A SIMPLE PROOF OF BORSUK'S CONJECTURE IN THREE DIMENSIONS

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1. Borsuk (1) made the following conjecture:

Every bounded set of points in Euclidean n-space E^n can be represented as the union of $n+1$ sets of smaller diameter.

Hadwiger (2, 3, 4) proved Borsuk's conjecture assuming the additional condition that the surface of the set is sufficiently smooth. On the other hand, a number of simple proofs have been supplied in the two-dimensional case (see, for example, Gale (5), where a stronger result is proved), as well as a complicated proof in the three-dimensional case (Eggleston (6)). In this note a simple proof is given for the conjecture in E^3. The proof is based on the idea (used also by Gale in E^2) of finding a suitable universal covering set (Deckel, couvercle, see Bonnesen & Fenchel (7), p. 87) for sets of diameter 1 in E^3 and partitioning the covering set in four parts, each of diameter less than 1.

2. As mentioned by Gale ((5), p. 225) every set of diameter 1 can be embedded in a regular octahedron, the distance between whose opposite faces is 1, i.e. whose diameter is $\sqrt{3}$. Starting from this octahedron, with vertices $A_1A_2A_3A_1'A_2'A_3'$, the triangle $A_1A_2A_3$ forming, say, the base and A_i' being the corner opposite to A_i, we find a smaller universal covering set in the following way:

Parallel to the plane a_k containing the vertices $A_iA_jA_i'A_j'$, where i, j, k are distinct, at distance $\frac{1}{2}$ from a_k intersect the octahedron by two planes b_k and b_k', which cut from the octahedron two square pyramids with A_k, A_k' as vertices. At least one of the pair of pyramids has no interior point in common with a set of diameter 1 inscribed in the octahedron, since the distance between their bases is 1. Repeating the procedure with all three pairs of opposite vertices we are able finally (changing the notation if necessary) to cut off the vertices A_1', A_2', A_3' and replace them by squares $B_iB_i'C_i'C_i$ ($i = 1, 2, 3$), the points B_i and B_i' being on the original face $A_1'A_2'A_3'$. The upper base of the new polyhedron is a hexagon with vertices $B_1B_1'B_2B_2'B_3B_3'$, angles $\frac{2}{3}\pi$ and sides $B_iB_i' = (\sqrt{3}-1)/\sqrt{2}$ and $B_1'B_2 = B_2'B_3 = B_3'B_1 = (2-\sqrt{3})/\sqrt{2}$. Its other faces, besides the base $A_1A_2A_3$ and the squares $B_iB_i'C_i'C_i$, are: three pentagons congruent to $B_1'B_2C_2A_3C_1'$ (the angle at A_3 being $\frac{1}{3}\pi$, the others $\frac{2}{3}\pi$; $A_3C_1' = A_3C_2 = 1/\sqrt{2}$) and three trapezoids congruent to $C_1C_1'A_3A_2$ (the angles at A_2 and A_3 being $\frac{1}{3}\pi$, the others $\frac{2}{3}\pi$). The diameter of the polyhedron is $\sqrt{2}$; it is the distance from A_i to the points B_i, B_i', C_i, C_i'. From its construction it is clear that the polyhedron is a universal covering set.

3. We shall show now that it is possible to divide the polyhedron described above (and drawn in Fig. 1) into four parts, each with diameter less than 1. Since the polyhedron is a universal covering set, this will prove Borsuk's conjecture in the case $n = 3$.

First of all we remark that it is sufficient to divide the *surface* of the polyhedron in four parts of diameter less than 1. Indeed, the sphere circumscribed to the polyhedron has a radius $< \frac{1}{2}\sqrt{3}$, since the sphere circumscribed to the original octahedron has radius $\frac{1}{2}\sqrt{3}$. Therefore, if the surface is subdivided into parts of diameter $d < 1, d \geqslant \frac{1}{2}\sqrt{3}$, the diameter of the convex hulls of the unions of these parts with the circumcentre of the polyhedron will also be $d < 1$.

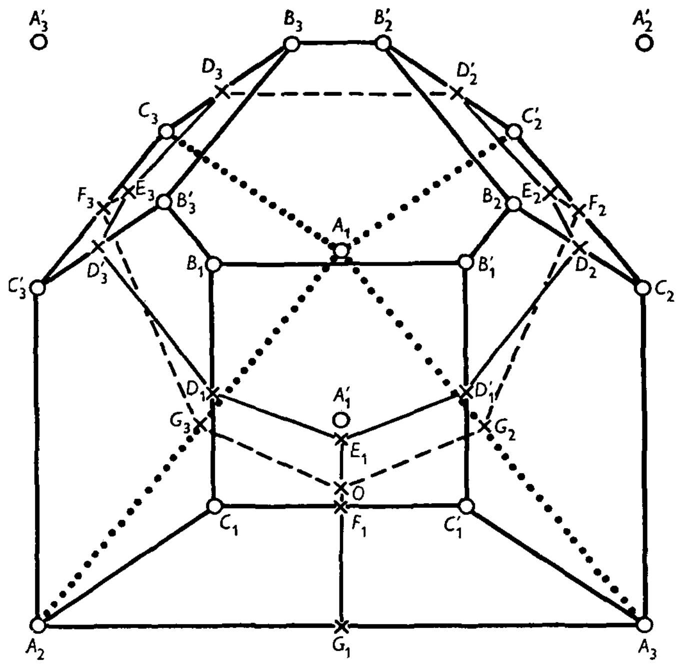

Fig. 1

Now to the subdivision of the surface of the polyhedron. The three (congruent) parts containing the vertices A_i will be denoted by $S_i, i = 1, 2, 3$. The fourth part will contain the hexagon $B_1B_1'B_2B_2'B_3B_3'$ and will be denoted by S_4. The boundary of S_1 is the closed polygon $OG_2F_2E_2D_2'D_3E_3F_3G_3O$, where O is the centre of $A_1A_2A_3$, G_2 the centre of the segment A_1A_3, F_2 the centre of C_2C_2', and similarly for G_3 and F_3; the location of the points E_i, D_i and D_i' will be specified later. The boundaries of S_2 and S_3 are, respectively, $OG_3F_3E_3D_3'D_1E_1F_1G_1O$ and $O_1G_1F_1E_1D_1'D_2E_2F_2G_2O$, while S_4 is bounded by $D_1E_1D_1'D_2E_2D_2'D_3E_3D_3'D_1$. The points D_i resp. D_i' are on the segments B_iC_i resp. $B_i'C_i'$, with $C_iD_i = C_i'D_i' = (15\sqrt{3}-10)/(46\sqrt{2})$. Then

$$D_1G_3 = D_1D_2' = (10+31\sqrt{3})/(46\sqrt{2}) = 0{\cdot}979.$$

The point E_i is on the segment joining F_i with the mid-point of B_iB_i', for $i = 1, 2, 3$, with $E_iF_i = (1231\sqrt{3}-1986)/(1518\sqrt{2})$. Then

$$E_3G_2 = E_3D_2 = (6{,}129{,}030-937{,}419\sqrt{3})^{\frac{1}{2}}/(1518\sqrt{2}) = 0{\cdot}9887,$$

and this is also the diameter of the sets $S_i, i = 1, 2, 3, 4$.

4. Applying essentially the same method as used above, the bound 0·9887 could be somewhat improved in two ways:

(i) By locating the points E_i, D_i and D_i' in such a fashion that $D_1G_3 = E_1G_3 = E_1D_3$.

(ii) By further truncating the polyhedron, e.g. by planes parallel to the plane $A_1G_1A_1'$ etc.

Both ways result in a very slight improvement, while considerably complicating the argument. I have found no way of proving by my method Gale's conjecture ((5), p. 224) according to which the bound should be $(3+\sqrt{3})^{\frac{1}{2}}/6 = 0{\cdot}888$.

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SOME SOLUTIONS OF DIOPHANTINE EQUATIONS

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I have recently carried out some direct searches for solutions of certain Diophantine equations of the form
$$f(x)+f(y)=f(z)+f(t),$$
where $f(x)$ is either a power of x or a binomial coefficient $\binom{x}{n}$ for fixed n. In this note I give a summary of the results obtained.

1. $f(x) = x^3$. This is the only part of the work where I know of a previous published table, that of Richmond (1). This gives solutions of the equation
$$x^3 \pm y^3 \pm z^3 \pm t^3 = 0$$
in positive integers less than 100. It is said to be 'possibly complete: there may be omissions, but not many'. I have checked only the entries with two of the cubes