# STRICTLY ANTIPODAL SETS*

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ABSTRACT

A subset $A$ of $E^3$ is called strictly antipodal provided that for every pair $X_1$, $X_2$ of points of $A$ there is a pair $H_1$, $H_2$ of parallel supporting planes of $A$ such that $H_i \cap A = \{X_i\}$. The main result asserts that a strictly antipodal set has at most five points. This strengthens a recent result of Croft [2].

1. **Introduction.** For a convex polyhedron $K$ let $v(K)$ denote the number of vertices of $K$. If $K_1$ and $K_2$ are convex polyhedra it is clear that $v(K_1 + K_2) \leqq v(K_1).v(K_2)$. It is easy to find examples showing that equality may hold for suitable $K_1, K_2$ in $E^3$; if $K_1, K_2, \subset E^2$, then

$$v(K_1 + K_2) \leqq v(K_1) + v(K_2).$$

More complicated, and unsolved in the general case, is the following related problem:

If $K$ is a convex polyhedron in $E^n$, with $v = v(K)$ vertices, how many vertices can $K^* = K + (-K)$ have? It is easily checked that, independent of $n$, $v(K^*) \leqq v(v-1)$. However, equality in this relation can take place only if $v$ is not too large with respect to $n$.

Let $f(n)$ denote the maximal $v$ such that there exists an $n$-dimensional convex polyhedron $K$ with $v = v(K)$ and $v(K^*) = v(v-1)$. It is easily seen that $f(2) = 3$. In Section 2 we shall prove the following result.

THEOREM. $f(3) = 5$.

As easy corollaries we shall obtain (in Section 3) a simple solution of a problem of Erdös [5] recently solved by Croft [2], as well as a number of results on families of translates of convex polyhedra in $E^3$. Some additional remarks and problems are also given in Section 3.

2. **Proof of the Theorem.** For an arbitrary set $A \subset E^n$ let a pair of points $X_1, X_2 \in A$ be called *strictly antipodal* if there exists a pair $H_1, H_2$ of (distinct) parallel supporting hyperplanes of $A$ such that $A \cap H_i = \{X_i\}$ for $i = 1,2$. A set $A$ is called strictly antipodal provided ever y two points of $A$ are strictly

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antipodal. Let $g(n)$ denote the maximal number of points in a strictly antipodal set $A \subset E^n$. The following assertion is obvious:

LEMMA. The set of vertices of a convex polyhedron $K$ is strictly antipodal if and only if $v(K^*) = v(K)\cdot(v(K) - 1)$.

If $A$ is a strictly antipodal set and $K$ its convex hull, then $A$ coincides with the set of vertices of $K$. Therefore, the lemma implies that $f(n) = g(n)$.

In order to prove the Theorem it is sufficient to show that the common value of $f(3)$ and of $g(3)$ is 5. Since $f(3) \geqq 5$ (see Section 3) and since every subset of a strictly antipodal set is strictly antipodal, we have only to show that no 6-pointed set in $E^3$ is strictly antipodal.

Assuming this to be false, let $A$ be a six-pointed strictly antipodal set in $E^3$, with convex hull $K$. Because of $f(2) = g(2) = 3$, all the faces of $K$ are triangles. Counting incidences and using Euler's formula it follows at once that $K$ must be a polyhedron of one of the two types represented in Figure 1 by their Schlegel diagrams.

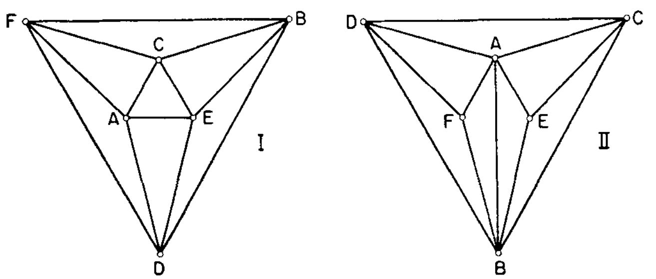


Figure 1

We first note that $K$ cannot have configuration II. Indeed, if there would exist such $K$, in view of the affine invariance of strict antipodality we could assume that $K$ has the form indicated in Figure 2. Since the segment $EF$ is not an edge of $K$, $\min\{a,\alpha\} < 1$. Without loss of generality we assume that $a < 1$, and we note

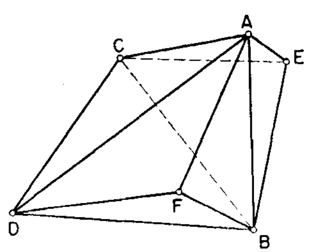


Figure 2

$$A = (1,0,1);\ B = (1,0,-1);\ C = (-1,1,0);$$

$$D = (-1,-1,0);\ E = (a,b,c);\ F = (\alpha,\beta,\gamma).$$

that $E$ is contained in the wedge formed by the planes through $ACD$ and $BCD$. Considering the sections of $K$ by the planes $z = 0$ resp. $z = c$ it is immediate that $C$ and $E$ are not strictly antipodal.

Thus we may assume, for the remaining part of the proof, that $K$ has configuration I; without loss of generality we may therefore assume that $K$ has the form indicated in Figure 3.

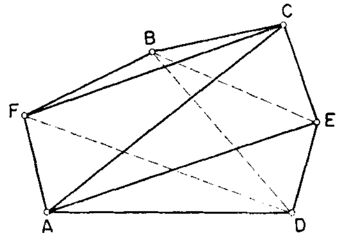


Figure 3

$$A = (-1,-1,0);\ B = (-1,1,0);\ C = (1,0,1);$$

$$D = (1,0,-1);\ E = (a,b,c);\ F = (\alpha,\beta,\gamma).$$

Since sufficiently small displacements of the points of a strictly antipodal set do not destroy strict antipodality, it follows that no generality is lost in assuming that no edge of $K$ is parallel to a face of $K$, and that $|b| + |c| \neq 1$, $|\beta| + |\gamma| \neq 1$. It follows that $K^* = K + (-K)$ will have 16 triangular faces (called $t$-faces in the sequel), and that all other faces of $K^*$ are parallelograms (called $p$-faces in the sequel). Using again a count of incidences and Euler's formula, it is easily found that $K^*$ has 20 $p$-faces.

We shall end the proof of the Theorem by examining the construction of $K^*$ and by showing that it cannot contain 20 $p$-faces.

Let $K_1$ denote the convex hull of $\{A,B,C,D,E\}$. Then $K_1^*$ is a polyhedron with the vertices $\pm(C-A) = \pm(2,1,1)$, $\pm(D-A) = \pm(2,1,-1)$, $\pm(C-B) = \pm(2,-1,1)$, $\pm(D-B) = \pm(2,-1,-1)$, $\pm(B-A) = \pm(0,2,0)$, $\pm(C-D) = \pm(0,0,2)$, $\pm(E-A) = \pm(1+a,1+b,c)$, $\pm(E-B) = \pm(1+a,-1+b,c)$, $\pm(E-C) = \pm(-1+a,b,-1+c)$, $\pm(E-D) = \pm(-1+a,b,1+c)$. Since $K$ is of type I, we have $a > 1$; this and the convexity of $K_1^*$ imply that $|b| + |c| < 1$. (Indeed, by considering the vertices $C-A$, $D-A$, $C-B$, $D-B$, $E-A$ and $E-B$ it follows from $a > 1$ that $|c| < 1$. Then, assuming without loss of generality that $c > 0$ and $b + c > 1$, a consideration of the vertices $C-D$, $C-B$, $E-B$, $E-D$, leads to the contradiction that $C-A$ is not a vertex of $K_1^*$). Projecting orthogonally the part of $K_1^*$ contained in the half-space $E^+ = \{(x,y,z)|\ x \geqq 0\}$ onto $x = 0$, we obtain a configuration of the type represented in Figure 4a. Denoting by $K_2$ the convex hull of $\{A,B,C,D,F\}$, the same reasoning applied to $K_2^*$ leads to a configuration of the type given in Figure 4b.

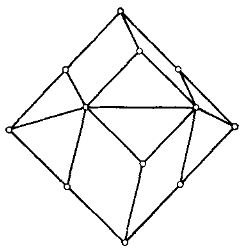

Figure 4a

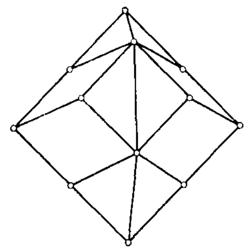

Figure 4b

Now, $K^*$ is the convex hull of $K_1^* \cup K_2^* \cup \{E - F, F - E\}$. Obviously, $E - F$ and $F - E$ are incident only to $t$-faces of $K^*$; therefore the number of $p$-faces of $K^*$ at most equals the number of $p$-faces in the convex hull $Q$ of $K_1^* \cup K_2^*$. But the latter number is at most 12. Indeed, superimposing Figures 4a and 4b, we observe that every $p$-face of $Q$ is a $p$-face of either $K_1^*$ or $K_2^*$, and that only one $p$-face of $Q$ contained in the half-space $E^*$ can be incident to each of the vertices $C - A$, $D - A$, $C - B$, $D - B$.

Thus $Q$ has at most four $p$-faces contained in $E^+$; by symmetry the same number of $p$-faces of $Q$ is contained in the half-space $E$. Together with the four $p$-faces parallel to the $x$-axis, this yields at most 12 $p$-faces for $Q$, and thus also for $K^*$, in contradiction to the former assertion that $K^*$ has 20 $p$-faces.

This completes the proof of the Theorem.

3. **Some related results and problems.**

i) Erdös [5] posed the problem of determining the maximal number $e(n)$ of points in $E^n$ such that all the angles determined by triplets of the points be acute. For $n = 3$ Croft [2] recently established that $e(3) = 5$. This results also from our Theorem and from the obvious assertion $e(n) \leqq g(n)$.

ii) The inequality $e(n) \geqq 2n - 1$ was established in [3] by means of the following example (reproduced here for the sake of completeness): Let $\{e_i\}_{i=1}^n$ be mutually orthogonal unit vectors in $E^n$. The $(2n - 1)$-pointed set $\{A, B_2, \cdots, B_n$, $C_2, \cdots, C_n$ satisfies Erdös' condition if, e. g., $A = e_1$, $B_k = \alpha_k e_1 + e_k$, $C_k = -\alpha_k e_1 - e_k$, $k = 2, 3, \cdots, n$, where all $\alpha_k$'s satisfy $0 < \alpha_k < 1$ and are different from each other.

iii) As mentioned (in part) in [3], it is easily shown that $g(n)$ is also the maximal number of members in any family $\mathscr{K}$ of translates of a convex body $K \subset E^n$, provided the family satisfies any of the following conditions:

(a) The intersection of any two members of $\mathscr{K}$ is a single point;

(b) The intersection of all members of $\mathscr{K}$ is a single point, which is also the only common point of any two members of $\mathscr{K}$;

(c) The intersection of any two members of $\mathscr{K}$ is $(n - 1)$-dimensional.

The same is true if in (a) or in (c) the attention is restricted to centrally symmetric $K$.

iv) The restriction of $\mathscr{K}$ to families of translates of one convex body is essential in iii). This is obvious in case of conditions (a) and (b); in the case of (c) the bound is 4 for $n = 2$ (while $g(2) = 3$); already for $n = 3$ it has been proved repeatedly (e.g. by Tietze, Besicovitch, Rado; see [4] for references to these and some related results) that there exists no finite bound. In [4] it is also pointed out that arbitrarily large families $\mathscr{K}$ in $E^3$, any two of whose members have a 2-dimensional intersection, are obtainable as the Schlegel-diagrams of the duals of 4-dimensional "neighborly polytopes" (see [7]). This was known, however, already to Brückner [1].

Nevertheless, the following question seems to be open even for $n = 3$: How many members can a family of centrally symmetric convex bodies in $E^n$ have, if every two have an $(n-1)$-dimensional intersection?

v) Among unsolved problems related to the Theorem of the present paper we mention:

(a) The determination of $e(n)$ and of $f(n) = g(n)$ for $n > 3$; in particular, is $e(n) = g(n)$ for all $n$?

(b) The determination of $h(k,n) = \max\{v(K + (-K)) \mid K \subset E^n, \ v(K) = k\}$ for $k \geqq 2n$, $n \geqq 3$.

Remark. The example ii) above implies $h(k,n) = k(k-1)$ for $n+1 \leqq k \leqq 2n-1$; for $k > n = 2$, we have $h(k,2) = 2k$. This follows from the observation that $h(k,n) = 2s(k,n)$, where $s(k,n)$ is the maximal number of strictly antipodal pairs of vertices for a convex polyhedron $K \subset E^n$ with $v(K) = k$, and the assertion $s(k,2) = k$. To prove this latter assertion assume that $s(k,2) > k$ for some $k$. Let $k_0$ be the minimal $k$ with this property and let $K$, with $v(K) = k_0$, be a $k_0$-gon such that more than $k_0$ pairs of vertices of $K$ are strictly antipodal. Then at least one vertex $V_0$ is antipodal to some three consecutive vertices $V_{i-1}$, $V_i$, $V_{i+1}$ of $K$; but then $V_i$ is easily seen to be strictly antipodal only to $V_0$. Thus the convex hull of the $k_0 - 1$ vertices of $K$ different from $V_i$ yields an example showing $s(k_0 - 1, 2) > k_0 - 1$, in contradiction to the minimality of $k_0$.

It is worth mentioning that $s(k,3) \geqq [k/2]\cdot[k(+1)/2] + 2$ for $k \geqq 4$, the difference in behavior between $s(k, 2)$ and $s(k,3)$ being similar to the jump in the number of times the diameter of a set is assumed in 3- and 4-dimensional sets (Erdös [6]). The above inequality is easily established by placing approximately half of the points on each of two suitable circular arcs.

vi) Klee [8] defined a pair of points $X_1$, $X_2$ of a set $A \subset E^n$ to be *antipodal* provided there exist distinct parallel supporting hyperplanes $H_1$, $H_2$ of $A$ such that $X_i \in A \cap H_i$, i $= 1,2$; he also asked about the maximal number of points in a set $A \subset E^n$ such that every two points of $A$ are antipodal. It was established in [3] that the required number is $2^n$. In analogy to the above definition of $s(k,n)$ one may ask what is the maximal number $a(k,n)$ of pairs of antipodal points in $k$-pointed sets in $E^n$. While the problem is open for $n \geqq 3$, it can be shown by arguments similar to those used above in connection with $s(k,2)$ that $a(k,2) = [3k/2]$.

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Added in proof (June 13, 1963):

The result of Croft [2] that $e(3) = 5$ was recently established also by Schütte, K., 1963, Minimale Durchmesser endlicher Punktmengen mit vorgeschriebenem Mindestabstand, *Math. Annalen*, **150**, 91–98.