THE FACES OF A REGULAR-FACED POLYHEDRON

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This paper is concerned with the problem of determining all *regular-faced polyhedra, i.e.*, those convex polyhedra in Euclidean 3-space which have as faces only regular polygons (not necessarily all of the same kind). Examples of such polyhedra are the 5 Platonic (regular) solids, the 13 Archimedean (semi-regular) polyhedra, and the infinite families of regular prisms and antiprisms. Each of these solids is *uniform, i.e.*, its faces are regular and its symmetry group is transitive on the vertices; it was shown by Kepler [**4**, pp. 114–127] that these are the only convex uniform polyhedra.

A *face angle* at a vertex V of a polyhedron is the angle at V of one of the faces meeting at V. The following properties of a convex polyhedron are well known: (a) Each face angle at a vertex is less than the sum of the remaining face angles at that vertex. (b) The sum of all the face angles at a vertex is less than 360°; the difference is called the *deficiency* at the vertex. (c) The sum of the deficiencies at all the vertices is 720°.

By means of these properties, Johnson [**2**] and Zalgaller [**5**] independently proved that if a regular-faced polyhedron has a face of 42 or more sides, it is either a prism or an antiprism. With an upper bound on the number of sides to a face and hence on the number of vertices of a regular-faced polyhedron that is not a prism or an antiprism, there follows

Theorem 1. *The number of nonuniform regular-faced polyhedra is finite.*

A number of nonuniform regular-faced solids (83, in fact) can be obtained by cutting certain uniform polyhedra by appropriate planes and by putting together various uniform polyhedra and pieces of uniform polyhedra. A regular-faced polyhedron is *elementary* if it is not the union of two regular-faced polyhedra that have a common face. Zalgaller [**5**, pp. 7–8] lists 9 elementary nonuniform polyhedra (each of which is part of a uniform polyhedron). Johnson [**3**] has found a total of 92 nonuniform regular-faced solids, of which 17 are elementary.

The aim of the present paper is to bring nearer the complete enumeration of convex polyhedra with regular faces, by establishing

Theorem 2. *The only polygons that may occur as faces of a regular-faced polyhedron, other than a prism or an antiprism, are triangles, squares, pentagons, hexagons, octagons, and decagons.*

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We shall prove Theorem 2 by showing that polygons different from those mentioned in the theorem can occur only in prisms and antiprisms.

It is easily verified that the minimal deficiency at a vertex of a regular-faced polyhedron having only faces with 3, 4, 5, 6, 8, or 10 sides is 6°, and, since the total deficiency is 720°, such a polyhedron has at most 120 vertices. This bound is the best possible, being attained for the Archimedean *truncated icosidodecahedron*, which has at each vertex a square, a hexagon, and a decagon.

A regular polygon of n sides will be denoted by the *Schläfli symbol* $\{n\}$. A vertex surrounded by faces $\{n_1\}$, $\{n_2\}$, ..., $\{n_k\}$, in cyclic order, will be said to be of type $(n_1.n_2.\ldots.n_k)$. If a polyhedron is uniform, all its vertices are of the same type,† and the solid itself may be denoted by the vertex-type symbol without parentheses. Thus, the truncated icosidodecahedron is $4.6.10$, the n-gonal prism is $4^2.n$, the n-gonal antiprism is $3^3.n$, etc. A vertex may also be classified as trivalent, tetravalent, or pentavalent, according as there are 3, 4, or 5 faces surrounding it.

LEMMA 1. *A regular-faced polyhedron that contains a face $\{n\}$, where $n \geqslant 12$, is either the n-gonal prism or the n-gonal antiprism.*

Proof. (i) An n-gonal face N, $n \geqslant 12$, cannot be incident with a k-gonal face K with $k \geqslant 6$. For, assuming such an incidence possible, let A be a vertex incident with N and K (see Fig. 1). Considering the face angles at A, we see that A is a trivalent vertex and that the third face incident with A is a triangle T. Let B be the other vertex of T incident with N, and C the other vertex of T incident with K. The vertex B must be trivalent (since the sum of the face angles of the additional faces would have to be greater than 120° and less than 150°), and therefore the third face must be a k-gon K′. It follows that vertex C is also trivalent, and this implies that $n = k$. But this is impossible, since the sum of the face angles at a vertex must be less than 360°. Thus

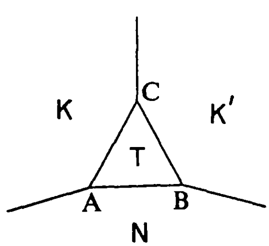

FIG. 1

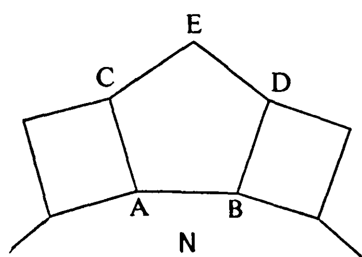

FIG. 2

† The converse is not true, however: there are two solids whose vertices are all of type (3.4^3), but only one is uniform. The existence of the extra figure [3, Table III, No. 37] has led Aškinuze [1] and, following him, several other Russian writers, *e.g.*, Zalgaller [5, p. 7], to claim that there are actually 14 " Archimedean " polyhedra.

(i) is proved, and for every polyhedron containing a face $\{n\}$, $n \geqslant 12$, each such face is incident only with faces having at most five sides. However,

(ii) An n-gonal face N, $n \geqslant 12$, cannot be incident with a pentagon. For, if N were incident with a pentagon, it would have to possess vertices of one of the types $(4.5.n)$ and $(3.5.n)$. A vertex of type $(4.5.n)$ is impossible, since (see Fig. 2) if A were such a vertex, the neighbouring vertex B would necessarily be of the same type; this means that the adjacent vertices of the pentagon, C and D, would have to be trivalent and therefore also of type $(4.5.n)$. But then the sum of the face angles at E, the opposite vertex of the pentagon, would be greater than 360°.

If a vertex A of type $(3.5.n)$ were to occur (see Fig. 3), then for the neighbouring vertex B we would have only the two possibilities $(3.5.n)$ or $(3^3.n)$. In the first case a contradiction is reached at the third vertex of the triangle, C, which on the one hand is not trivalent but on the other hand allows no regular polygon to fit as the fourth face. But the second case is also impossible, since convexity would be violated at B (see Fig. 4). (Note that $\angle CBD$ would be 108°.)

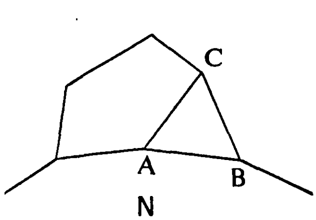

FIG. 3

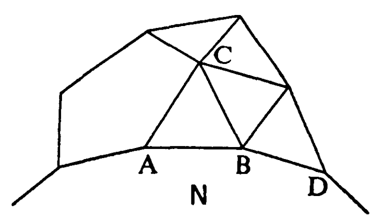

FIG. 4

It follows that every n-gonal face, $n \geqslant 12$, is incident only with triangles and squares.

(iii) If a regular-faced polyhedron has an n-gonal face N, $n \geqslant 12$, incident with a square, it is the n-gonal prism. For, a vertex incident with a $\{4\}$ and an $\{n\}$ must be trivalent if $n \geqslant 12$, and, by (i) and (ii), the third face has at most four sides. A triangle cannot occur since then the sum of two of the face angles would not exceed the third. Thus every vertex of N is of type $(4^2.n)$, and the edges of the squares opposite to those of N form another regular n-gon N′. If N′ is not a face of the polyhedron, then at each vertex of N′ there must be, in addition to the two squares incident with N, at least two other faces with angles totalling more than 150°. As these faces would be incident with N′ in a regular-faced polyhedron obtained by cutting the given polyhedron along the plane of N′, they can consist only of triangles and squares. Thus if the angle sum of the additional faces is to exceed 150°, it must be at least 180°, making the sum of all the face angles at the vertex at least 360°. Therefore N′ is a face, and the polyhedron is the n-gonal prism.

(iv) If a regular-faced polyhedron has an n-gonal face N, $n \geqslant 12$, incident only with triangles, it is the n-gonal antiprism. For, all the vertices of N must be of type $(3^3.n)$. Let the vertices of the triangles opposite the edges of N be $V_1, V_2, \ldots, V_n$ (see Fig. 5). Let α_i be the angle $V_{i-1}V_iV_{i+1}$, where $V_i = V_j$ if $i \equiv j \pmod{n}$. Then, if $\alpha_i = \alpha_{i+1}$ for some i, all the angles α_i are equal and $V_1V_2\ldots V_n$ is a regular n-gon N′. As in (iii), N′ must be a face of the polyhedron, which is therefore the n-gonal antiprism. Thus it may be assumed that $\alpha_i \neq \alpha_{i+1}$ for any i. As is easily seen, $\alpha_i = \alpha_{i+2}$ for all i; if $\alpha_i \neq \alpha_{i+1}$, n must be even. Also, $120° < \alpha_i < 180°$ and $\max\{\alpha_i, \alpha_{i+1}\} > 150°$ for all i. Without loss of generality, we may assume $\alpha_1 < \alpha_2$. If V_1 were tetravalent, say of type $(3^3.k)$, then the inequality $\alpha_1 = 180°.(k-2)/k < \alpha_2$ implies that V_2 would have to be of type $(3^3.k.h)$. Since $\alpha_1 > 120°$, *i.e.*, $k > 6$, this would yield at V_2 a sum of face angles greater than 360°. It follows that V_1, and consequently every V_i, is pentavalent. They cannot be of types (3^5) or $(3^4.4)$, since $\alpha_2 > 150°$. The only remaining possibility is that all the vertices V_i are of type $(3^4.5)$. The next tier of vertices U_3, U_5, etc., are then of type $(3.5.3.5)$. The edges W_2W_4, W_4W_6, etc., are equi-inclined to the edges V_2V_3, V_4V_5, etc., and thus form a regular polygon $\{\frac{1}{2}n\}$. The vertices W_2, W_4, etc., are tetravalent, since $\angle W_2W_4W_6$ is at least 120° for $n \geqslant 12$. The additional face must have angles less than 132°; *i.e.*, n is either 12 or 14. But the impossibility of these two cases is readily established (either by computation or by constructing cardboard models).†

This completes the proof of Lemma 1.

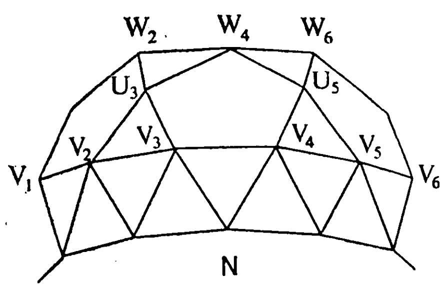

FIG. 5

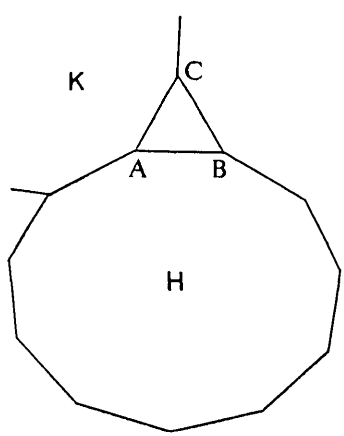

FIG. 6

† The above construction is possible for $n = 10$, in which case it yields the *gyro-elongated pentagonal rotunda* [3, Table III, No. 25].

LEMMA 2. *A regular-faced polyhedron that contains a face* $\{11\}$ *is either the hendecagonal prism or the hendecagonal antiprism.*

Proof. (i) A hendecagonal face H cannot be incident with a k-gonal face K with $k \geq 7$. For, any vertex A incident with both H and K would be trivalent, the third face being a triangle (see Fig. 6). By (i) in the proof of Lemma 1, we may assume that $k \leq 11$. Obviously, neither of the other vertices of the triangle, B and C, can be trivalent, unless $k = 11$ and the other face incident with the edge BC is also an $\{11\}$. But even this is impossible since no Archimedean solid 3.11^2 exists; thus we may assume that both B and C are at least tetravalent. Since the largest face that may occur in a pentavalent vertex is a pentagon, B and C must be tetravalent, the two additional faces at B being a triangle and a square, and the additional faces at C being either a triangle and a square or (if $k = 7$) a triangle and a pentagon. But, as is easily verified, the faces fit only if $k = 11$, both B and C then being incident with two $\{3\}$'s, a $\{4\}$, and an $\{11\}$. The two possibilities for the arrangement of these faces are represented in Figs. 7 and 8. In the case of Fig. 7, no single

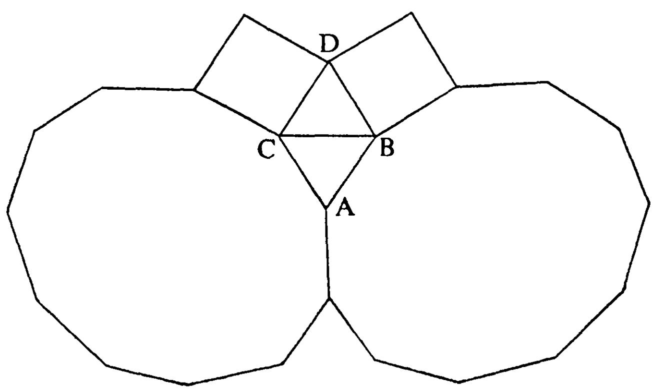

FIG. 7

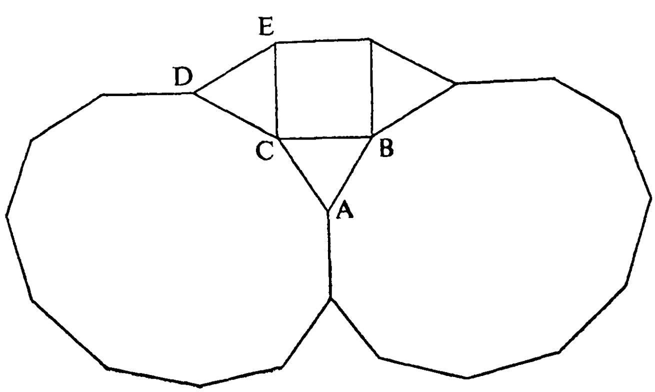

FIG. 8

face nor any combination of faces fits at D. In the case of Fig. 8, although the free angle at D is equal to the angle of an $\{11\}$, the vertex cannot be completed with a hendecagon, since then there would be no way to complete vertex E. The only other possibility would be to close up D with a triangle and a square, but neither of the two conceivable arrangements allows vertex E to be completed.

Thus a hendecagonal face can be incident only with triangles, squares, pentagons, and hexagons, and the only types of vertices that need to be considered are:

$$(4.6.11),\ (3.6.11),\ (3.4.11),\ (4^2.11),$$

$$(3.4.3.11),\ (3^2.4.11),\ (4.5.11),\ (3.5.11),\ (3^3.11).$$

We shall complete the proof of Lemma 2 by showing that only vertices of types $(4^2.11)$ and $(3^3.11)$ actually occur, the polyhedron being the hendecagonal prism in the former case and the hendecagonal antiprism in the latter.

(ii) A vertex of type $(4.6.11)$, $(3.6.11)$, or $(3.4.11)$ cannot occur. For, if a vertex of type $(4.6.11)$ were to occur at A (see Fig. 9), then the free angle at B would be 120°; for convexity, B must be trivalent, the third face being a hexagon. Vertex C is also necessarily trivalent, and, the free angle being 90°, the third face is a square. Therefore, hexagons and squares would have to alternate around a hendecagon, which is absurd. Thus a vertex of type $(4.6.11)$ cannot occur.

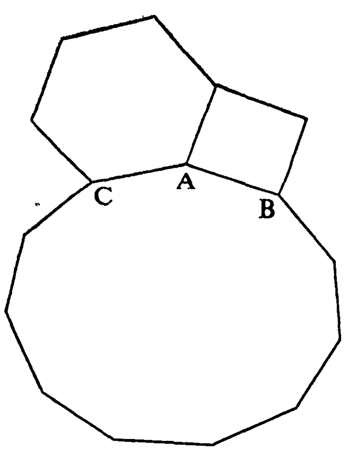

FIG. 9

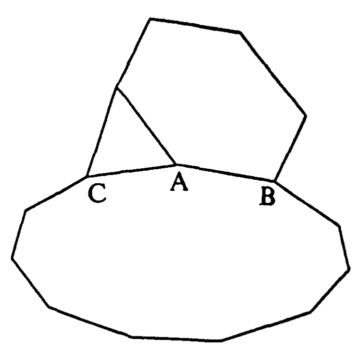

FIG. 10

Similarly, if A (see Fig. 10) were a vertex of type $(3.6.11)$, vertex B would have to be of the same type. For C the possibilities would be $(3.6.11)$, $(3.4.3.11)$, or $(3^2.4.11)$, but both tetravalent types of vertices are ruled out by convexity considerations. Thus hexagons and triangles would have to alternate around a hendecagon; *i.e.*, a vertex of type (3.6.11) cannot occur.

Analogously it can be shown that a vertex of type (3.4.11) cannot occur.

(iii) If a regular-faced polyhedron has a vertex of type $(4^2.11)$, the polyhedron is the hendecagonal prism. For, if A (see Fig. 11) is a vertex of type $(4^2.11)$, a neighbouring vertex B is either of type $(4^2.11)$ or of type $(3^2.4.11)$; the latter case, however, is easily seen not to be convex. Therefore, all the vertices of the $\{11\}$ are of type $(4^2.11)$, and the opposite edges of the squares form another $\{11\}$. This second $\{11\}$ must be a face of the polyhedron, since otherwise two faces—necessarily a triangle and a square—would have to be added at every vertex, but, with an odd number of edges, this is impossible. Thus the polyhedron is the hendecagonal prism.

(iv) A vertex of type (3.4.3.11) or $(3^2.4.11)$ cannot occur. To show this, we consider a vertex A of type (3.4.3.11) and distinguish two cases:

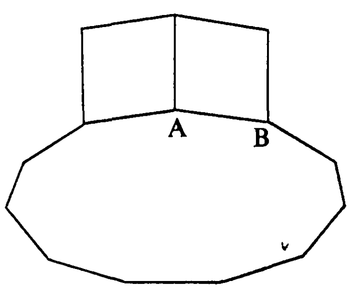

FIG. 11

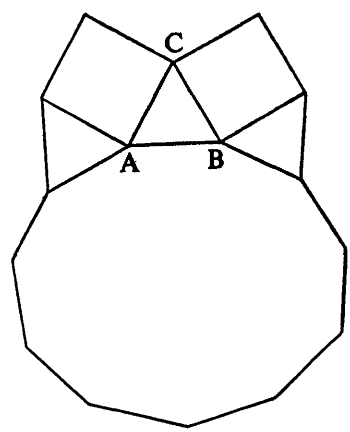

FIG. 12

(a) A neighbouring vertex B (see Fig. 12) is also of type (3.4.3.11). Then vertex C is tetravalent. The fourth face cannot be larger than a pentagon, but it is easily seen that the free angle at C is greater than 108°, so that there is no regular polygon that fits. Hence no two vertices of type (3.4.3.11) are neighbouring.

(b) Neither of the neighbouring vertices is of type (3.4.3.11). Then, since the free angle at B (see Fig. 13) is greater than 108° and vertices of type $(3.k.11)$ with $k \geqslant 6$ are impossible, B cannot be trivalent and must be either of type $(3^3.11)$ or of type $(3^2.4.11)$. The first is impossible because of the requirement of convexity. Thus we are left only with the possibility that both neighbours of A are of type $(3^2.4.11)$.

Now consider a vertex V of type $(3^2.4.11)$ (see Fig. 14). The free angle at the neighbouring vertex U being less than 120° but greater than 108°, U cannot be trivalent and is therefore also of type $(3^2.4.11)$. Likewise, vertex W is necessarily tetravalent, either of type $(3^2.4.11)$ or of

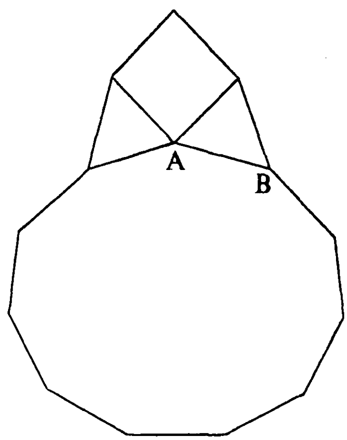

FIG. 13

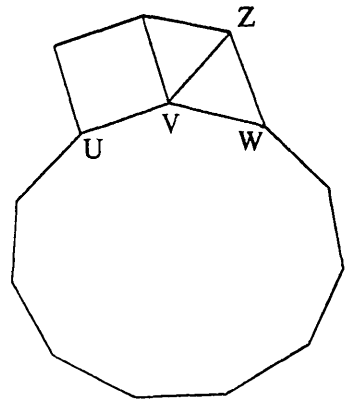

FIG. 14

type $(3.4.3.11)$. The former case, however, is impossible, since it would yield at vertex Z a free angle greater than 168° ($= 60° + 108°$), so that Z could be closed up neither with a single face of 11 or fewer sides nor with any pair of faces; thus W is of type $(3.4.3.11)$.

It follows from the above that a vertex of type $(3.4.3.11)$ is adjacent to two vertices of type $(3^2.4.11)$, and a vertex of type $(3^2.4.11)$ is adjacent to one vertex of the same type and one of type $(3.4.3.11)$. But since 11 is not a multiple of 3, this is impossible.

(v) No vertex of type $(4.5.11)$ can occur since all the vertices of the $\{11\}$ would have to be of the same type, and this is obviously impossible.

(vi) The only remaining types of vertices are $(3^3.11)$ and $(3.5.11)$. We remark first that if all the vertices of a hendecagon are of type $(3^3.11)$, the resulting belt of triangles is rigid (as it is for every polygon with an odd number of sides).

Now consider a face $\{11\}$ with vertices of types $(3.5.11)$ and $(3^3.11)$. Not all of them can be of type $(3.5.11)$ (since 11 is odd), and if all are of type $(3^3.11)$, the polyhedron is the hendecagonal antiprism. If there were vertices of both kinds, then (see Fig. 15) since $\angle ABF = \angle CDH = \angle DCF = 108°$, we could delete each pentagon from the belt around the hendecagon and introduce triangles like CDG and DGH (see Fig. 16). The dihedral angle along DG being the same as that along BE, convexity would not be destroyed, and by symmetry CFG is also an equilateral triangle. Thus it is possible to replace each pentagon in the belt by three triangles; in other words, we obtain from the original belt another one,

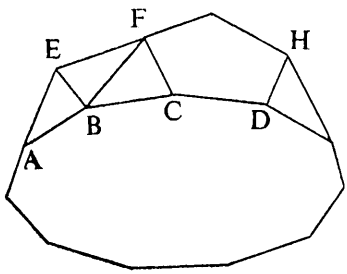

FIG. 15

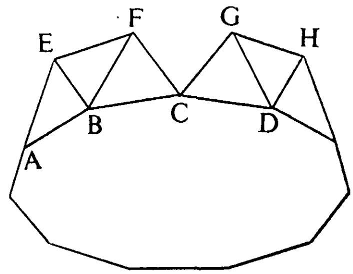

FIG. 16

consisting of triangles only. As noted above, the latter belt is rigid; *i.e.*, it is the belt of the antiprism. But in the hendecagonal antiprism the angles like ABF are not 108°. Hence a combination of vertices of types $(3.5.11)$ and $(3^3.11)$ is impossible.

This completes the proof of Lemma 2.

LEMMA 3. *A regular-faced polyhedron that contains a face* $\{7\}$ *or* $\{9\}$ *is either a prism or an antiprism.*

Proof. There are 32 *a priori* possible types of vertices that include a $\{7\}$ or a $\{9\}$. We shall show that all but four cannot actually occur in a regular-faced polyhedron, the four exceptions leading to the heptagonal and enneagonal prisms and antiprisms.

(i) As in part (ii) of the proof of Lemma 2, in some cases it can be shown that two different kinds of faces must alternate around a polygon with an odd number of sides, which is impossible. This reasoning applies to the following types of vertices:

$$(3.4.7),\ (3.4.9),\ (3.6.9),\ (4.5.9),\ (4.6.7),\ (4.6.9),$$
$$(4.7^2),\ (4.7.8),\ (5^2.7),\ (5^2.9),\ (5.6.7).$$

The necessity of continuing with certain kinds of faces follows from limitations on the free angles and (in some cases) from the fact that certain *a priori* possible continuations yield nonconvex solid angles.

(ii) If there is a vertex of type $(4^2.7)$ or $(4^2.9)$, it is easily seen that all the vertices of the $\{7\}$ or $\{9\}$ are of the same type and that the $\{7\}$ or $\{9\}$ formed by the opposite edges of the squares must be a face, so that the polyhedron is a prism.

(iii) Vertices of type $(3.6.7)$ are impossible. For, the other two vertices of the triangle must be tetravalent, but then the inclinations to the $\{6\}$ and the $\{7\}$ are different, and no combinations of faces will fit at both vertices. The same reasoning applies to vertices of types $(3.7.8)$, $(3.7.9)$, $(3.7.10)$, $(3.8.9)$, and $(3.9.10)$.

(iv) If a vertex of type (3.7^2) were present, it can be shown that each of the other five vertices of either heptagon would have to be incident with one square and two triangles. But the whole configuration, *i.e.*, the belt around one of the heptagons, being rigid, one finds that the belt does not close up. The same reasoning applies to (3.9^2).

(v) Vertices of types $(3.5.7)$ and $(3^3.7)$ or types $(3.5.9)$ and $(3^3.9)$ are dealt with in the same manner as in part (vi) of the proof of Lemma 2: vertices $(3^3.7)$ and $(3^3.9)$ occur only in antiprisms; the other two types do not occur.

(vi) If one of the vertices of a face $\{9\}$ is of type $(3.4.3.9)$ or $(3^2.4.9)$, then all are of these types, and there would necessarily be two adjacent ones of the first kind. However, there is no way to complete the third vertex of the triangle incident with these two vertices. A similar argument rules out vertices of types $(3.4.3.7)$ and $(3^2.4.7)$ and of types $(3.5.3.7)$ and $(3^2.5.7)$.

(vii) If a vertex of type $(4.5.7)$ were to occur, then the neighbouring vertices of the $\{7\}$ would have to be of the same type, since vertices of types $(3^2.4.7)$ and $(3^2.5.7)$ cannot occur. Thus squares and pentagons must alternate around a heptagon, which is impossible.

This completes the proof of Lemma 3 and thus also of Theorem 2.

Zalgaller *et al.* [6] have determined all the regular-faced polyhedra that have at least one trivalent vertex, as well as those having only pentavalent vertices. This enables us to prove

THEOREM 3. *Every nonuniform regular-faced polyhedron has at least four triangular faces.*

Proof. If every vertex is trivalent, the solid is uniform. All the nonuniform regular-faced polyhedra with one or more trivalent vertices being known, it can be directly verified that each of them has at least four triangular faces. If a polyhedron has no trivalent vertices, then by Euler's formula it has at least eight triangular faces.

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