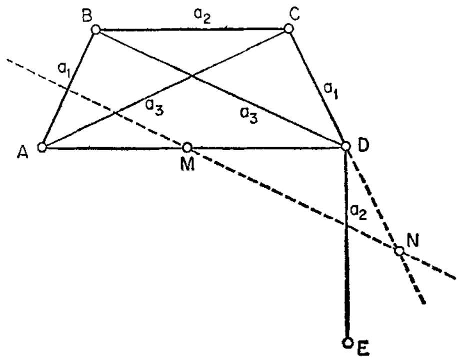

Fig. 1

PROOF. (See Fig. 1.) Let the perpendicular bisector of AB meet AD in M and CD in N. Since $AD > BD$, M is between A and D. We now claim that $a_2 \geqq MD \geqq ND$. In fact, $\measuredangle ABM = \measuredangle BAM \geqq 60°$ and hence $\measuredangle CBM \leqq 60°$. Thus $\measuredangle CBM + \measuredangle BCD \leqq 180$ and $a_2 \geqq MD$.

Furthermore, $\measuredangle BAM \geqq 60°$ implies that $\measuredangle DMN \leqq 30°$ which together with $\measuredangle CDM \geqq 60°$ leads to the conclusion that $DNM \geqq 30°$. Thus $MD \geqq ND$.

It is now clear that E cannot be in the triangle MDN and hence is in the open half plane, π^+, having line MN as edge and containing point A. By the same argument π^+ contains all the vertices of the polygon with the exception of B, C and D.

Since $BP > AP$ for all points P in π^+, the diameter from A must be AD. ■

The last two paragraphs of the argument for case V of Theorem 6 should be replaced by the following (compare Fig. 2): We may suppose $P_0P_2 = P_1P_3 = a_3$.

CASE 1. $\measuredangle P_0P_1P_2 \leqq 120°$ or $\alpha = 180° - \measuredangle P_0P_1P_2 \geqq 60°$.

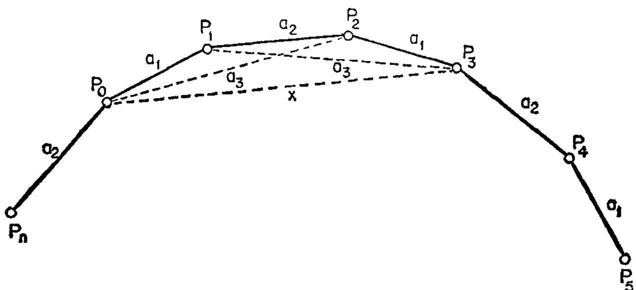

Fig. 2

By the comparison lemma $P_2P_5 \geqq x$. Lemma 6′ implies that P_0P_3 is a diameter of the polygon and hence $P_2P_5 = x$, $P_2P_4 = P_3P_5 = a_3$. It can now be inferred by induction that all second order diagonals are of equal length and the polygon is cyclic.

However it is a little bit more revealing to note that, since diametral segments must intersect, $P_4P_1 = x$. Then from quadrilaterals $P_0P_1P_2P_3$ and $P_2P_3P_4P_5$ we conclude that $x = 2a_1 \cos\alpha + a_2 = 2a_2 \cos\alpha + a_1$ and that $\alpha = 60°$. Now the polygonal angles at P_1, P_2, P_3, P_4 are all 120° and an easy calculation shows that $P_0P_5 = a_2$. Thus the polygon is a quasiregular hexagon.

CASE 2. $\sphericalangle P_0P_1P_2 > 120°$.

Lemmas 2′ and 1 show that $P_2P_n > x$. If $P_2P_4 = x$ then $a_1 + a_2 \gtreqqless x$ and from Lemma 3 it follows that $\sphericalangle P_0P_1P_2 < 120°$. Since $P_2P_5 \geqq x$ by the comparison lemma, $P_2P_5 = x$. Thus $P_2P_4 = P_3P_5 = a_3$. As above it follows by induction that all second order diagonals are of length a_3 and the polygon is cyclic.

REFERENCE

1. B. Grünbaum and L. M. Kelly, *Metrically homogeneous sets.* Israel J. Math., **6** (1968), 183–197.

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