METRICALLY HOMOGENEOUS SETS

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ABSTRACT

A subset of a metric space is called metrically homogeneous if the set of distances from a chosen point of the subset to all the other points of the subset is independent of the chosen point. The main result of the paper is a complete characterization of the compact, metrically homogeneous subsets of the Euclidean plane.

1. **Introduction.** If $\mathscr{S}_1$ and $\mathscr{S}_2$ are two subsets of a distance space, $\rho(\mathscr{S}_1, \mathscr{S}_2)$ is the set of distances $\rho(x,y)$, $x \in \mathscr{S}_1$, $y \in \mathscr{S}_2$. A set $\mathscr{S}$ such that $\rho(P, \mathscr{S}) = \rho(Q, \mathscr{S})$ for any two points P and Q of $\mathscr{S}$ is a *metrically homogeneous set.*

Since the problem of characterizing such sets is even more general than the difficult and important problem of determining those subsets of a space whose group of self-isometries is simply transitive, its solution in general settings should not be expected to be easy. We succeed here only in completely characterizing the compact metrically homogeneous subsets of the Euclidian spaces E_2; however, much of our preliminary analysis is valid in any strictly convex two dimensional real Banach space.

Curves of constant width and the vertex sets of regular polygons are quite clearly metrically homogeneous. A convex cyclic polygon with alternate sides equal is called *quasi regular* and its vertex set is still another example of a metrically homogeneous set in E_2. Indeed, if we insist that the set of distances emanating from each point have the same "repetitions" then the smooth curves of constant width together with the vertex sets of regular and quasi regular polygons are the only *metrically strictly homogeneous sets* in E_2.

However, if one of the vertices be omitted from the vertex set of an odd sided regular polygon, the residual set is metrically homogeneous and suggests the ultimate characterization theorem to the effect that such sets are either arcs of curves of constant width, vertex sets of quasi regular polygons or suitably truncated vertex sets of regular polygons.

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Surprisingly, it follows from this characterization that each finite metrically homogeneous subset of the plane has a non-trivial group of isometries. Already in Euclidean 3-space there exist finite metrically homogeneous sets lacking this property.

The stimulus for this investigation was a question posed by J. Conway at the East Lansing conference on Combinatorial Geometry. The present form of the proof of case V of the main theorem is due in essence to Mr. William Webb, to whom we are grateful for several useful ideas.

2. **Preliminaries and a few general theorems.** Generally points will be denoted by capitals. PQ is the distance from P to Q while $\overline{PQ}$ is the segment defined by P and Q.

B_2 is a two dimensional Banach space over the reals with a strictly convex norm. The *diameter* of a set $\mathscr{S}$ relative to a point $P \in \mathscr{S}$ is $\text{lub}\{PX \mid X \in \mathscr{S}\}$. If the diameter of $\mathscr{S}$ relative to each point of $\mathscr{S}$ is constant and finite, the set is said to be of *constant diameter*. If PQ is a diameter of $\mathscr{S}$, $\overline{PQ}$ is a *diametral segment* of $\mathscr{S}$. It is well known that in E_2 a set of constant diameter is the boundary of a set of constant width, and conversely. In the sequel, d will always denote the diameter of the set considered.

The main result of the present section is the characterization of infinite compact metrically homogeneous sets in E_2 (Theorem 4).

THEOREM 1. *Any two diametral segments of a set $\mathscr{S}$ in B_2 intersect.*

Proof. Suppose $AB = d$, $DC = d$ and $A\ B\ C\ D$ are the vertices of a convex quadrilateral (see Fig. 1). Let the diagonals $\overline{AC}$ and $\overline{BD}$ intersect in E. Then

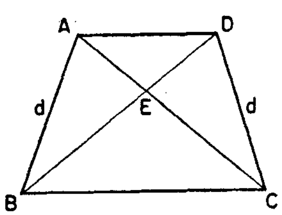


Fig. 1

$AE + EB > d$, $DE + EC > d$ hence $AC + DB > 2d$. Since $AC \leq d$ and $DB \leq d$, this is a contradiction.

If $A\ B\ C\ D$ is not convex then one of the points, say D, is in the triangle formed by the other three points (see Fig. 2). Let $\overline{DC}$ intersect $\overline{AB}$ in E and let F be a point collinear with C and D and such that $FE = (y/x)EC$. Then $FA = (y/x)a$ and from the triangle inequality we have $FA + AC > FC = FE + EC$, or $(y/x)a + b$

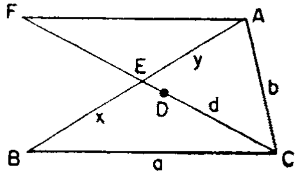


Fig. 2

$>(y/x)d + d$, hence $y(a-d) + bx > xd$ or $y(a-d) > x(d-b)$. This is a contradiction, and the proof of Theorem 1 is completed.

THEOREM 2. *If $\mathcal{S}$ is a compact set of constant diameter in B_2 then $\mathcal{S} = \text{ext conv}\,\mathcal{S}$.*

Proof. Assuming the theorem false, there is a point $P \in \mathcal{S}$ which is in the relative interior of a segment $\overline{XY} \subset \text{conv}\,\mathcal{S}$.

Let $Q \in \mathcal{S}$ be diametral to P (i.e. such that $\overline{PQ}$ is a diameter of $\mathcal{S}$). The strict convexity of the norm in B_2 implies that $2PQ < QX + QY$, thus $PQ < \max\{QX, QY\}$. Since the diameter of conv $\mathcal{S}$ is the same as the diameter of $\mathcal{S}$, this is a contradiction, and the theorem is proved.

We note that Theorem 2 implies the existence of a natural cyclic order for the points of any set of constant diameter (and in particular for any metrically homogeneous set). This observation will be used throughout the sequel.

The next result will also be used very frequently; we shall refer to it as the "monotonicity theoıem".

THEOREM 3. *If P, Q, X, Y are different points of a set $\mathcal{S}$ of constant diameter in B_2, such that PQ is a diameter of $\mathcal{S}$, $\overline{QX} \cap \overline{PY} \neq \emptyset$ and $PY < PQ$, then $PY > PX$.*

Proof. (Compare Fig. 3). Let Y^* be a point diametral to Y, and let $E = \overline{YP} \cap \overline{Y^*X}$. (The point E exists by virtue of the axiom of Pasch, and convexity.) Since $YE + EY^* > Y^*Y = d$ and $Y^*E + EX = Y^*X \leq d$, we have $YE > EX$. Hence $PY = EP + EY > EP + EX > PX$, as claimed.

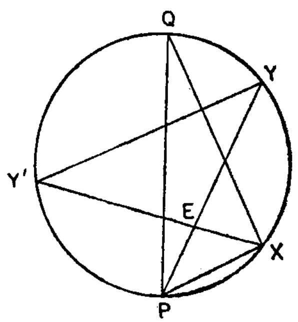


Fig. 3

A little more descriptively, the montonicity theorem asserts that as the point X moves on the boundary of conv $\mathscr{S}$ away from P, the distance PX is monotone strictly increasing till X encounters the first point diametral to P. PX is then constant as X moves through the points diametral to P and then it is monotone strictly decreasing as X returns to P.

Heppes [1] has recently characterized curves of constant width by the property that each chord of the curve is maximal among the chords of at least one of the arcs determined by the chord. This is another way of viewing the monotonicity property.

Definition. The *midpoint* of an arc $\mathscr{A}$ of a closed convex curve $\mathscr{C}$ in B_2, with endpoints A and B, is the (unique) point M of $\mathscr{A}$ such that $MA = MB$. If $\mathscr{A}$ contains points P, Q, R such that $PQ = QR = d$, the diameter of $\mathscr{C}$, then $\mathscr{A}$ is a *major arc* of $\mathscr{C}$. Otherwise it is a *minor arc.* $\mathscr{C}$ itself is included in the class of *major arcs.*

Theorem. 4. *A compact metrically homogeneous subset $\mathscr{S}$ of B_2 is either a finite set or a major arc of a curve of constant diameter.*

Proof. Recall that $\mathscr{S}$ is a closed subset of the boundary $\mathscr{C}$ of conv $\mathscr{S}$ and assume that $\mathscr{S}$ is infinite. Each point of $\mathscr{S}$ is then an accumulation element of $\mathscr{S}$. Suppose $P \in \mathscr{C} - \mathscr{S}$ and let A and B be the end points of the maximal arc of $\mathscr{C}$ in $\mathscr{C} - \mathscr{S}$ containing P. A and B are, of course, in $\mathscr{S}$. We now propose to show that A and B are endpoints of a major arc of $\mathscr{C}$ lying in $\mathscr{S}$.

To this end let $\varepsilon = \min\left\{\frac{AB}{2}, \frac{d}{2}\right\}$, where d is the diameter of $\mathscr{S}$ and hence also of $\mathscr{C}$, and let $\mathscr{B} = \mathscr{B}(\varepsilon, A)$ be the sphere of radius ε centered at A. Let $\{X_i\}$ be a sequence such that $X_i \in \mathscr{S} \cap \mathscr{B}$ and $\lim X_i = A$. Now suppose that E is any point in $\mathscr{S} \cap \mathscr{B}$ and consider $Y_i \in \mathscr{S} \cap \mathscr{B}$ such that $Y_iX_i = EA$. It is clear that the sequence $\{Y_i\}$ tends to E from the same side as $\{X_i\}$ tends to A, i.e. $Y_i E A B$ holds in the cyclic order defined on $\mathscr{C}$. Now define Z_i as an element of $\mathscr{S}$ such that $AZ_i = EX_i$. It follows easily that $\{Z_i\}$ tends to E from the other side, i.e. $EZ_i AB$ holds. Thus an arbitrary point of $\mathscr{S} \cap \mathscr{B}$ is a two-sided accumulation element of $\mathscr{S}$.

Let $\widehat{AB}$ be the arc of $\mathscr{C}$ which does not contain P and suppose Q is a point of $(\mathscr{C} - \mathscr{S}) \cap \mathscr{B} \cap \widehat{AB}$. At least one of the end points of the maximal arc of $\mathscr{C}$ in $\mathscr{C} - \mathscr{S}$ which contains Q is in $\mathscr{S} \cap \mathscr{B}$, and is a one-sided accumulation element of $\mathscr{S}$, contradicting the above. Hence such Q does not exist, and A is the end point of an arc of $\mathscr{C}$ lying in $\mathscr{S}$. Let $\mathscr{A}$ be the component of $\mathscr{S}$ containing A, with midpoint M, and suppose F its other end point. Then $\mathscr{A}$ must be a major arc since otherwise there is clearly a point Z in $\mathscr{C} - \mathscr{S}$ such that for no point $X \in \mathscr{S}$ is $MX = MZ$, while there is a point $Y \in \mathscr{S}$ with $AY = MZ$. This contradicts the homogeneity of $\mathscr{S}$ and proves that $\mathscr{A}$ is a major arc of $\mathscr{C}$.

If $F \neq B$ then there would be two disjoint major arcs of $\mathscr{C}$ in $\mathscr{S}$ which is clearly impossible. Hence $\mathscr{A} = \widehat{AB}$ and the proof of Theorem 4 is completed.

A finite metrically homogeneous set in B_2 is the vertex set of a convex polygon and we seek to characterize such polygons; for brevity, we shall say that such a polygon is metrically homogeneous. Let $a_1 < a_2 < \cdots < a_m = d$ be the numbers in the distance set. If the vertices of a metrically homogeneous polygon, labelled in cyclic order, are $P_0, P_1, \cdots, P_n$ then it is easy to see that if $P_iP_j = a_1$ then P_i and P_j are adjacent.

3. Some lemmas. We list now a sequence of lemmas which will be needed in the proof of Theorem 5. Thus far our results have been valid in a strictly convex B_2. We cannot proceed much further without recourse to additional structure so we shift the locale at this point to E_2.

LEMMA 1. *If the convex quadrilaterals* $A\ B\ C\ D$ *and* $A'\ B'\ C'\ D'$ *satisfy* $AB = A'B', BC = B'C', CD = C'D', AC \geqq A'C'$ *and* $BD \geqq B'D'$, *then* $AD \geqq A'D'$ *with equality if and only if* $AC = A'C'$ *and* $BD = B'D'$.

Proof. Consider a third quadrilateral $A''B''C''D''$ with $A''B'' = AB$, $B''C'' = BC$, $C''D'' = CD$, $A''C'' = AC$ and $B''D'' = B'D'$. Then $\measuredangle BCD \geqq \measuredangle B''C''D''$, and $\measuredangle ACB = \measuredangle A''C''B''$. Thus $\measuredangle ACD \leqq \measuredangle A''C''D''$ and since $AC = A''C''$ and $DC = D''C''$, it follows that $AD \geqq A''D''$. A similar argument shows that $A''D'' \geqq A'D'$ and hence $AD \geqq A'D'$. It is now easy to see that equality occurs only if $AC = A'C'$ and $BD = B'D'$.

LEMMA 2. *If the quadrilaterals* $ABCD$ *and* $A'B'C'D'$ *satisfy* $AB = CD = p = B'C' < A'B' = C'D' = q = BC$, $AC = BD = r \leqq r' = A'C' = B'D'$, $AD = s$, $A'D' = s'$, *then* $s' > s$.

Proof. Again consider an auxilliary quadrilateral $A''B''C''D''$ with $A''B'' = q$, $B''C'' = p$, $C''D'' = q$, $A''C'' = B''D'' = r$, $A''D'' = s''$. Quadrilaterals $ABCD$ and $A''B''C''D''$ have congruent circumcircles and it is clear that the arc $ABCD$ is less than or equal to the arc $A''B''C''D''$. Thus $s'' > s$ and, by Lemma 1, $s' \geqq s''$.

LEMMA 3. *If the quadrilateral* $ABCD$ *satisfies* $AB = CD = p$, $BC = q$, $AD = s$, $AC = BD = r$ *and* $p + q > s$, *then* $\measuredangle ABC < 120°$.

Proof. From the theorem of Ptolemy we have $p^2 + qs = r^2$, thus $p^2 + q(p+q) > r^2$. Hence $p^2 + pq + q^2 > p^2 + q^2 - 2pq \cos \measuredangle ABC$, and therefore $\cos \measuredangle ABC > -\frac{1}{2}$, i.e., $\measuredangle ABC < 120°$.

LEMMA 4. *If the quadrilaterals* $ABCD$ *and* $A'B'C'D'$ *satisfy* $AB = BC = CD = a$, $AC = BD = b$, $AD = c$, $A'B' = B'C' = a$, $C'D' = b$, $A'C' = B'D' = d$ *and* $A'D' = c$, *then* $\measuredangle B'C'D' < 90°$.

Proof. Ptolemy's theorem implies that $ac + a^2 = b^2$ and $ac + ab \geqq d^2$. Thus $ab - a^2 \geqq d^2 - b^2 = a^2 + b^2 - 2ab \cos \measuredangle B'C'D' - b^2$ and hence

$ab(1 + 2\cos \sphericalangle B'C'D') \geqq 2a^2$. Therefore $b(1 + 2\cos \sphericalangle B'C'D') \geqq 2a > b$, i.e. $\cos \sphericalangle B'C'D' > 0$ or $\sphericalangle B'C'D' < 90°$.

LEMMA 5. *If quadrilaterals ABCD and A'B'C'D' satisfy* $AB = CD = A'B' = B'C' = a$, $BC = C'D' = b$, $AC = BD = B'D' = c$ *and* $AD = A'D' = d$, *then* $A'B'C'D'$ *is cyclic.*

Proof. Consider C^*, the reflection of C in the perpendicular bisector of BD. Quadrilateral $ABCD$ is certainly cyclic and C^* is on its circumcircle. But $A'B'C'D'$ is congruent to ABC^*D, hence the assertion.

LEMMA 6. *No metrically homogeneous polygon in* E_2 *contains consecutive vertices* A,B,C,D,E *with* $AB = CD = p, BC = DE = q, AC = BD = r, AD = CE = s$ *and* $s > r > q \geqq p$.

Proof. (Compare Fig. 4). From Lemma 1 we conclude that $BE > AD = s$ and that the diameter d is greater than s. The diametral segment from A must

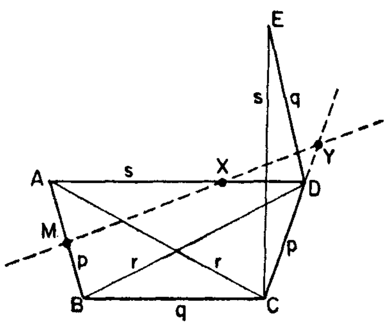
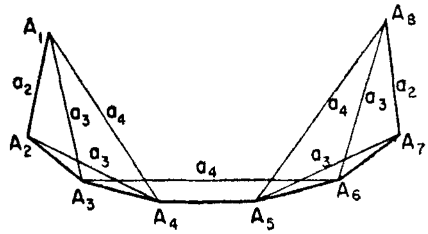


Fig. 4

intersect the perpendicular bisector of the segment $\overline{AB}$. From Lemma 3 it follows that $\sphericalangle ABC = \sphericalangle BCD < 120°$. Now a simple calculation shows that the perpendicular bisector of the segment $\overline{AB}$ intersects $\overline{AD}$ in X and the line CD in Y such that $q > XD \geqq YD$. It follows that the only points of the polygon on the opposite side of the line MX from A are B, C, and D, and none of the segments $\overline{AD}$, $\overline{AC}$, or $\overline{AB}$, is diametral. This contradicts the homogeneity assumption, and thus completes the proof.

LEMMA 7, *No metrically homogeneous set* $\mathcal{S}$ *in* E_2 *contains points* $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$, such that $A_3 A_5 = A_1A_2 = A_7A_8 = A_4A_6 = a_2$, $A_2A_3 = A_3A_4 = A_4A_5 = A_5A_6 = A_6A_7 = a_1$, $A_1A_3 = A_2A_4 = A_5A_7 = A_6A_8 = a_3$ *and* $A_1A_4 = A_3A_6 = A_5A_8 = a_4$.

Proof. (Compare Fig. 5). From Lemma 4 we have $\sphericalangle A_1A_2A_3 = \sphericalangle A_8A_7A_6 < 90°$. Suppose $A_1A_8 = a_1$. A_2A_7 is clearly greater than A_1A_8 and hence an



Fig, 5

examination of quadrilaterals $A_1A_4A_5A_8$ and $A_2A_4A_5A_7$ shows that $A_2A_6 > A_8A_4$. Hence $A_8A_4 < d$, thus $A_8A_3 = d$. But then no diametral segment from A_5 intersects A_8A_3, which is a contradiction. Hence there is a point A_0 of $\mathscr{S}$ such that $A_0A_1 = a_1$ and $A_0 \neq A_i$ for $i = 1,2,\cdots,8$. If $A_0A_2 = a_3$ then $\sphericalangle A_0A_1A_2 < 90°$ which is clearly impossible. An easy montonicity argument shows that $A_0A_i \neq a_3$ for $i = 3, 4, 5, 6, 7, 8$. Simlarly $A_0A_i \neq a_2$, $i = 2,3,\cdots,8$. Hence $A_0A_p = a_3$ and $A_0A_q = a_2$ for some $q > p > 8$, and it is clear that $A_pA_q \geqq a_1$. Hence $\sphericalangle A_0A_qA_p < 90°$. A similar argument shows that $\sphericalangle A_8A_sA_t < 90°$ for suitable s,t; but then the polygon conv $\mathscr{S}$ would contain four acute angles which is impossible. This completes the proof of Lemma 7.

LEMMA 8. *No metrically homogeneous set in E_2 contains the eight point configuration $A_1, A_2 \cdots A_8$ with*

$$A_1A_2 = A_3A_4 = A_4A_5 = A_5A_6 = A_7A_8 = a_1$$
$$A_2A_3 = A_6A_7 = A_3A_5 = A_4A_6 = a_2$$
$$A_1A_3 = A_2A_4 = A_5A_7 = A_6A_8 = a_3$$
$$A_1A_4 = A_8A_5 = a_4 \leqq A_3A_6.$$

Proof. (Compare Fig. 6). The perpendicular bisector of $\overline{A_5A_6}$ intersects $\overline{A_5A_8}$ in Y and the line A_7A_8 in X. A simple calculation shows that A_8X and

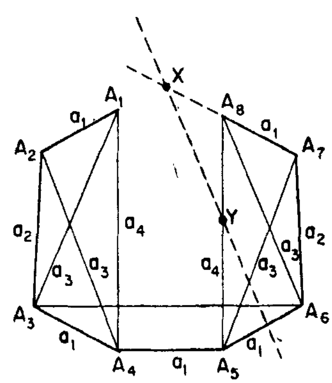


Fig. 6

A_8Y are both less than a_2. A point $\bar{A}_5$, diametral to A_5, must lie in the triangle A_8XY and hence $\bar{A}_5A_8 = a_1$, i.e. $\bar{A}_5$ is adjacent to A_8. Similarly for $\bar{A}_4$. Now, if $\bar{A}_5 = A_1$ it is clear that $A_3A_7 > A_5A_1 = d$, which is impossible. If $\bar{A}_5 = \bar{A}_4$ then $\bar{A}_5A_i = a_3$ has no solution, contradicting the homogeneity. But then the diametral segments $\overline{A_5\bar{A}_5}$ and $\overline{A_4\bar{A}_4}$ are disjoint, which is again impossible. Hence in all cases we have a contradiction and the lemma is proved.

4. Characterization of finite, metrically homogeneous sets in E_2. The following theorems provide information on metrically homogeneous polygons, which leads to their complete characterization in Theorem 7.

THEOREM 5. *The length of a side of a metrically homogeneous polygon cannot exceed* 2.

Proof. Suppose that the vertices of the polygon labelled in cyclic order, are $P_0, P_1, \cdots, P_n$ and that $P_nP_0 = a_k > a_2$. Let $P_0P_t = d$ while $P_0P_x < d$ for $x < t$.
We propose now to show that for all $i \leqq t$

(1) $P_0, P_1, \cdots, P_i$ are concyclic;

(2) $P_\alpha P_\beta = a_{|\alpha-\beta|}$ for $\alpha, \beta \leqq i, |\alpha - \beta| \leqq \left[\frac{i+1}{2}\right]$;

(3) $P_nP_i \geqq \min\{a_{i+k}, d\}$.

Suppose, in fact, that these conditions are met for $i \leqq m < t$. If $m = 2r$ consider the point P_x such that $P_rP_x = a_{r+1}$. It is clear from (2) that $x > 2r$.

If $d \leqq a_{r+2}$ then monotonicity and the fact that $P_0P_r = a_r$ while $P_0P_t = a_{r+2}$ imply that either $P_0P_i = d$ for some $i < t$, an impossibility, or that $2r < t \leqq r + 2$. This is possible only if $r < 2$, i.e., $P_0P_3 = d$ hence $a_k = a_3$. But it is easy to show that $a_k = d = a_3$ is an impossibility. Hence $d > a_{r+2}$. If $x > 2r+1$ then $P_rP_y = d$ for some y, $r < y < x$. This in turn implies that $P_rP_n < P_rP_x = a_{r+1}$. But this is impossible since $P_rP_n \geqq \min\{a_{r+k}, d\} \geqq a_{r+2}$. So $x = 2r+1$. We now show that $P_{m+1}P_m, P_{m+1}P_{m-1}, \cdots, P_{m+1}\,P_r$ is a strictly monotone increasing sequence. This follows from the fact that $P_{m+1}P_r = a_{r+1} \neq d$ and diametral segments must intersect. Since $P_{m+1}P_r = a_{r+1}$ it follows that $P_{m+1}P_i = a_{m+1-i}$, for $r \leqq i < m + 1$.

It is immediate that $P_0P_1 \cdots P_{m+1}$ are concyclic since the triangle $P_{m+1}P_mP_{m-1}$ is congruent to the triangle $P_{m-2}P_{m-1}P_m$. This implies that $P_\alpha P_\beta = a_{|\alpha-\beta|}$ for $\alpha, \beta \leqq m + 1$, $|\alpha - \beta| \leqq \left[\frac{2r+2}{2}\right] = r + 1$. Finally $P_nP_{m+1} \geqq \min\{d, a_{k+m+1}\}$ since otherwise we would have $P_nP_m = d$ while no diametral segment from P_{m+1} meets the diametral segment $\overline{P_nP_m}$.

Now suppose that $m = 2r + 1$ and let P_x be such that $P_rP_x = a_{r+2}$. From the previous case we know that $d > a_{r+2}$ and it follows that $P_rP_n > a_{r+2}$. Hence $x = n$. If $x > m + 1$, there must be a y such that $P_rP_y = d$, where $r < y < x$. But this means that $P_rP_n < a_{r+1}$, a contradiction. Hence $x = m + 1$. Exactly as in the previous case we conclude that $P_{m+1}P_i = a_{m+1-i}$ for $r \leqq i < m + 1$, $P_0, P_1, \cdots, P_{m+1}$ are concyclic, and $P_\alpha P_\beta = \alpha_{|\alpha-\beta|}$ for $\alpha, \beta, \leqq m + 1$, $|\alpha - \beta| \leqq \left[\frac{2r+3}{2}\right] = r + 1 < r + 2$.

This completes the proof of Theorem 5.

THEOREM 6. *A metrically homogeneous polygon in E_2 is cyclic.*

Proof. The proof is somewhat detailed and is organized into six main cases, the first and most involved being that in which the polygon has a side of length 2 followed by at least five successive sides of length 1, a five-plus run as we call it. The remaining cases are concerned in order with the four, three, two and one-runs respectively and that in which the polygon is equilateral. In each analysis the monotonicity lemma and various comparison lemmas allow us to obtain considerable information about the combinatorial structure of the polygon, while at strategic junctures a euclidean calculation or two, as represented by Lemmas 6–8, allow us to rule out unwanted structure and conclude that the polygon is not only cyclic but highly regular.

Case I. Let the vertices of the polygon be labelled $P_0, P_1, \cdots, P_n$ in cyclic order and suppose that $P_0P_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = a_1$ and $P_nP_0 = a_2$. We also assume that $d > a_3$ and $n > 5$. The excluded cases are dealt with separately.

It is immediate that $P_1P_3 = a_2$. For suppose $P_1P_x = a_2$. If $x = n$ then the monotonicity lemma implies $d = a_2$, which contradicts the assumptions; hence $P_1P_n > a_2$. Supposing $x > 3$, let i be such that $P_1P_i = d$. Then $i > x$ is impossible by the monotonicity lemma, since it would imply $a_1 = P_1P_2 < P_1P_3 < P_1P_x = a_2$. But $x > i \geqq 3$ is also impossible, since it would imply $a_2 = P_1P_x \geqq P_1P_n \geqq a_3$. Since $i \neq x$, we have exhausted all the cases, and established $P_1P_3 = a_2$.

Subcase (i): $P_0P_2 = a_2$. Consider P_x such that $P_2P_x = a_3$. If $x = n$ then, by monotonicity, $P_nP_1 = d$. Since $d > a_3$, monotonicity implies $P_nP_5 = 0$, contradicting the assumption $n > 5$. Hence $x < n$ and $P_2P_n > a_3$. Suppose $x > 5$. Monotonicity again implies that $P_2P_i = d$ for some $i > x$, and thus $P_2P_4 = a_1$. This is impossible, hence $x = 4$ or $x = 5$.

Suppose $x = 5$, *i.e.*, $P_2P_5 = a_3$. Then (by monotonicity) $P_2P_4 = P_3P_5 = a_2$ and by congruence (Lemma 2) $P_0P_3 = P_2P_5 = P_1P_4 = a_3$.

Now, if $P_2P_n = a_4$, we claim that $P_1P_n = a_3$. Indeed, if not, monotonicity would imply $P_1P_n = d > a_3$, and if $d > a_4$, no diametral segment from P_2 would intersect $\overline{P_1P_n}$. If $d = a_4$, then $P_2P_6 = 4$. Now, if $n \neq 6$, it follows that $P_3P_n = a_4$. Noting that the points $P_0, P_1, P_2, P_3, P_4, P_5$ are concyclic, we see that P_n is the center of the circle through them (since $P_nP_1 = P_nP_2 = P_nP_3 = a_4$), — a manifest impossibility since $P_nP_0 = a_2$. On the other hand, if $n = 6$ the same concyclicity argument implies $P_6P_3 < a_4$; hence $P_3P_6 = a_3$, $P_4P_6 = a_2$, and $P_5P_6 = a_1$.

Thus the points $P_0, \cdots, P_6$ are all concyclic and are 7 of the 8 vertices of a regular octagon. But then $a_4 = P_1P_6 < P_2P_6 = a_4$. The contradiction establishes our assertion that $P_1P_n = a_3$; by congruence we also have $P_1P_5 = a_4$.

If, on the other hand, $P_2P_n \neq a_4$ then $P_2P_n > a_4$, and it follows that $P_2P_6 = a_4$, $P_3P_6 = a_3$, $P_4P_6 = a_2$, $P_5P_6 = 1$. Then, by congruence, we have $P_1P_5 = a_4$.

Therefore in any event, regardless of the assumption about P_2P_n, we have $P_1P_5 = P_0P_4 = a_4$.

Now, if the polygon fails to be cyclic, there exist three vertices P_i, P_j, P_k such that either $P_iP_j = P_jP_k = a_1$ and $P_iP_k > a_2$, or $P_iP_j = a_1$. $P_jP_k = a_2$ and $P_iP_j > a_3$.

In the first case any triangle $P_xP_yP_z$ such that $P_xP_y = P_yP_z = a_1$ and $P_xP_z = a_2$ has $\measuredangle P_xP_yP_z < 120°$ by Lemma 1. Note now that the points $P_0, P_1, \cdots, P_5$ are concyclic and hence each of the circular arcs $\widehat{P_0P_1}$, $\widehat{P_1P_2}, \cdots, \widehat{P_4P_5}$ is at least 60°. This means that $P_0P_5 = a_1$, which is easily seen to be impossible.

In the second case, if $P_xP_yP_z$ is such that $P_xP_y = a_1$, $P_yP_z = a_2$ and $P_xP_z = a_3$ then $\measuredangle P_xP_yP_z < 120°$, again by Lemma 1. In this case we have the circular arcs $\widehat{P_0P_1}, \widehat{P_1P_2}, \cdots, \widehat{P_4P_5}$ are at least 40° and it follows that $P_0P_6 \leqq a_2$. This, too, is easily seen to be impossible.

This completes the proof in subcase (i).

Subcase (ii): $P_0P_2 \neq a_2$. Let x be such that $P_2P_x = a_2$. If $x = n$ then $P_2P_0 = d$, and no diametral segment from P_n would intersect $\overline{P_2P_0}$; hence $x < n$. If $x > 4$, then y such that $P_2P_y = d$ would satisfy $3 < y < x$; it would follow that $P_2P_n \leqq a_1$, which is clearly impossible. Hence $x = 4$, i.e., $P_2P_4 = a_2$.

If $P_0P_2 > a_3$, then $P_2P_5 = a_3$. Indeed, let z be such that $P_2P_z = a_3$; $z = 0$ would imply that $P_0P_2 = d$, and then the diametral segment from P_3 would not intersect $\overline{P_0P_2}$. Hence $z \neq 0$. If $z > 5$, it would follow that $P_2P_y = d$ for some y such that $4 < y < z$; but then we would have $P_2P_0 < a_3$, which is a contradiction. Therefore $z = 5$, as claimed. Now $P_3P_5 = a_2$; otherwise we would have $P_5P_3 = d$ and this diametral segment would not intersect any diametral segment from P_2. Then $P_1P_4 = a_3$ follows by congruence; thus P_1, P_2, P_3, P_4, P_5 are concyclic, and from the lemmas we infer that the arcs $\widehat{P_1P_2}, \widehat{P_2P_3}, \widehat{P_3P_4}, \widehat{P_4P_5}$ are at least 60° each; hence $P_1P_5 \leqq a_2$, which is clearly impossible.

Therefore we may assume that $P_0P_2 = a_3$. Let x be such that $P_3P_x = a_3$; then $x = 0$ implies $d = a_3$, against our assumptions. If $x > 6$ we would have $P_3P_y = d$ for some y with $4 < y < x$; but then it would follow that $P_3P_1 \leqq a_1$ which is impossible. Hence $x = 5$ or $x = 6$.

Suppose first that $x = 6$; then $P_5P_6 = a_1$, and it readily follows that $P_3P_5 = P_4P_6 = a_2$. By congruence we have $P_1P_4 = a_3$, and P_1, P_2, P_3, P_4, P_5, P_6 are concyclic. The lemmas imply that the arcs $\widehat{P_1P_2}$, $\widehat{P_2P_3}, \cdots, \widehat{P_5P_6}$ are at least 60° each; hence $P_6P_0 = a_1$, which is impossible.

Suppose therefore that $x = 5$. Then $P_1P_4 = a_3$ is impossible by Lemma 5, hence we may assume that $P_1P_4 \geqq a_4$. By the monotonicity lemma it follows that $P_1P_n = a_3$. The comparison lemma shows that if $P_0P_3 > a_4$ then $P_3P_6 = a_4$, and then that $P_5P_6 > a_1$; hence $P_5P_6 = a_2$. But then $P_2P_x = a_4$; indeed, if we had $P_2P_x = a_4$ for some x with $x < n$, it would follow that $P_2P_n \leqq a_3$, contradicting $P_0P_2 = a_3$. Now, $P_1P_4 > a_4$ is impossible, since it would imply $P_1P_{n-1} = a_4$, which contradicts Lemma 5. Hence we may assume $P_1P_4 = a_4$. But then we have the configuration of Lemma 7, which is impossible in a metrically homogeneous polygon.

Hence $P_0P_2 \neq a_2$ is impossible, and the proof in subcase (ii) is completed.

Turning now to the still remaining special cases, we first observe that if $n = 5$ the only possible configuration is that of six vertices of a regular heptagon, which is clearly cyclic. If $n > 5$ but $d = a_3$, then necessarily $P_1P_3 = a_2$, $P_1P_4 = a_3 = P_0P_3$. If we had $P_0P_2 = a_3$, it would follow that $P_2P_4 = a_2$ contradicting the comparison lemma. Hence $P_0P_2 = a_2$, and similarly $P_2P_4 = a_2$. Then clearly $P_2P_5 = a_3$, $P_3P_5 = a_2$, and $P_2P_n = a_3$. By monotonicity, also $P_nP_3 = a_3$. Therefore $P_0, P_1, \cdots, P_5$ are concyclic, and P_n is the center of the circle through them. But since $P_nP_0 = a_2 \neq a_3 = P_nP_1$, this shows that $n > 5$ and $d = a_3$ are incompatible.

This completes the proof of Theorem 6 in Case I.

Case II. We consider now a 4-run, satisfying $P_0P_1 = P_1P_2 = P_2P_3 = P_3P_4 = a_1$ and $P_nP_0 = P_4P_5 = a_2$. It is immediate that $P_1P_3 = a_2$ and that at least one of P_0P_2 and P_2P_4 is a_2. Without loss of generality we shall assume that $P_0P_2 = a_2$. If $d = a_3$, it follows at once that $P_nP_1 = P_3P_5 = a_3$. Since diametral segments must intersect, this would imply $n = 5$ which is impossible (since $P_nP_x = a_1$ would have no solution). Thus we may assume $d > a_3$.

We consider first the possibility $P_2P_4 > a_2$. By monotonicity we have $P_2P_5 \geqq a_4 \leqq P_2P_n$, hence $P_2P_4 = a_3$. Now, $P_3P_0 = a_3$ is impossible by Lemma 5 applied to P_0, P_1, P_2, P_3, P_4. Therefore $P_3P_5 = a_3$. Since $P_0P_3 \geqq a_4$, it follows from the comparison lemma that $P_1P_4 \geqq a_5$; hence $P_1P_n = a_3$. Moreover, we must have $P_1P_{n-1} = a_4$, and so $P_{n-1}P_0 = a_3$. The comparison lemma shows that $P_2P_5 > P_2P_n$. Hence $P_2P_n = a_4$. Lemma 7 implies that the points P_{n-1}, P_n, P_0, P_1, P_2 are concyclic; since P_0, P_1, P_2, P_3 are clearly concyclic or well, it follows that P_{n-1}, P_n, P_0, P_1, P_2, P_3 are on one circle. But this is impossible since the arc $\widehat{P_0P_3} = \widehat{P_0P_1} + \widehat{P_1P_3}$ should equal the arc $\widehat{P_nP_1} = \widehat{P_nP_0} + \widehat{P_0P_1}$, while $P_0P_3 \geqq a_4 > a_3 = P_1P_n$. Thus the assumption $P_2P_4 > a_2$ leads to a contradiction.

On the other hand $P_2P_4 = a_2 = P_0P_2$ is also impossible. Indeed, without loss of generality we may assume $P_nP_2 = a_3$, and the monotonicity then implies $a_2 < P_nP_1 < a_3$.

This completes the proof in Case II.

Case III. Next we consider a 3-run, satisfying $P_0P_1 = P_1P_2 = P_2P_3 = a_1$

and $P_nP_0 = P_3P_4 = a_2$. The case in which $d \leqq a_3$ is easily disposed of, so we assume $d \geqq a_4$. Clearly $P_1P_3 = P_2P_0 = a_2$, and either $P_2P_4 = a_3$ or $P_2P_n = a_3$. But $P_2P_n = a_3$ would imply $P_1P_n = d$, and this diametral segment could not intersect any diametral segment from P_2. Hence $P_2P_4 = a_3$, and similarly $P_1P_n = a_3$.

Assume now $P_0P_3 = a_3$; then the points $P_n, P_0, P_1, P_2, P_3, P_4$ are concyclic. If the entire polygon is not cyclic, points P_i, P_j, P_k exist such that either $P_iP_j = P_jP_k = a_1$ and $P_iP_k > a_2$, or $P_iP_j = a_1$, $P_jP_k = a_2$ and $P_iP_k > a_3$. In the first case, by Lemma 3 applied to the points P_0, P_1, P_2, P_3, we have $\measuredangle P_pP_1P_2 < 120°$. This implies that the arc $\widehat{P_0P_1}$ is more than 60°, and that the arc $\widehat{P_nP_0 \cdots P_5}$ is greater than 360°, — which is impossible. In the second case, Lemma 3 applied to $P_2P_3P_4P_5$ implies $\measuredangle P_2P_3P_4 < 120°$, which in turn means that the arc $\widehat{P_2P_3}$ is greater than 40°, and that the arc $\widehat{P_nP_0 \cdots P_4}$ is greater than 280°; hence $P_nP_4 = a_1$, which is easily seen to be impossible. Thus, in the present case, the polygon is cyclic, as required.

Assume next that $P_0P_3 \geqq a_4$; it follows at once that $P_3P_5 = a_3 = P_0P_{n-1}$ and that $n - 1 > 5$. Also, $P_0P_3 < P_1P_{n-1} = P_2P_5$, since otherwise the configuration of Lemma 8 would result. Thus $P_2P_5 = P_1P_{n-1} > a_4$, and therefore $P_2P_n = a_4 = P_1P_4$. If $d = a_5$ it would follow that $\overline{P_2P_5}$ and $\overline{P_1P_{n-1}}$ are disjoint diametral segments; hence $d > a_5$. Next we claim that $P_2P_x = a_5$ implies $x = 5$; indeed, $P_2P_{n-1} = a_5$ would imply $P_{n-1}P_1 \leqq a_4$, while $P_2P_y = a_5$ for some y with $5 < y < n - 1$ would imply $P_2P_{n-1} \leqq a_4$, both of which are impossible. Hence $P_2P_5 = P_1P_{n-1} = a_5$, and therefore $P_0P_3 = a_4$. Now, $P_3P_i = a_5$ implies $i = 6$ or $i = 7$; indeed, assuming $i > 7$ we would have $P_3P_j = d$ for some j with $5 < j < i$, hence $P_2P_{n-1} \leqq a_5$ which is impossible.

Suppose first that $i = 7$, i.e., $P_3P_7 = a_5$. It follows at once that $P_5P_6 = P_6P_7 = a_1$, $P_3P_6 = a_4$, $P_4P_6 = P_5P_7 = a_2$, hence $P_3P_7 = P_0P_4 = P_nP_3 = a_5$. But the comparison lemma applied to the quadrilaterals $P_2P_3P_4P_5$ and $P_0P_2P_3P_4$ shows that $P_0P_4 > P_2P_5$. Hence $i \neq 7$.

Now suppose $i = 6$. Then, by the comparison lemma, $P_5P_6 = a_1$. Points P_2, P_3, P_4, P_5 are concyclic, as are P_2, P_3, P_5, P_6; thus P_2, P_3, P_4, P_5, P_6 are concyclic, hence $P_4P_6 < P_3P_5 = a_3$, i.e., $P_4P_6 = a_2$. Then it follows that P_1 and P_0 are concyclic with the previously mentioned points. But this is impossible since it would imply that $a_4 = P_0P_3 = P_2P_4 = a_3$. Hence $P_0P_3 > a_3$ is not possible, and the proof in case III is completed.

Case IV. The assumption of a 2-run $P_0P_1 = P_1P_2 = a_1$, $P_nP_0 = P_2P_3 = a_2$, is easily seen to be impossible, since $d > a_2$ and $P_1P_x = a_2$ has no solution.

Case V. We turn now to the case of 1-runs, assuming $P_0P_1 = P_2P_3 = \cdots = P_{n-2}P_{n-1} = a_1$ and $P_1P_2 = P_3P_4 = \cdots = P_nP_0 = a_2$, and $n \geqq 5$.

We first show that a configuration Q, consisting of four points A, B, C, D with

$AB = CD = a_1$, $AC = BD = a_3$, must occur among the vertices of the polygon. Indeed, suppose that Q does not occur. Without loss of generality we may assume that $P_0P_2 = a_3$, and then $P_1P_3 = a_4$ (otherwise either P_{n-1}, P_n, P_0, P_1, or P_0, P_1, P_2, P_3, would be in configuration Q); hence $P_3P_5 = a_3$. Now, $P_4P_7 = a_4$ would yield Q (with points P_4,P_5,P_6,P_7), while $P_4P_6 = a_4$ would imply $P_4P_2 = a_3$ and Q would occur (with P_2,P_3,P_4,P_5). Therefore we necessarily have $P_4P_2 = a_4$ and $P_4P_6 = a_3$. Continuing (by induction) we see that $P_1P_3 = P_2P_4 = P_5P_7 = P_6P_8 = \cdots = P_{n-2}P_n = P_{n-1}P_0 = a_4$, and $P_3P_5 = P_4P_6 = P_7P_9 = \cdots = P_nP_1 = P_0P_2 = a_3$. It is immediate that the polygon is a $(4k)$-gon, and that $P_0,P_4,P_8,\cdots,P_{n-3}$, and $P_1,P_5,P_9,\cdots,P_{n-2}$, are vertices of congruent regular k-gons with a common center; hence the two sets of points are on a circle (with center 0, and radius R). Similarly, $P_2,P_6,P_{10},\cdots,P_{n-1}$, and $P_3,P_7,P_{11},\cdots,P_n$, are vertices of congruent regular k-gons inscribed in a circle of center 0 and radius r. Since $P_0P_4 > P_2P_6$ it follows that $R > r$.

We propose now to show that these two concentric but unequal circles circumscribe congruent triangles. This being obviously impossible, the existence of configuration Q will be established. We shall prove that the triangles $P_0P_1P_4$ and $P_2P_3P_7$ are congruent; to do this, we must ascertain some additional distances. Note, first, that if $d = a_4$ then $n \leqq 5$ and the impossibility is easily checked. Next, $P_4P_1 > P_4P_7$ implies $P_4P_7 = a_5 = P_6P_9 = P_2P_5 = \cdots$, and $d > a_5$. Since $P_4P_8 > P_4P_1$, it follows that $P_4P_1 = a_6$ and $d > a_6$. The fact that $P_4P_0 = P_4P_8$ means that both must equal a_7. Similarly $P_6P_2 = P_6P_{10} = a_6$ since $P_6P_3 < P_1P_4 = a_6$. Finally, $P_6P_1 > P_1P_5 = P_0P_4 = a_7$, and since $P_6P_2 < P_1P_5 = a_7$, it follows that $P_6P_{11} = a_7 = P_2P_7$. Hence the two triangles mentioned above are congruent, and we have proved that configuration Q must occur.

Therefore we may assume $P_0P_2 = P_1P_3 = a_3$. From the fact that $P_0P_3 < P_3P_6$ we see $P_3P_6 > a_4$. If $P_3P_0 \neq a_4$ then $P_0P_3 \geqq a_5$ and $P_3P_5 = a_4$; also, $P_2P_n \geqq P_0P_3 \geqq a_5$, which implies $P_2P_4 = a_4$. Similarly, $P_1P_{n-1} \geqq P_0P_3 \geqq a_5$, hence $P_1P_n = a_4$. Now $P_2P_x = a_5$ has no solution since $P_2P_5 > P_3P_0 \geqq a_5$ and, by Lemmas 1 and 2, $P_2P_n > P_3P_0 \geqq a_5$. Thus P_3P_0 necessarily equals a_4.

Now, by Lemma 6, $P_2P_4 = a_3$ and, since $P_2P_n > P_0P_3 = a_4$, we have $P_2P_5 = a_4$ and $P_3P_5 = a_3$. It follows by induction that all second order diagonals have length a_3, and thus the polygon is cyclic.

Case VI. There remains only the case in which $P_0P_1 = P_1P_2 = \cdots = P_nP_0 = a_1$. We may assume that $P_0P_2 = P_1P_3 = a_2$. If $P_0P_3 = a_3$, then $P_2P_4 = a_2$ (otherwise the configuration of Lemma 6 would occur). Now $P_1P_4 = a_3$ by congruence, and it follows by induction that all second order diagonals have length a_2. Hence the polygon is cyclic, as required.

On the other hand, if $P_0P_3 \neq a_3$ then $P_0P_3 \geqq a_4$. But $P_2P_5 \geqq P_0P_3$, and similarly $P_2P_n \geqq a_4$. Hence $P_2P_4 = a_3$, $P_3P_5 = a_3$, and $P_1P_n = a_3$. Then $P_2P_x = a_4$

has no solution, since $P_2P_5 > P_2P_n > P_0P_3 \geqq a_4$ by the comparison lemma. The contradiction reached completes the proof in Case VI.

Thus Theorem 6 is proved.

In order to characterize finite metrically homogeneous sets conveniently, we need the following:

DEFINITION. If the set of vertices of a convex polygon P may be represented as $A - B$, where each of A and B is the set of all vertices of some regular polygon, $A \supset B$, card $A = 2k(m+1)$, and card $B = m$, then P is called an *evenly truncated regular polygon* of type (k, m).

Note that a quasi regular polygon can, but need not, be an evenly truncated regular polygon.

Now we are ready for the characterization theorem:

THEOREM 7. *A finite set in E_2 is metrically homogeneous if and only if it is the vertex set of a regular, quasi regular, or evenly truncated regular polygon.*

Proof. It is easily checked that the sets mentioned are metrically homogeneous. Suppose therefore that $P_0, P_1, \cdots, P_n$ are the vertices of a metrically homogeneous polygon P, labeled in the cyclic order. By Theorem 6, if all edges of P have length a_1, all the vertices of P are concyclic and thus P is regular. If among the edges of P of P there is no k-run with $k \geqq 2$, but some edge has length a_2, then all edges of P of length a_1 belong to 1-runs; since all the 1-runs are congruent, and the points of each are concyclic, all vertices of P are concyclic and P is quasi regular. Thus we are left with the case in which P contains some k-run, $k \geqq 2$. By Theorem 6, each m-run, $m \geqq 1$, contains some three points P_i, P_j, P_k such that $P_iP_j = a_1$, $P_jP_k = a_2$, and $P_iP_k = a_3$. Since all the vertices of each m-run are concyclic, this implies that the radius of the circle is independent of m. Therefore all points of neighboring runs (which share an edge of length a_2) are concyclic, and thus all vertices of P belong to the same circle.

In order to complete the proof we shall show that each run is of odd length, and that all runs have the same length.

Indeed, assuming $P_0P_1 = P_1P_2 = \cdots = P_{2k-1}P_{2k} = a_1$ and $P_nP_0 = P_{2k}P_{2k+1} = a_2$, there is no solution x to $P_kP_x = P_0P_{k+1} = a_{k+1}$; hence P is not metrically homogeneous.

In order to show that all runs have same length, assume that $P_nP_0 = a_2$, $P_0P_1 = P_1P_2 = \cdots = P_{2k-2}P_{2k-1} = a_1$, and $P_{2k-1}P_{2k} = P_{2k+2m-1}P_{2k+2m} = a_2$, with $k > m$. Then there is no x such that $P_{k+m}P_x = P_0P_{k+m+1} = a_{k+m+1}$, hence P is not metrically homogeneous.

This completes the proof of Theorem 7.

As an immediate consequences we have:

COROLLARY 1. *A compact subset of E_2 is strictly metrically homogeneous if and only if it is either a smooth curve of constant width, or else the vertex set of a regular or quasi regular polygon.*

COROLLARY 2. *Each finite metrically homogeneous subset of E_2 has a non-trivial group of self-isometries.*

REMARK. The structure of metrically homogeneous subsets of E_3 seems to be much more complicated. Even Corollary 2 fails in E_3; the simplest example known to us consists of 10 points, obtained by omitting from the 12 vertices of the Archimedean solid with 4 triangles and 4 hexagons as faces, two vertices which belong to the same hexagon but are neither neighbors nor diametrally opposite.

It is well possible that the characterization of finite strictly metrically homogeneous sets in E_3 is not completely hopeless. In particular, it may be conjectured that the analogue of Corollary 2 is valid in E_2 if restricted to strictly metrically homogeneous sets.

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