ADDITION AND DECOMPOSITION OF CONVEX POLYTOPES(1,2)

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ABSTRACT

A new addition of convex polytopes is defined and the possibility of representing each polytope as a sum of "standard" polytopes is established

Introduction. Vector addition of convex bodies is very useful in different investigations. However there are problems for which vector addition, though appropriate in the plane case, becomes unsuitable in higher dimensions. An example is the question of decomposability of polytopes. In the plane every polygon is the sum of finitely many summands of a simple type (segments and triangles), and every convex set is the limit, in the Hausdorff metric, of finite sums of triangles; both assertions fail to have analogues in higher dimensions.

Blaschke in [3, p. 112] suggested another composition process for convex bodies which involves the pointwise addition of the products of the principal radii of curvature as functions of the outer normal in the case of sufficiently smooth convex bodies. Even earlier Minkowski [10, pp. 116–117], discussed a corresponding process of composition for polyhedral bodies. Cf. also [4, p. 124]. In the present paper we shall define a new addition for convex polytopes which is a modification of that mentioned by Minkowski and which may be considered as a special case of the addition of generalized curvature functions. Our approach is elementary except that, in section 4, we discuss the generalization of this addition to arbitrary convex sets.

The new addition is such that the known results on planar decomposition in terms of vector addition have valid analogues in higher dimensions. Further, the addition coincides with vector addition in the case of non degenerate polygons and so planar theorems similar to the usual ones are included in a natural way.

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In section 1 we define the new addition which rests on Minkowski's theorem about the determination of a convex polytope by the directions and areas of its faces of maximal dimension. Some related notions are discussed. Sections 2 and 3 contain decomposition results, while section 4 is devoted to variants and generalizations of the addition.

1. **The new addition of convex bodies.** Let P be a k-dimensional convex polytope in Euclidean d-space E^d such that the origin is in the relative interior of P. Let E^k be the k-space spanned by P and let $f(P) \geqq k+1$ denote the number of $(k-1)$-dimensional faces of P. With each such face F_i, $1 \leqq i \leqq f(P)$, we associate a vector N_i as follows:

(i) If $k=1$, i.e. if P is a segment with end points F_1 and F_2, we set $N_i = (-1)^i(F_2 - F_1)$.*

(ii) If $k \geqq 2$, then $N_i \in E^k$, its direction is that of the outer normal to F_i and its length $|N_i|$ equals the $(k-1)$-dimensional content of F_i.

This definition associates a system $\mathcal{N}(P) = \{N_i \mid 1 \leqq i \leqq f(P)\}$ of vectors with every polytope P containing the origin in its relative interior. We extend this association. Clearly if two translation-equivalent polytopes P_1 and P_2 contain the origin as a common relative interior point, then $\mathcal{N}(P_1) = \mathcal{N}(P_2)$. Hence, for any convex polytope P, we may define $\mathcal{N}(P)$ to be $\mathcal{N}(P-Q)$ where Q is any relative interior point of P.

A system $\mathcal{V} = \{V_i \mid 1 \leqq i \leqq n\}$ of non-zero vectors in E^k is called *equilibrated* if $\sum_{i=1}^n V_i = 0$ and if no two members are positively proportional. $\mathcal{V}$ is called *fully equilibrated in* E^k when it is equilibrated and E^k is the span of $\mathcal{V}$.

The following result is well-known and easily proved, cf. [10]:

(M1) If P is a convex polytope in E^d, then $\mathcal{N}(P)$ is equilibrated; moreover if $0 \in \operatorname{int} P$ and P spans E^k, then $\mathcal{N}(P)$ is fully equilibrated in E^k.

Minkowski's existence and uniqueness theorem is much deeper, cf. [10], [11] and also [2], [4]:

(M2) If $\mathcal{V}$ is a fully equilibrated system of vectors in E^k, $k \geqq 2$, there exists a convex polytope P, unique to within a translation, such that $\mathcal{V} = \mathcal{N}(P)$. Clearly P is k-dimensional.

The above facts make it possible to define a composition of convex polytopes called #-addition. More precisely, we define #-addition between the classes of translation-equivalent polytopes (however, see remark 2 of section 4) in the following way:

Let $P_j, j = 1, 2$, be convex polytopes of dimension k_j in E^d and let the indexing of their associated systems

$$\mathcal{N}(P_j) = \{N_i^{(j)} \mid 1 \leqq i \leqq n_j = f(P_j)\}$$

* We do not distinguish a point and the radius vector of the point.

be chosen so that the vectors $N_i^{(1)}, N_i^{(2)}$ are positively proportional for i satisfying $1 \leqq i \leqq n_0 \leqq n$ while no other pair of vectors from $\mathcal{N}(P_i) \cup \mathcal{N}(P_2)$ are positively proportional. Then the system

$$\mathscr{V} = \{V_i \mid 1 \leqq i \leqq n_1 + n_2 - n_0\}$$

defined by

$$V_i = \begin{cases} N_i^{(1)} + N_i^{(2)}, & \text{for } 1 \leqq i \leqq n_0, \\ N_i^{(1)}, & \text{for } n_0 < i \leqq n_1, \\ M_{i-n_1+n_0}^{(2)}, & \text{for } n_1 < i \leqq n_1 + n_2 - n_0, \end{cases}$$

is equilibrated since each $\mathcal{N}(P_j)$ is equilibrated. Moreover, the linear span of $\mathscr{V}$ is of dimension k which satisfies $k \geqq \max(k_1, k_2)$. Hence $\mathscr{V}$ is fully equilibrated in some E^k. According to Minkowski's theorem (M2) there exists a convex polytope P in this E^k such that $\mathscr{V} = \mathcal{N}(P)$ and P is unique to within a translation. We define

$$P = P_1 \# P_2.$$

It is convenient to define an associated multiplication by a scalar factor λ. If $\lambda = 0$, we define $\lambda \times P$ to be a point; otherwise $\lambda \times P$ is that convex polytope for which $\mathcal{N}(\lambda \times P) = \{\lambda N_i \mid N_i \in \mathcal{N}(P)\}$. Again, by (M2), the existence and uniqueness up to a translation are assured. Clearly $(-1) \times P$ is the image of P under a central reflection and, if P is k-dimensional for $k \geqq 2$, then $\lambda \times P = \pm |\lambda|^{1/(k-1)} P$ where the last is the usual scalar multiplication associated with vector addition and the indeterminate sign is that of λ. For $k = 1$, $\lambda \times P = |\lambda| P$.

The properties of $\#$-addition and its associated $\times$-multiplication which are listed below are easily verified:

$$P_1 \# P_2 = P_2 \# P_1,$$

$$P_1 \# (P_2 \# P_3) = (P_1 \# P_2) \# P_3,$$

$$\lambda \times (P_1 \# P_2) = \lambda \times P_1 \# \lambda \times P_2,$$

$$(\lambda_1 \lambda_2) \times P = \lambda_1 \times (\lambda_2 \times P),$$

$$(\lambda_1 + \lambda_2) \times P = \lambda_1 \times P \# \lambda_2 \times P \text{ when } \lambda_1 \lambda_2 \geqq 0.$$

We shall also use the notation

$$\mathop{\#}_{i=1}^{n} P_i = P_1 \# P_2 \# \cdots \# P_n.$$

REMARK. In the plane the vector sum $P_1 + P_2$ of two non-degenerate polygons has the property that the length of any edge e is the sum of the lengths of those edges of P_1 and P_2 which have the same outer normal as e. Hence $P_1 + P_2 = P_1 \# P_2$ in the plane.

2. **A decomposition theorem.** Our first decomposition result is essentially a geometric formulation of the algebraic fact that an equilibrated system of vectors is a superposition of minimal equilibrated systems.

THEOREM 1. *Every convex polytope P is expressible in the form*

$$(*) \qquad P = \mathop{\#}_{i=1}^{m} P_i,$$

where each P_i is a simplex. Further, if P is d-dimensional and $f(P) = n \geqq d + 1$, then there is a representation (*) *with $m \leqq n - d$.*

Proof. We use induction. The assertion is obvious for $d = 1$ and also for $d > 1$ when $n = d + 1$. Thus we may assume from here on that $d > 1$, $n > d + 1$. Without loss of generality we may assume further that the origin 0 is in the relative interior of P.

Let C be the convex hull of $\mathcal{N}(P)$ and let $N_{i_0} \in \mathcal{N}(P)$. Then, for a suitable $\alpha_0 > 0$, we have $-\alpha_0 N_{i_0} \in \operatorname{bdry} C$ and so, by Carathéodory's theorem,

$$-\alpha_0 N_{i_0} = \sum_{\nu=1}^{d} \alpha_\nu N_{i_\nu}, \quad \alpha_\nu \geqq 0.$$

In other words, some positive combination of at most $d + 1$ vectors from $\mathcal{N}(P)$ is zero:

$$(1) \qquad \sum_{\nu=0}^{d} \alpha_\nu N_{i_\nu} = 0.$$

We suppose the indexing to be chosen so that $\alpha_\nu \neq 0$ when $0 \leqq \nu \leqq d_0$ where $d_0 \leqq d$ and $\alpha_\nu = 0$ for $\nu > d_0$. We also assume that the vectors N_{i_ν} and weights α_ν were such as to make d_0 minimal. Then $-\alpha_0 N_{i_0}$ is a relative interior point of the $(d_0 - 1)$-simplex determined by the points N_{i_ν}, $1 \leqq \nu \leqq d_0$.

Set

$$\alpha = \max_{0 \leqq \nu \leqq d_0} \alpha_\nu,$$

and

$$\beta_\nu = \alpha_\nu / \alpha \text{ for } 0 \leqq \nu \leqq d_0;$$

then

$$0 < \beta_\nu \leqq \max_{0 \leqq \nu \leqq d_0} \beta_\nu = 1.$$

By equation (1):

$$\sum_{v=0}^{d_0} \beta_v N_{i_v} = 0,$$

and therefore the system $\mathscr{N}_0 = \{\beta_v N_{i_v} \,|\, 0 \leqq v \leqq d_0\}$ is equilibrated. Hence the system $\mathscr{N}_1$ obtained from

$$\{(1-\beta_v) N_{i_v} \,|\, 0 \leqq v \leqq d_0\} \cup \{N_i \,|\, i \notin \{i_v \,|\, 0 \leqq v \leqq d_0\}\}$$

by omitting the null vectors is also equilibrated. By (M2), $\mathscr{N}_0$ and $\mathscr{N}_1$ each determines a polytope $P^{(0)}$ and $P^{(1)}$ respectively. The dimension of $P^{(j)}$ is $d_j \leqq d$. The number of (d_0-1)-faces of $P^{(0)}$ is d_0+1. Thus $P^{(0)}$ is a simplex.

Let the number of (d_1-1)-faces of $P^{(1)}$ be n_1. We observe that $n - n_1 \geqq d - d_1 + 1$. For, if $q \geqq 0$ is the number of $\beta_v \neq 1$, then

$$n = n_1 + 1 + d_0 - q.$$

On the other hand, the q non-zero vectors in $\{(1-\beta_v) N_{i_v} | 0 \leqq v \leqq d_0\}$ are linearly independent since $\mathscr{N}_0$ is associated with a simplex $P^{(0)}$. Thus the dimensionality of the intersection of the spaces E^{d_0} and E^{d_1} is at least q and so

$$d \leqq d_1 - d_0 - q.$$

This establishes the observation. On applying the induction assumption to $P^{(1)}$, we find that $P^{(1)}$ is representable as the $\#$-sum of no more than $n_1 - d_1$ simplices. Therefore P is decomposable into at most $1 + n_1 - d_1 \leqq n - d$ simplices as claimed.

REMARK 1. The vector-sum analogue of Theorem 1 is well known for polygons (see [14]); however, even in E^3, there are infinitely many types of polyhedra P which are indecomposable under vector addition, i.e. such that, if $P = P_1 + P_2$, then each P_i must be homothetic to P. See Gale [8], Shephard [12].

REMARK 2. Since every d-dimensional convex polytope may be approximated in the Hausdorff metric arbitrarily well by polytopes, no d of whose outer normals are linearly dependent, the proof of Theorem 1 yields immediately the following

THEOREM 2. *Every d-dimensional convex polytope P may be approximated in the Hausdorff metric arbitrarily well be finite $\#$-sums of d-simplices. If P has n faces of dimension $d-1$, the number of summands need be no more than $n-d$.*

It is remarkable that the analogous assertion for vector-sums is false for $d \geqq 3$; in E^3 even the regular octahedron is not the limit of finite vector sums of simplexes, cf. Asplund [9, p. 264], Shephard [13].

REMARK 3. We omit the simple proof of the following result which has no analogue in the case of vector-addition for $d \geqq 3$. We let $[x]$ be the greatest integer not larger than x.

Every centrally symmetric polytope P is a #-sum of parallelotopes. More precisely, if P is k-dimensional and has 2m faces, $m \geqq k$, then P is representable as a sum of $-[-m/p]$ p-dimensional parallelotopes where $1 \leqq p \leqq k$.

3. **A representation theorem.** In a certain sense, #-addition seems more natural if the summands and the sum all have the same dimensions; in other words if the systems $\mathcal{N}(P_i)$, $\mathcal{N}(P)$ are all fully equilibrated in the same E^k. Following this idea, one is led to the question as to whether every polytope in an E^k may be represented as a #-sum of k-simplices in E^k, or other "standard" polytopes of the dimension k. Without loss of generality, we may take $k = d$.

The example of the cube shows that simplices alone will not suffice for the purpose. Indeed for every representation of the d-cube P as $P = P_1 \# P_2$ with P_1 and P_2 d-dimensional, we have

$$f(P_1) = f(P_2) = fP).$$

Thus the bound $2d$ in our next theorem is the best possible.

THEOREM 3. *Every d-polytope P is representable in the form $P = \#_{i=1}^{m} P_i$ where each P_i is a d-polytope with $f(P_i) \leqq 2d$.*

Proof. We prove the theorem by induction. The assertion is trivially true for the cases $d = 1$ and $d > 1, f(P) \leqq 2d$. Thus we may assume $d > 1, f(P) > 2d$.

The vectors in $\mathcal{N}(P)$ span E^d, i.e. the origin 0 belongs to the interior of the convex hull of the points $\{N_i \mid 1 \leqq i \leqq f(P)\}$. By a Carathéodory type theorem on the interior points of the convex hull of a set, (see e.g. [6], Theorem 3.2) there exists a subset I of $\{1, 2, \cdots, f(P)\}$ which contains at most $2d$ integers and is such that 0 is in the interior of the convex hull of $\{N_i \mid i \in I\}$. Therefore, for suitable $\alpha_i > 0$, the system $\mathcal{N}_1 = \{\alpha_i N_i \mid i \in I\}$ is fully equilibrated in E^d. Obviously we may assume that the α_i are such that $\max_{i \in I} \alpha_i = 1$. Let $\mathcal{N}_2 = \{M_j \mid j \in J\}$ be the system obtained from

$$\{(1 - \alpha_i) N_i \mid i \in I\} \cup \{N_i \mid i \notin I\}$$

by the omission of zero-vectors.

Since $\mathcal{N}(P)$ and $\mathcal{N}_1$ are fully equilibrated, $\mathcal{N}_2$ is equilibrated. Let P_1 and P_2 be polytopes such that $\mathcal{N}(P_1) = \mathcal{N}_1$, $\mathcal{N}(P_2) = \mathcal{N}_2$.

If $\mathcal{N}_2$ is fully equilibrated, that is if P_2 is d-dimensional, then the proof is completed by induction since $f(P_2) < f(P)$.

Suppose, however, that $\mathcal{N}_2$ is not fully equilibrated; let E^k, where $1 \leqq k \leqq d-1$, be the space spanned by $\mathcal{N}_2$. Since P_2 is k-dimensional, by the inductive assumption it is representable in the form

$$P_2 = \mathop{\#}_{s=1}^{q} R$$

where each R_s is k-dimensional and $f(R_s) \leqq 2k$. But

$$P = P_1 \# P_2 = \mathop{\#}_{s=1}^{q} \left[R_s \# \left(\frac{1}{q} \times P_1 \right) \right];$$

thus the theorem will be proved if we establish it in the case that $f(P_2) \leqq 2k$, i.e. provided J has at most $2k$ elements.

Let p be that projection of E^d onto E^{d-k} which carries E^k onto 0. The projection of a fully equilibrated system is fully equilibrated. Thus $\{p(N_i) \mid i \in I\}$ is fully equilibrated in E^{d-k}; possibly some $p(N_i)$ are zero vectors and have to be omitted. As before, there exists a set $I_0 \subset I$ which has at most $2(d-k)$ integers, as well as a collection of positive numbers $\beta_i \leqq \alpha_i/2$ such that

$$\sum_{i \in I_0} \beta_i p(N_i) = 0$$

where $\{p(N_i) \mid i \in I_0\}$ is fully equilibrated in E^{d-k}. Now the vector

$$N = \sum_{i \in I_0} \beta_i N_i$$

is in E^k. Since $\mathcal{N}_2$ is fully equilibrated in E_k, there is a $\beta, 0 < \beta \leqq 1$, for which

$$-\beta N = \sum_{j \in J} \gamma_j M_j$$

where $\gamma_j \geqq 0$ and $0 < \max_{j \in J} \gamma_j = \gamma < 1$. Consequently, the two systems obtained by the deletion of any zero vectors from the two systems

$$\mathcal{N}_1^* = \{\beta\beta_i N_i \mid i \in I_0\} \cup \{(1-\gamma-\gamma_i)M_j \mid j \in J\}$$

and

$$\mathcal{N}_2^* = \{(\alpha_i - \beta\beta_i N_i \mid i \in I_0\} \cup \{\alpha_i N_i \mid i \in I - I_0\} \cup \{(\gamma - \gamma_j)M_j \mid j \in J\}$$

are both fully equilibrated in E and each contains less than $f(P)$ vectors. Hence, the inductive assumption may be applied to the d-polytopes $P_1{}^*$ and $P_2{}^*$ corresponding to the systems $\mathcal{N}_1{}^*$ and $\mathcal{N}_2{}^*$. Clearly $P = P_1{}^* \# P_2{}^*$ and so this completes the proof of Theorem 3.

4. REMARKS. (1) By a slight change in definitions, $\#$-addition *in the plane* may be made to coincide with vector addition of polygons and segments. Only the definition of $\mathcal{N}(P)$ for the case of a segment P needs to be modified to read: If P is a segment, $\mathcal{N}(P)$ is a pair of opposite vectors, each of length equal to that of P and perpendicular to the carrier line of P. The definition of $\mathcal{N}(P)$ for proper polygons and the definition of $\#$-addition in terms of the associated equilibrated systems remains unchanged. Then it is easily seen that

$$P_1 + P_2 = P_1 \# P_2$$

for all proper or improper polygons P_1 and P_2. Theorems 1 and 3 remain valid and become only reformulations of well-known results. In the first case: Every polygon is a vector sum of segments and triangles. In the second case the representation involves triangles and quadrangles.

(2) Let $a(P)$ denote a point valued function, defined for all convex polytopes P, which is translation invariant, that is

$$a(P + X) = a(P) + X$$

for any point X. For example, take $a(P)$ to be the area centroid of P. Minkowski's theorem (M2) implies that if $\mathcal{V}$ is an equilibrated system there exists a unique convex polytope P for which $\mathcal{N}(P) = \mathcal{V}$ and $a(P)$ is a preassigned point. Let $f(P) = n$ and

$$w(P) = \sum_{i=1}^{n} |N_i|.$$

Then an addition of convex polytopes (as opposed to classes of translation-equivalent polytopes) can be defined by taking the composite of P_1 and P_2 to be that translate of $P_1 \# P_2$ for which $a(P)$ has the value

$$[w(P_1)\,a(P_1) + w(P_2)\,a(P_2)]/[w(P_1) + w(P_2)].$$

This composition is commutative and associative.

(3) The $\#$-addition is a specialization to convex polytopes of an operation which may be defined for all convex bodies. For simplicity we discuss only the case of d-dimensional bodies in E^d.

The *area function* $S_K(\omega)$ of a convex body K is a non-negative, totally additive set-function defined for the Borel sets ω of the surface $\Omega = \{\zeta\}$ of the unit sphere of E^d by the following condition. $S_K(\omega)$ is the $(d-1)$-dimensional content of the set of boundary points of K each of which has a supporting hyperplane with outer normal in ω.

Minkowski's theorem (M2) has the following generalization, (see [1], [5], [7]):

A non-negative, totally additive set-function $\phi(\omega)$ over the Borel sets of Ω is the area function $S_K(\omega)$ of a convex body K if and only if ϕ is positive for each open hemisphere and

$$\int_\Omega \zeta\phi(d\Omega) = 0$$

i.e. the centroid of the mass-loading of Ω specified by ϕ is at the centre of Ω. Moreover, if ϕ satisfies these conditions, K is determined to within a translation.

Now if K_1 and K_2 are convex bodies in E^d, it follows that there exists a convex body $K_1 \# K_2$ unique to within a translation, which has area function $S_{K_1}(\omega) + S_{K_2}(\omega)$.

This #-addition of general convex bodies reduces to the composition process suggested by Blaschke which we mentioned at the beginning of this paper if the bodies are sufficiently smooth. It may have applications to different problems. As a minor instance, we mention the following. The #-sum of convex bodies of constant brightness is clearly of constant brightness. Now the only such convex bodies which have been explicitly described are the sphere and a special figure of revolution K_0, see [3, p. 153]. The existence of bodies of constant brightness which are *not* figures of revolution is assured: for we may let K_0 and K_0' be two bodies of the type described by Blaschke which are not coaxial. Then it is easily shown that $K_0 \# K_0'$ is not a figure of revolution, but it is a body of constant brightness. One more observation along these lines: a central-symmetric body of constant brightness must be a sphere. This gives the curious result that, if K is of constant brightness, $K \# (-K)$ is a sphere. The analogous result for vector addition is: if K is of constant width, then $K + (-K)$ is a sphere.

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