Common Secants for Families of Polyhedra

By

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1. Introduction. A family $\mathscr{K}$ of sets in E^n is said to have property $\mathscr{S}$ if there exists a hyperplane (i.e. an $(n-1)$-dimensional linear variety) intersecting every member of $\mathscr{K}$. The family $\mathscr{K}$ is said to have property $\mathscr{S}(k)$ if every k-membered subfamily of $\mathscr{K}$ has property $\mathscr{S}$.

The question under what conditions on the family $\mathscr{K}$ does $\mathscr{S}(k)$ imply $\mathscr{S}$ was posed by VINCENSINI [6]. The first positive result in that direction was the following result of SANTALÓ [5]:

For families of parallelotopes in E^n, with edges parallel to the coordinate axes, $\mathscr{S}(2^{n-1}(n+1))$ implies $\mathscr{S}$.

Except for $n = 2$, SANTALÓ's paper and VALENTINE [7] are the only references known to us dealing with this problem. The case $n = 2$ and the analogous problem about straight lines transversal to members of a family of sets in E^n, received much greater attention. See Chapter 5 of DANZER-GRÜNBAUM-KLEE [1] for an account of the known results and references.

In the present paper we shall prove a generalization of SANTALÓ's theorem to families of polyhedra *related* (see definition below) to any given polyhedron.

The method of proof is an extension of that used in [2] for $n = 2$. Our proof is somewhat related to SANTALÓ's in as much as in both proofs HELLY's theorem on intersections of convex sets (see [1]) is highly relevant. SANTALÓ uses a variant of RADON's proof of HELLY's theorem, while in the present paper the use of HELLY's theorem itself permits a simpler reasoning and greater generality.

We also show (Theorem 3) that the limitation to polyhedra in Theorem 1 is necessary, at least for $n = 2$.

2. Statement of results. A convex cone C, with vertex at the origin 0, is called the *associated cone* of a (convex) polyhedron $P \subset E^n$ with respect to the vertex v of P if $v + C$ is the *conical extension* of P from v (i.e. $v + C$ is the union of all the half-lines with endpoint v which contain at least one point of P different from v). A polyhedron P' is *related* to a polyhedron P provided each associated cone of P' is an intersection of associated cones of P. (Note that P' may be related to P without P being related to P'.) A family $\mathscr{P}$ is related to P if each member of $\mathscr{P}$ is related to P.

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For example, SANTALÓ's families of parallelotopes are related to the unit cube. Families of translates, or homothets, of a given polyhedron P are related to P.

Using the notion of a family related to a polyhedron, we may formulate our results as follows.

Theorem 1. *For families $\mathscr{P}$ related to a centrally symmetric convex polyhedron $P \subset E^n$ with $2p$ vertices, $\mathscr{S}(p(n+1))$ implies $\mathscr{S}$.*

Theorem 2. *For every positive integer k there exists a $t = t(k, n)$ such that for families $\mathscr{P}$ related to a convex polyhedron $P \subset E^n$ with k vertices, $\mathscr{S}(t)$ implies $\mathscr{S}$. Moreover, we may take $t(k, n) \leqq \binom{k}{2} \cdot (n+1)$.*

The restriction to polyhedra in Theorem 1 is natural, as shown by

Theorem 3. *Let a centrally symmetric convex body $K \subset E^2$ have the property that $\mathscr{S}(t)$ implies $\mathscr{S}$ for families of (positive) homothets of K, where t depends on K only. Then K is a polygon.*

3. A lemma. Before proving the theorems we shall establish a lemma, for the formulation of which we have to introduce some notation.

Let C be a proper convex cone in E^n, with vertex 0. Let K be a convex set in E^n such that, for suitable $x_1, x_2 \in K$, we have $K \subset (x_1 + C) \cap (x_2 - C)$. This condition is fullfilled, e.g., if C is an associated cone of a polyhedron P and K is related to P. We denote by $\mathscr{H}$ the set of all hyperplanes in E^n having translates which support C. We map the set $\mathscr{H}$ onto an n-dimensional Euclidean space R^n in the following fashion. Let $y \in \operatorname{int} C$ be a fixed point; in the hyperplane Y through 0, perpendicular to the vector y, we choose n points z_i, $1 \leqq i \leqq n$, with linear hull Y, and $\sum_{i=1}^{n} z_i = 0$. Every $H \in \mathscr{H}$ intersects each of the lines $L_i = z_i + Ry$ in a unique point $z_i + \alpha_i y$. Then $H \to \alpha(H) = (\alpha_1, \ldots, \alpha_n)$ is the mapping of $\mathscr{H}$ onto R^n we need, the α_i being the coordinates of H. In this notation we have

Lemma 1. *The set $\{\alpha(H) : H \in \mathscr{H}, H \cap K \neq \emptyset\} \subset R^n$ is convex.*

Proof of the lemma. Let $H_0, H_1 \in \mathscr{H}$; the family of hyperplanes

$$\{H_\lambda : \alpha(H_\lambda) = (1-\lambda)\alpha(H_0) + \lambda\alpha(H_1), 0 \leqq \lambda \leqq 1\},$$

the "segment" with endpoints H_0 and H_1, is easily seen to be part of the pencil of hyperplanes determined by H_0 and H_1. Moreover, for at least one i, $L_i \cap H_0 \cap H_1 = \emptyset$. Let S be the segment determined by $L_i \cap H_0$ and $L_i \cap H_1$. Then $H_\lambda \cap S \neq \emptyset$ for all λ, $0 \leqq \lambda \leqq 1$. (See Fig. 1 for the case $n = 2$ to which the general case reduces at once.)

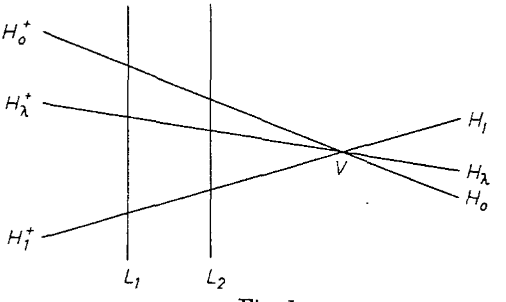

Fig. 1

Let $K \cap H_0 \neq \emptyset$, $K \cap H_1 \neq \emptyset$. If $H_0 \cap H_1 = \emptyset$ then $H_\lambda \cap K \neq \emptyset$ for $0 \leqq \lambda \leqq 1$ on account of the convexity of K. If, on the contrary, $V = H_0 \cap H_1 \neq \emptyset$, then V is an $(n-2)$-dimensional variety which divides each H_λ into closed half-hyperplanes H_λ^+ and H_λ^-, the notation being such that $H_\lambda^+ \cap S \neq \emptyset$. If $H_i^+ \cap K \neq \emptyset$ for $i = 0{,}1$, or if $H_j^- \cap K \neq \emptyset$ for $i = 0{,}1$, again $K \cap H_\lambda \neq \emptyset$ by the convexity of K. But $K \cap H_0^+ = = K \cap H_1^- = \emptyset$, or $K \cap H_0^- = K \cap H_1^+ = \emptyset$ is impossible since $K \subset (x_1 + C) \cap (x_2 - C)$ and both H_0 and H_1 are parallel to supporting hyperplanes of C. This ends the proof of Lemma 1.

4. Proof of Theorem 1. Let the vertices of P be $\pm x_i$, $1 \leqq i \leqq p$, and let C_i be the associated cone of P with respect to x_i, $1 \leqq i \leqq p$. Let $\mathscr{H}_i$ be the family of all hyperplanes in E^n having translates which support C_i. Clearly $\bigcup_{i=1}^{p} \mathscr{H}_i$ is the set of all hyperplanes in E^n. Assuming that there is no hyperplane in E^n intersecting all the members of $\mathscr{P}$, it follows that for each i, $1 \leqq i \leqq p$, no hyperplanes in $\mathscr{H}_i$ intersect all the members of $\mathscr{P}$. According to the above lemma, the set $\{\alpha(H) : H \in \mathscr{H}_i, H \cap P_\nu \neq \emptyset\}$ is convex for every $P_\nu \in \mathscr{P}$, and it follows from HELLY's theorem on intersections of convex sets that some $n+1$ members of $\mathscr{P}$ do not have a common secant belonging to $\mathscr{H}_i$. Therefore there exists a subfamily $\mathscr{P}' \subset \mathscr{P}$, containing at most $p(n+1)$ members and such that the members of $\mathscr{P}'$ do not have a common secant belonging to $\mathscr{H}_i$ for $i = 1, 2, \ldots, p$; in other words, the members of $\mathscr{P}'$ do not have any common secant. This ends the proof of Theorem 1.

5. Proof of Theorem 2. The existence assertion of Theorem 2 is an obvious consequence of Theorem 1 and

Lemma 2. *A convex polyhedron $K \subset E^n$ is related to its "differences body" ("vector-body") $K^* = K + (-K)$.*

The proof of Lemma 2 is immediate: For a vertex v of K, let v_j, $1 \leqq j \leqq m$, be all those vertices of K^* for which there exists vectors x_j, such that $x_j + K \subset K^*$ and $x_j + v = v_j$. Obviously, the associated cone of K with respect to v is then the intersection, for $1 \leqq j \leqq m$, of the associated cones of K^* with respect to v_j.

The estimate of $t(k, n)$ follows from the trivial observation that P^* has at most $2\binom{k}{2}$ vertices.

6. Proof of Theorem 3. Assume that $K = -K$ is not a polygon. Then it is possible to find t points $p_1, \ldots, p_t$ of K with the following properties:

(i) each p_i is an exposed point of K, i.e. there exists a line L_i supporting K and such that $K \cap L_i = \{p_i\}$.

(ii) $p_i + p_j \neq 0$, $1 \leqq i, j \leqq t$.

(iii) L_i is not parallel to L_j for $i \neq j$, $1 \leqq i, j \leqq t$.

Let us define $p_{t+i} = -p_i$ for $1 \leqq i \leqq t$, $p_{2ht+i} = p_i$ for $h = 0, \pm 1, \pm 2, \ldots$, $1 \leqq i \leqq 2t$, and correspondingly for the supporting lines L_i. We assume, moreover, that the notation is arranged in such a fashion that p_{i+1} follows immediately after p_i in the positive direction on the boundary of K.

Let H_i^+ denote that closed half-plane determined by L_i which contains K, and let H_i^- denote the other closed half-plane determined by L_i. We put $C_i = H_i^- \cap \cap H_{i+t-1}^+ \cap H_{i+t+1}^+$. Then C_i is a closed convex set whose boundary consists of two half-lines (contained in L_{i+t-1} resp. L_{i+t+1}) and one segment of L_i. (Cf. Fig. 2. where C_1 is shaded.)

Let $K_i = x_i + \alpha_i K$ be that (uniquely determined) set homothetic to K which is inscribed into C_i, i.e. $K_i \subset C_i$, $K_i \cap L_i \neq \emptyset$, $K_i \cap L_{i+t-1} \neq \emptyset$ and $K_i \cap L_{i+t+1} \neq \emptyset$. (For a very similar construction see HADWIGER-DEBRUNNER [4], p. 17.) Then, obviously, $K_i \cap L_{i+j} \neq \emptyset$ for all j with $0 \leqq j \leqq t-1$ and $t+1 \leqq j \leqq 2t-1$. Thus for each $2t-1$ members of $\mathscr{K} = \{K_i : 1 \leqq i \leqq 2t\}$ there exists a straight line intersecting them all. But there is no straight line intersecting all the members of $\mathscr{K}$. Indeed, if L were a common secant for the members of $\mathscr{K}$, due to the symmetry of the family $\mathscr{K}$ (viz. $K_i = -K_{i+t}$), the line $-L$ would also be a common secant. Then $L^* = L + (-L)$ would be a common secant, too. But if L^* is a common secant for K, it is a fortiori a common secant for the family $\mathscr{C} = \{C_i : 1 \leqq i \leqq 2t\}$. However, the only common secants (through the origin) of all the members of $\mathscr{C}$ are easily seen to be those straight lines which, for some i, intersect $C_i \cup C_{i+t-1}$ in the single point $C_i \cap C_{i+t-1} = L_i \cap L_{i+t-1}$. But such a line misses both K_i and K_{i+t-1} by the choice of the exposed points $p_i \in K$ and supporting lines L_i. Thus $\mathscr{S}(2t-1)$ does not imply $\mathscr{S}$ for families of homothets of K. Since t was arbitrary, this ends the proof of Theorem 3.

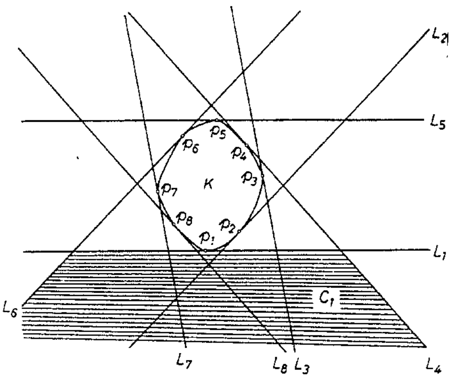

Fig. 2

7. Remarks and Problems. (i) In case $n = 2$, the number $2^{n-1}(n+1)$ of SANTALÓ's theorem is the smallest possible [6], even if only families of translates of a square are considered [3]. This is the only case in which the number $(n+1)p$ of Theorem 1 is known to be the best possible. From the proof of Theorem 3 it follows easily that, in case $n = 2$, one may not substitute $\mathscr{S}(2h-1)$ for $\mathscr{S}(3h)$ in Theorem 1.

(ii) The bound given for $t(k, n)$ in Theorem 2 is probably much too high. For $n = 2$ it can easily be reduced to $3k$. But even in that case, and for P a triangle (i.e. $k = 3$), the least value for $t(k, n)$ is not known.

(iii) Let $v(k, n)$ denote the maximal number of vertices of P^* for polyhedra $P \subset E^n$ having k vertices. As easily seen $v(k, 2) = 2k$, but in the case $n \geqq 3$ no upper bound for $v(k, n)$, better than the trivial $2\binom{k}{2}$, seems to be known[1]).

(iv) It would be interesting to know whether Theorem 3 remains true if only families of translates of K are considered, or if K is not assumed to have a center of symmetry. Is it possible to extend Theorem 3 to higher dimensions?

[1]) Cf. B. GRÜNBAUM, Strictly antipodal sets (to appear in Israel Bull. of Math.).

(v) The subject of the present paper may be put in the following "applied" setting. Assume a linear relationship between some physical variables (e.g. between x and y) is to be tested. Experimental results are obtained as pairs (x_i, y_i), each afflicted with errors Δx_i, Δy_i estimated in a suitable form (e.g., $|\Delta x_i| \leqq \alpha_i$, $|\Delta y_i| \leqq \beta_i$; or $|\Delta x_i| + |\Delta y_i| \leqq \gamma_i$, etc.). To check whether a linear relationship is compatible with the experimental results, within the assumed limits of error, one does not have to consider the whole set of data at once; it is enough to verify the possibility of accomodating linearly subsets of data consisting of a predetermined number of measurements, this number depending on the assumed behaviour of the errors.

(vi) The interpretation in (v) leads quite naturally to the following type of problems, presented here in geometric language and in the simplest case, $n = 2$.

Let a family $\mathscr{K} = \{x_i + K\}$ of translates of a convex body $K = -K \subset E^2$ be given, such that every m-membered subfamily of K has a common secant. What is the smallest positive $\lambda = \lambda(K, m)$ such that, for every $\mathscr{K}$ satisfying the above conditions, the family $\mathscr{K}' = \{x_i + \lambda K\}$ has a common secant.

Even in the simplest cases (e.g., $m = 3$ and K is a circle, or a square) no results seem to be available.

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