A PROOF OF ROGERS' CONJECTURE ON PAIRS OF CONVEX DOMAINS

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1. *Introduction*

Rogers [7] established the following result:

Let K_1, K_2 be centrally symmetric convex bodies in the plane; unless both K_1 and K_2 have ellipses as boundaries, there exists an affine transformation T such that $TK_2 \subset K_1$, and $\operatorname{bd} K_1 \cap \operatorname{bd} TK_2$ is the union of four non-empty disjoint closed sets.

In the general case, *i.e.* if the sets K_i are not assumed to be centrally symmetric, Rogers conjectured that an analogous result holds, with "four" replaced by "three" in the last part of the theorem.

The aim of the present paper is to establish this conjecture of Rogers. We shall prove the following

Theorem. *Let K_1, K_2 be plane convex bodies; unless both $\operatorname{bd} K_1$ and $\operatorname{bd} K_2$ are ellipses, there is an affine transformation T such that $TK_2 \subset K_1$ and $\operatorname{bd} K_1 \cap \operatorname{bd} TK_2$ has at least three connected components.*

In §2 we shall prove the theorem for the special case $K_1 = K_2$; the general case is established in §§ 3 and 4. Contained in §5 are some remarks on other characterizations of ellipses.

2. *Proof of the theorem in case $K_1 = K_2$*

If $K_1 = K_2$ (or if K_1 is an affine image of K_2), the assertion of the theorem follows from well-known results.

Indeed, if $K = K_1 = K_2$ is centrally symmetric, our assertion is only a weakened version of a special case of Rogers' theorem. Therefore we may assume that K is not centrally symmetric.

Let $\sigma(K)$ denote the Minkowski measure of symmetry of K. [For various definitions of $\sigma(K)$, its properties, and a list of references, see §6.1 of [4]; there the Minkowski measure of symmetry of K is denoted by $F_1(K)$.] Our assumptions imply that $\frac{1}{2} \leqslant \sigma(K) < 1$. Let the origin O be the σ-critical point of K. Taking $TK_2 = -\sigma(K).K$, our theorem follows from the result of Neumann [6].

3. *Proof of the theorem in the general case*

In §§ 3 and 4 we shall assume that K_1 and K_2 are not affinely equivalent. Our proof shall be simplified by the following

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LEMMA 1. *Let K_1, K_2 be plane convex bodies such that $C = K_2 \cap \mathrm{bd}\, K_1$ has at least three connected components. Let, moreover, components C_0, C_1, C_2 of C, points p_1, p_2 and lines L_1, L_2, L exist such that:*

(i) *$p_1 \in C_0 \subset \mathrm{bd}\, K_2$ and $V \cap K_2 \subset V \cap K_1$ for some open set $V \supset C_0$;*
(ii) *L_1 contains p_1 and is a supporting line of K_2;*
(iii) *L_2 is a supporting line of K_1 parallel to L_1, $L_2 \cap K_2 = \phi$, and K_2 is contained in the strip determined by L_1 and L_2;*
(iv) *$p_2 \in L_2 \cap K_1$;*
(v) *L is determined by p_1 and p_2;*
(vi) *$C_1 \cap L = C_2 \cap L = \emptyset$; C_1 and C_2 are on opposite sides of L.*

Then there exists an affine transformation T such that $TK_2 \subset K_1$ and $\mathrm{bd}\, K_1 \cap \mathrm{bd}\, TK_2$ has at least three connected components.

Proof of Lemma 1. Let $C^{(i)}$ be the part of $C \setminus C_0$ contained in that half-plane determined by L which contains C_i, $i = 1, 2$. Let q_i be the point of $C^{(i)}$ nearest to p_2 in the sense that the open arc $\widehat{q_i\, p_2}$ does not meet C. Without loss of generality we shall assume that p_1 is the origin O, that L_1 is one of the coordinate axes and $L^{(i)}$ the other, $i = 1$ or 2, where $L^{(i)}$ is the line determined by p_1 and q_i. We consider affine transformations $T_i^\alpha(p) = (\alpha x, y)$, where (x, y) are the coordinates of the point p in the coordinate system determined by L_1 and $L^{(i)}$, and $\alpha > 0$. We shall call T_i^α a *shrinking* [resp. *stretching*] *with base $L^{(i)}$ in the direction L_1* provided $\alpha < 1$ [resp. $\alpha > 1$].

Each transformation T_i^α, $i = 1, 2$, $0 < \alpha < 1$, is area-diminishing. Unless $K_2 \subset K_1$, at least one of the T_i^α can be applied, leading to a pair K_1, $T_i^\alpha K_2$ which satisfies the assumptions of Lemma 1 (with the same points p_1, p_2 and lines L_1, L_2, L, and with arc-distance from C_0 to $C \setminus C_0$ not decreased). An application of Blaschke's selection theorem yields therefore such an affine transform K_2^* of K_2 for which the assumptions of Lemma 1 are satisfied *and* $K_2^* \subset K_1$. But this proves the assertion of Lemma 1.

The proof of the theorem shall therefore be completed if, for given K_1 and K_2, we succeed in finding affine transforms of K_1 and K_2 which satisfy the assumptions of Lemma 1. In the present section we shall establish the existence of such transforms in the case that either K_1 is not strictly convex, or that K_2 fails to be smooth. In §4 we shall perform the same task assuming that K_1 is strictly convex and that K_2 is smooth.

Let a segment S, with midpoint p_1, be contained in $\mathrm{bd}\, K_1$, and let q be an exposed point of K_2 (see Straszewicz [8]). There obviously exists an affine transformation T_0 such that $p_1 = T_0(q)$ and $T_0(K_2 \setminus \{q\}) \subset \mathrm{int}\, K_1$. Then a suitable stretching of $T_0(K_2)$, with any line through p_1 and a point of $\mathrm{int}\, T_0(K_2)$ as base, in the direction of the carrierline of S, yields a transform K_2^* of K_2 such that K_1 and K_2^* satisfy the conditions of Lemma 1.

If $\operatorname{bd} K_2$ contains a point q through which pass two different supporting lines of K_2, we choose a smooth point $p_1 \in \operatorname{bd} K_1$ and continue as above, using the supporting line to K_1 at p_1 in place of the line S.

4. *Proof of the theorem* (*end*)

Before completing the proof of the theorem, we shall establish a lemma whose idea (in case B is an ellipse) goes back to Behrend [**1**].

LEMMA 2. *Let A, B be plane convex bodies, $B \subset A$, such that $T(B) \subset A$ for an affine transformation T implies* area $T(B) \leqslant$ area B. *Then* $G = \operatorname{bd} A \cap \operatorname{bd} B$ *is a global set on* $\operatorname{bd} A$.

Here a closed subset G of $\operatorname{bd} A$ is called *global* provided

(i) G is not a pair of *antipodal points* of A (*i.e.* points contained in a pair of parallel supporting lines of A);

(ii) G is not contained in a *small arc* of $\operatorname{bd} A$, where a small arc of $\operatorname{bd} A$ is an arc contained in the interior of an arc of $\operatorname{bd} A$ determined by a pair of antipodal points of A.

Proof of Lemma 2. If G were contained in the interior of the arc of $\operatorname{bd} A$ determined by the antipodal points r_1, r_2 of A, we assume, without loss of generality, that $r_1 = (0, 1)$, $r_2 = (1, 1)$, that $x = 0$ and $x = 1$ are supporting lines of A containing r_1 resp. r_2, that G is contained in the half-plane $y \leqslant 1$ and that $y = 0$ is a supporting line of B. Then, clearly, a suitable stretching T, with the x-axis as base, in direction of the y-axis, satisfies $T(B) \subset A$ although T is area-increasing. Thus G is not contained in any small arc of $\operatorname{bd} A$.

If G were reduced to a pair of antipodal points p_1, p_2, with parallel lines L_1, L_2 supporting A at p_1, p_2, we assume, without loss of generality, that $p_i = \big((-1)^i, 0\big)$, while L_i is the line $x = (-1)^i$. Then, for $\lambda < 1$ sufficiently close to 1, the area-preserving transformation

$$T(x, y) = (\lambda x, \lambda^{-1} y)$$

can be shown to yield $T(B) \subset \operatorname{int} A$, *i.e.* A contains affine images of B of area greater than that of B. This completes the proof of Lemma 2.

Now we are ready to return to the proof of the theorem. Let $T_0 K_2$ be an affine transform of K_2 contained in K_1 and having maximal possible area. (Its existence follows from Blaschke's selection theorem.) Without loss of generality we assume $T_0 K_2 = K_2$. According to Lemma 2, $G = \operatorname{bd} K_1 \cap \operatorname{bd} K_2$ is a global set on $\operatorname{bd} K_1$. If G has three or more connected components, the theorem is established. There remain to be considered only the two following cases:

(i) G is connected; (ii) G has two connected components.

In case (i), G is an arc of $\mathrm{bd}\,K_1$. (Note that, since K_1 and K_2 are not affinely equivalent, G cannot coincide with $\mathrm{bd}\,K_1$.) Since G is global on $\mathrm{bd}\,K_1$, there exist parallel supporting lines L_i of K_1, and points $g_i \in G$, such that $L_i \cap G = g_i$ for $i = 1, 2$. Moreover, the lines L_i may be chosen in such a way that only one of the supporting lines of K_1, parallel to the line g_1, g_2, intersects G. Let this supporting line be L, and let M be the line parallel to L_i and passing through the (unique) point $L \cap G$. Then a suitable stretching of K_2, with base L, in direction M, yields a set $K_2{}^*$ which, together with K_1, satisfies the assumptions of Lemma 1. This establishes the theorem in case (i).

In case (ii) let G consist of the two arcs G_1 and G_2; note that at least one of the arcs, say G_1, is not reduced to a point. Let $c_i \in G_i$ be points on parallel supporting lines M_i of K_1; let $p_1 \in G_1$, $p_1 \neq c_1$, and let L be the line determined by c_1 and p_1. We shrink K_2, with base L in direction M_1, to obtain a set $K_2{}^*$. For a sufficiently small shrinking, the set $G^* = K_2{}^* \cap \mathrm{bd}\,K_1$ consists of at least three connected components: one reduced to c_1, another containing (or reduced to) p_1, and one near c_2, on the arc $\widehat{p_1 c_2}$. If p_1 and the component C_0 of G^* containing it satisfy the assumptions of Lemma 1, we are through. Otherwise, we shrink $K_2{}^*$, with the line determined by c_1 and the point of G^* (arcwise) nearest to c_2 as base, in direction M_1, to obtain a set $K_2{}^{**}$ such that K_1 and $K_2{}^{**}$ satisfy the assumptions of Lemma 1.

This completes the proof of the theorem.

5. *Remarks*

(i) As a corollary of our theorem and of the theorem of Rogers' we obtain the following strengthening of a result of Süss [9]:

If C is a plane convex curve different from an ellipse, it is possible to find affine images C_1 and C_2 of C, $C_1 \neq C_2$, such that $C_1 \cap C_2$ contains at least six points. Moreover, if C is centrally symmetric, C_1 and C_2 may be chosen in such a fashion that $C_1 \cap C_2$ contains at least eight points.

Indeed, taking as K_1 and K_2 the convex hull of C, one has to apply to the affine image TK_2 of K_2 obtained by our theorem, or by Rogers' theorem, a homothetic expansion with centre at the centroid of TK_2 and with a suitably small ratio $\lambda > 1$.

The first half of the above version of Süss' theorem may also be obtained directly from an application of our Lemma 2, with A a circular disc, and B a suitable affine image of the convex hull of C. Then C_1 may be taken as $\mathrm{bd}\,B$, while C_2 is obtained from C_1 by an appropriately small rotation of C_1 about the centre of A.

(ii) Lemma 2, and arguments closely related to it, may be used to obtain very short proofs of various known results. As an example we mention the theorem of Bertrand [2] and Brunn (see [3; p. 143]): *Ellipses are the only convex curves with straight "Schwerlinien".* (A "Schwerlinie" is the locus of midpoints of parallel chords.) Indeed, it is very easily established that a convex curve C with straight "Schwerlinien" has a centre of symmetry; then, if C were not an ellipse, consider the maximal inellipse E of C. Let p_1 and p_2 be two points on $C \cap E$ such that the (open) *small arc* determined by them on C does not meet E, and let p_0 be a point on that arc of C. Consider the chord $p_2 p_3$ of C parallel to $p_0 p_1$. Since the "Schwerlinien" of a centrally symmetric C pass through its centre, and since p_0 is outside E, it follows that p_3 is inside E, thus contradicting $p_2 \in C$.

Other applications of related ideas are given in [5].

(iii) It may be conjectured that both Rogers' and our theorems generalize to higher dimensions in the following form:

Let K_1, K_2 be [centrally symmetric] n-dimensional convex bodies, $n \geqslant 2$; unless both are ellipsoids, there is a non-singular affine transformation T such that $TK_2 \subset K_1$ and $\text{bd}\, K_1 \cap \text{bd}\, TK_2$ has at least three [four] connected components.

In order to see that the bounds 3 respectively 4 cannot be improved even for $n \geqslant 3$, it is sufficient to take for K_2 a solid n-dimensional sphere, and for K_1 a solid half-sphere, or a spherical zone, respectively.

(*Added December* 15, 1963.) A weaker version of the above conjecture is obtained if the transformation T is allowed to be singular. The following reasoning, supplied by the referee, proves the conjecture in this formulation.

We recall a result, proved for $n = 3$ by T. Kubota ("Einfache Beweise eines Satzes über die konvexe, geschlossene Fläche", *Science Reports of the Tôhoku Imperial University*, 1st Series, 3 (1914), 235–255) and for general n by H. Buseman (*The geometry of geodesics*, New York, 1955, page 91):

If p is an inner point of a convex body K and each two-dimensional section of K by a plane through p is an ellipse, then K is an ellipsoid.

By the theory of polar reciprocal convex bodies this is equivalent to the result (proved in the case $n = 3$ by T. Kubota, *loc. cit.*, and also by W. Blaschke and G. Hessenberg, "Lehrsätze über konvexe Körper", *Jber. Deutsch. Math.-Verein.*, 26 (1917), 215–220):

If all the two-dimensional orthogonal projections of a convex body K are ellipses, then K is an ellipsoid.

We suppose that K_1 and K_2 are not both ellipsoids. It follows that we can choose a two-dimensional section S_1 of K_1 and a two-dimensional projection P_2 of K_2, which are not both ellipses. It follows from the main theorem that we can choose an affine transformation T_0 so that $T_0 P_2 \subset S_1$

and $\operatorname{bd} S_1 \cap \operatorname{bd} T_0 P_2$ has at least three connected components.

The corresponding result for centrally symmetric bodies follows in the same way from Rogers' theorem.

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