LONGEST SIMPLE PATHS IN POLYHEDRAL GRAPHS

B. GRÜNBAUM *and* T. S. MOTZKIN

1. A graph (= one-dimensional simplicial complex) G shall be called d-polyhedral if the nodes and edges of G may be identified with the vertices and edges of a convex polyhedron P in Euclidean d-space.

Tutte [**8**] constructed a 46-node 3-polyhedral graph H (reproduced also in Berge [**2**]) of valence† 3 which does not admit a Hamiltonian circuit (= closed path passing exactly once through each node); this example disproved Tait's [**7**] conjecture on the existence of a Hamiltonian circuit in any 3-polyhedral graph of valence 3. Balinski [**1, 1′**] mentions as unsolved the question whether every n-polyhedral graph admits a simple path passing through all the nodes of G.

In the present note we shall show that the answer to Balinski's problem is negative, and establish a stronger result on the maximal length of simple paths in some classes of graphs.

For any graph G with $n(G)$ nodes let $p(G)$ denote the maximal number of nodes contained in a simple path, and let

$$p(n, d) = \min\{p(G): \ G \text{ is } d\text{-polyhedral and } n(G) = n\};$$

$$p^*(n, d) = \min\{p(G): \ G \text{ is } d\text{-polyhedral, of valence } d, \text{ and } n(G) = n\}.$$

A theorem of Dirac [**3**; Theorem 5] implies that $p^*(n, d) > c \log n$, where $c > 0$ depends on d only, the same estimate holding also if one of the endpoints of the path is preassigned.

We shall prove the following results:

THEOREM 1. *There exists an* $\alpha < 1$ *such that* $p^*(n, 3) < 2n^\alpha$; *e.g.*, $\alpha = 1-2^{-19}$.

THEOREM 2. *There exists an* $\alpha < 1$ *such that* $p(n, d) < 2(d-2)n^\alpha$ *for* $d \geqslant 3$; *e.g.*, $\alpha = 1-2^{-19}$.

2. In the proof of these theorems we shall use the following facts:

(A) Let G be a connected planar graph (with no 1- or 2-circuits), which has at least four nodes, and is such that:

(1) each edge belongs to two different faces;

Received 19 September, 1960. Presented to the American Mathematical Society on January 25, 1961. The preparation of this work was sponsored in part by the Office of Naval Research, U.S.A.

† Number of edges incident at a node. " Degree " is also used, but the term seems overworked; the chemical term " valence " suggested by one of the authors (see, *e.g.*, [**4**]) was recommended by the International Colloquium on Graph Theory, Dobogókő, 1959.

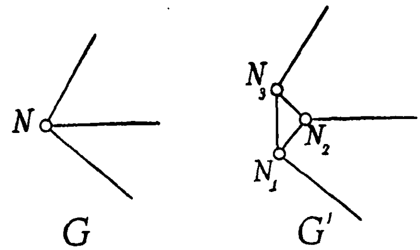

Fig. 1.

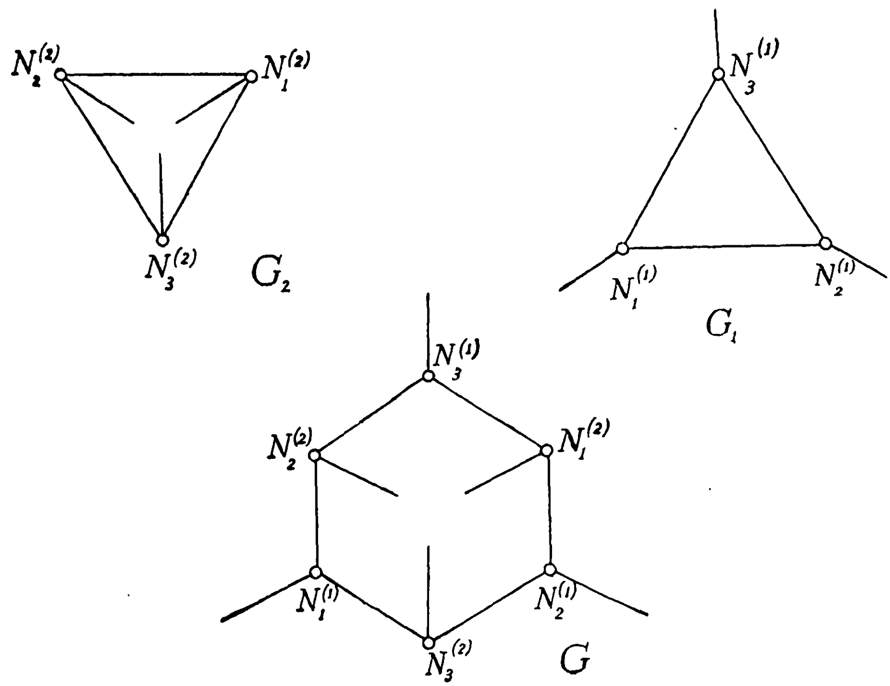

Fig. 2.

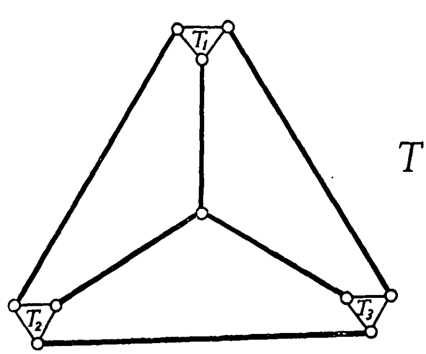

Fig. 3.

(2) for each node N and each face F containing N there exist exactly two edges containing N and contained in F;

(3) if the different nodes N_1 and N_2 both belong to the different faces F_1 and F_2, then N_1 and N_2 determine an edge which belongs to F_1 and F_2;

then G is 3-polyhedral.

Statement (A) is easily derived from Steinitz's "Fundamentalsatz der konvexen Typen" [**6**; p. 192]. As an inspection of the following proof of Theorem 1 shows, the assumptions of (A) will be fulfilled for the graphs T, H^*, H^{**}, G_n and $G^{(n)}$, and therefore these graphs are 3-polyhedral. [The analogue of (A) for d-polyhedral graphs, $d > 3$, seems to be unknown.]

(B) If G is a graph, N a node of degree 3 of G, and if G' is obtained from G be deleting N and introducing three nodes N_1, N_2, N_3, forming a 3-circuit, then $p(G') \leqslant p(G)+2$. (Fig. 1.)

(C) If G_1 and G_2 are graphs containing 3-circuits $N_1^{(1)}$, $N_2^{(1)}$, $N_3^{(1)}$, resp. $N_1^{(2)}$, $N_2^{(2)}$, $N_3^{(2)}$, and G is obtained by *merging* these two 3-circuits to a 6-circuit (Fig. 2), then $p(G) \leqslant p(G_1)+p(G_2)$.

(D) Any simple path P containing all the nodes of the graph T (Fig. 3) *passes* through one of the triangles T_i (*i.e.* no endpoint of P is in that T_i), and omits one of the three heavy edges issuing from that T_i.

The statements (B), (C) and (D) are easily verified, and we omit the proofs. (For (B) and (C) the discussion is similar to that in Tutte [**8**].)

3. We now turn to the proof of Theorem 1.

Starting from Tutte's graph H which admits no Hamiltonian circuit we obtain the graph H^* (Fig. 4) by replacing a node of H (arbitrarily chosen) by a 3-circuit. As in (B), it is clear that H^* admits no Hamiltonian circuit. Now we merge, by the procedure described in (C), three copies of H^* with T, one copy at each of the three 3-circuits of T represented by light edges in Fig. 3, and obtain a new graph H^{**} (with 154 nodes). Because of (D), *no simple path in* H^{**} *contains all the nodes of* H^{**}. [If for a d-polyhedral ($d \geqslant 3$) graph G we have $p(G) < n(G)$, then the nodes of G may not be covered by two simple, disjoint circuits. For H^{**} it is easily seen that its nodes may not be covered even by three simple, disjoint circuits; they may be covered by four such circuits.]

Next, we take six copies H_i^{**}, $i = 1, 2, 3, 4, 5, 6$, of H^{**}, replace one node by a 3-circuit in H_1^{**} and in H_6^{**}, two nodes by two 3-circuits in H_i^{**}, $i = 2, 3, 4, 5$, and merge these six graphs into a new graph G_1. Then G_1 has 944 vertices and, according to (C), $p(G_1) \leqslant 938$.

We proceed by induction to construct a sequence of graphs G_k, with $n_k = n(G_k)$ nodes and such that every simple path in G_k contains at most

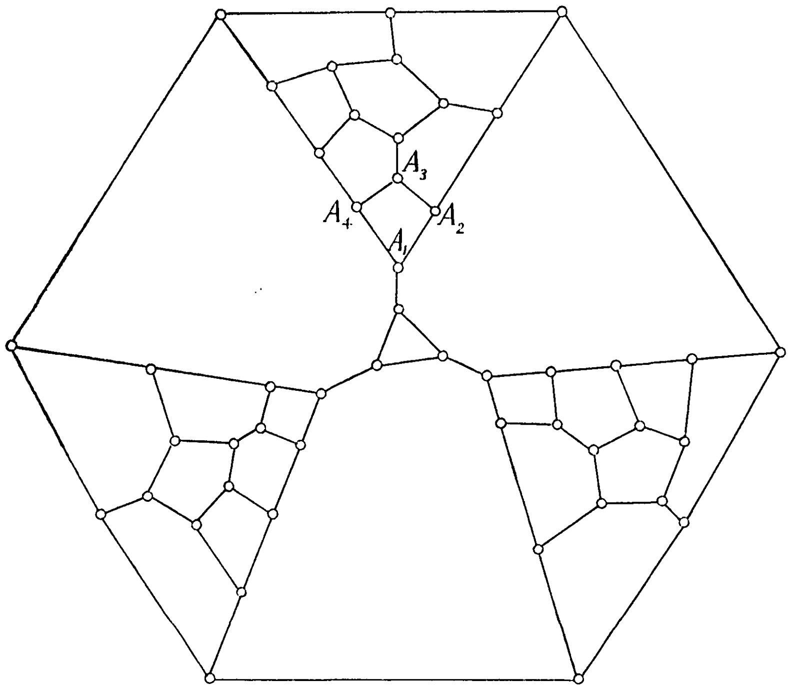

Fig. 4.

$p_k = p(G_k)$ nodes, where the order of magnitude of p_k is indicated in Theorem 1.

We first replace each node of G_k by a 3-circuit, thus obtaining a graph G_k^* with $n(G_k^*) = 3n_k$. Then we take n_k copies of G_k, replace in each of them one node by a 3-circuit, and merge these graphs with G_k^* along the 3-circuits of G_k^*. The resulting graph is G_{k+1}. Obviously

$$n_{k+1} = n_k(n_k+5).$$

Since each simple path in G_k fails to contain some $n_k - p_k$ (or more) nodes, the same number of copies of G_k, used in the construction of G_{k+1}, will be completely missed by any simple path in G_{k+1}; in each of the remaining p_k (or less) copies of G_k, the simple path in G_{k+1} determines a simple path in G_k. Therefore, taking into account the additional nodes introduced with the 3-circuits, it follows that each simple path in G_{k+1} omits at least $(n_k - p_k)(n_k+5) + p_k(n_k - p_k)$ nodes, and thus $p_{k+1} \leqslant p_k(p_k+5)$. [We note that the construction may be arranged in such a way as to ensure $p_{k+1} = p_k(p_k+5)$.] Let $\beta > 0$ be such that

$$\frac{p_1+5}{n_1} < \frac{1}{(n_1+5)^{\beta}}.$$

(In view of $n_1 = p_1 + 6 = 944$, we may take, *e.g.*, $\beta = 2^{-17}$.) By induction it follows that

$$\frac{p_k}{n_k} < \frac{p_k+5}{n_k} < (n_k+5)^{-\beta} \text{ for all } k > 1.$$

Indeed,

$$\frac{p_{k+1}}{n_{k+1}} \leqslant \frac{p_k(p_k+5)}{n_k(n_k+5)} < \left(\frac{p_k+5}{n_k}\right)^2 < (n_k+5)^{-2\beta} < (n_{k+1}+5)^{-\beta}.$$

To complete the proof we have still to construct graphs $G^{(n)}$ for all even integers n different from the n_k. Let $m_k = n_k+4$; then

$$m_{k+1} = m_k(m_k-3) \text{ and } p_k+5 < m_k^{1-\beta}.$$

Any even integer $n > m_1$ can be (uniquely) expressed in the form $n = 2(q_0-2)+\Sigma_{i=1}^k q_i m_i$, with $0 \leqslant q_i < m_i-3$ for $1 \leqslant i \leqslant k$, $q_k \geqslant 1$ and $0 \leqslant 2q_0 < m_1$. We obtain a graph $G^{(n)}$ with n nodes by taking q_i copies of G_i, for $i = 1, \ldots, k$, replacing some of their nodes by 3-circuits and merging them, and by replacing in the resulting graph q_0 nodes by 3-circuits. Using the easily verified relations

$$(m_{s+1}+3)p_{s+1} \geqslant m_1 + \sum_{i=1}^{s} m_i(p_i+2)$$

and

$$p(n) \leqslant p(G^{(n)}) \leqslant \sum_{i=1}^{k} q_i(p_i+2)+2q_0-4$$

we obtain

$$\frac{p(n)}{n} \leqslant \frac{2q_0-4+\Sigma_{i=1}^k q_i(p_i+2)}{2q_0-4+\Sigma_{i=1}^k q_i m_i} \leqslant \frac{q_k(p_k+2)+(m_{k-1}-3)p_{k-1}}{q_k m_k}$$

$$\leqslant \frac{p_k+2}{m_k} + \frac{p_{k-1}}{m_{k-1}} \leqslant 2m_k^{-\frac{1}{2}\beta} < 2m_{k+1}^{-\frac{1}{4}\beta} < 2n^{-\frac{1}{4}\beta}.$$

This ends the proof of Theorem 1, with $\alpha < 1-2^{-19}$. (A slightly better bound, $\alpha < 1-2^{-16}$, can be obtained if instead of T one uses the graph obtained by truncating all the vertices of a simplex.)

4. Before proving Theorem 2, we observe the following immediate consequence of Theorem 1.

Let $p_r(n)$ denote the maximal number of nodes of $G^{(n)}$ contained in the union of r mutually disjoint, simple paths. Then for each r we have $p_r(n) < 2rn^{\alpha}$.

The proof of Theorem 2 is now trivial. Let $P_3^{(n)}$ be a convex polyhedron in E^3 whose vertices and edges form a graph isomorphic with $G^{(n)}$. For $d > 3$ we construct by induction a d-dimensional polyhedron $P_d^{(n)}$

with $n+d-3$ vertices, by taking the convex hull of $P_{d-1}^{(n)}$ and a point outside the E^{d-1} spanned by $P_{d-1}^{(n)}$. For the graph $G_d^{(n)}$, determined by the vertices and edges of $P_d^{(n)}$, we have

$$p(n+d-3,\, d) \leqslant p(G_d^{(n)}) \leqslant d-3+p_{d-2}(n) < d-3+2(d-2)n^{\alpha}$$
$$< 2(d-2)(n+d-3)^{\alpha}.$$

This completes the proof of Theorem 2.

5. *Remarks.* (i) Tutte [**10**] proved that every 4-connected *planar* graph (with at least two edges) has a Hamiltonian circuit. Balinski [**1′**] proved that every d-polyhedral graph is d-connected. Our Theorem 2 shows, therefore, that Tutte's result may not be generalised from planar to d-polyhedral graphs, despite the higher degree of connectivity of these graphs.

(ii) It would be interesting to find closer bounds for $p^*(n, 3)$ than $c_1 \log n < p^*(n, 3) < c_2 n^{\alpha}$, given by Dirac's [**3**] and our results. Possibly $p^*(n, d) = o(n^{\alpha})$ for every $\alpha > 0$.

(iii) The polyhedra $P_3^{(n)}$ we constructed contain as faces k-gons such that $k \to \infty$ for $n \to \infty$. Does an estimate $p(G) > c\,n(G)$ hold for some $c > 0$ and 3-polyhedral graphs G of valence 3 for which the corresponding polyhedra have faces of bounded orders? Is $p(G) = n(G)$ if G has valence 3 and if all the faces of the polyhedron are k-gons, $k \leqslant 6$, or if paths may pass through diagonals of the faces? (*Cf.* [**11**].)

(iv) Among the many unsolved problems related to paths in graphs or in polyhedral graphs, we mention also:

(1) Is it possible to generalise Dirac's logarithmic lower bound to d-polyhedral graphs, or to planar graphs of connectivity $\geqslant 3$?

(2) If N_1, N_2 are nodes of a d-polyhedral graph G, let $p(N_1, N_2)$ denote the maximal number of nodes of G contained in a simple path with endpoints N_1 and N_2. It is easily seen, even if $N_1 \neq N_2$ is assumed, that $p(N_1, N_2)$ may depend on N_1 and N_2. [*E.g.*, in the graph H^* (Fig. 4), $p(A_1, A_3) = n(H^*) = p(A_2, A_4)+1$.] How does

$$\min_{N_1, N_2 \varepsilon G} p(N_1, N_2)$$

depend on $n(G)$; what are the bounds, in terms of $n(G)$, for

$$\Sigma_{N_1 \neq N_2} p(N_1, N_2)?$$

Does the average of $\max_{N_1, N_2 \varepsilon G} p(N_1, N_2)$ over all non-isomorphic (d-polyhedral) graphs G with n nodes tend to 0 for $n \to \infty$? For what fraction of non-isomorphic graphs of order n is $p(G) = n$?

(3) How does the minimal number $m(G)$ of disjoint, simple paths needed to cover all the nodes of G depend on $n(G)$? From Theorem 1 follows that for some graphs $m(G) > [n(G)]^{\beta}$ for some fixed $\beta = 1-\alpha > 0$. Conceivably $\max_{n(G)=n}\big(m(G).p(G)\big) > n^{1+\gamma}$ for some $\gamma > 0$.

(4) Let $\delta(N_1, N_2)$ denote the distance between N_1 and N_2 (*i.e.* minimal number of nodes contained in a path connecting the nodes N_1 and N_2 of G). From Balinski's [1'] result and Whitney's [**12**] theorem it follows that $\delta(N_1, N_2) \leqslant 2+\dfrac{n(G)}{d}$ for d-polyhedral graphs G. This result is the best possible one, as is shown by the following example (described, for simplicity, for $d = 3$; completely analogous examples may be given for $d > 3$). We take $3n+2$ points on the unit sphere, namely the two poles and the points with (spherical) coordinates $\left(\dfrac{k\pi}{2n}; \dfrac{2m\pi}{3}\right)$ where $m = 0, 1, 2$ and $k = 0, 1, \ldots, n-1$. The convex hull of these points has $3n+2$ vertices, and the distance between the poles is $n+2$.

On the other hand, if the d-polyhedral graph G is required to be of valence d, a better estimate of $\delta(N_1, N_2)$ should be possible. *E.g.*, for $d = 3$, one might conjecture $\delta(N_1, N_2) \leqslant 2+\dfrac{n(G)}{4}$; this bound is reached for n-sided prisms.

(5) One may also consider paths passing through some nodes more than once. A result of Petersen [**5**; §5] may be stated as follows: For every graph G of valence $2k$ there exists a closed path passing through each edge once and through each node k times. Using the criterion of Tutte [**9**; Theorem V] and Balinski's [**1'**] result on the degree of connectivity of polyhedral graphs, it follows easily that for every $(2d-1)$-polyhedral graph of valence $2d-1$ there exists a closed path passing d times through each node. One might inquire, *e.g.*, whether the last statement is true for all $(2d-1)$-polyhedral graphs; how many nodes have to be passed more than once by a path containing all the nodes; what fraction of the $n(G)$ nodes can be reached by paths containing no node more than k times; and for which smallest $k = k(n, d)$ every d-polyhedral graph with n nodes can be covered by a path that passes through no node more than k times.

(v) Another class of related problems concerns the nature of trees, or other special graphs, containing all the nodes of a given (polyhedral) graph. The existence of such trees is obvious for all connected graphs; it is easy to see, using Theorem 1, that already for 3-polyhedral graphs the number of, and the valences at, the branching points cannot be simultaneously bounded. Provided the valences of branching points are bounded (as is the case, *e.g.*, in d-polyhedral graphs of valence d) how does the minimal number of branching points of trees containing all the nodes of G increase with increasing $n(G)$? Does there exist, for every 3-polyhedral graph G, a connected graph G^* containing all the nodes of G and such that each node of G is of valence $\leqslant 3$?

Added in proof.

1. The main result of Whitney [11] is that there exists a Hamiltonian path in every polyhedral graph G with the property that the corresponding polyhedra P have only triangular faces, G being such that every 3-circuit corresponds to a face of P. Whitney also established by an example that the last condition may not be dropped. Starting from a 60-faced polyhedron derived from the icosahedron, it is not hard to produce examples of polyhedral graphs G_n, with n nodes, whose corresponding polyhedra have only triangular faces, such that $p(G_n) < 2n^{\alpha}$, for some fixed $\alpha < 1$.

2. J. Chuard claimed to have proved Tait's conjecture on the existence of Hamiltonian paths in polyhedral graphs of valence 3 (and thus also to have affirmatively solved the four-colour problem); see (i) "Une solution du problème des quatre couleurs", *Verh. Internat. Math.-Kongr., Zürich*, 1932, Vol. 2, 199–200, and (ii) "Les réseaux cubiques et le problème des quatre couleurs", *Mémoires de la Société Vaudoise des Sciences Naturelles*, No. 25, Vol. 4 (1932), 41–101. Doubts in the validity of Chuard's claim were raised in a review of (ii) by Pannwitz in the *Jahrbuch über die Fortschritte der Math.*, 58 (1932), 1204.

References.

1. M. L. Balinski, "An algorithm for finding all vertices of convex polyhedral sets", *Princeton ONR Report*, May 1959, p. 23.

1′. ———, "On the graph structure of convex polyhedra in n-space", *Pacific J. Math.*, 11 (1961), 431–434†.

2. C. Berge, *Théorie des graphes et ses applications* (Dunod, Paris, 1958).

3. G. A. Dirac, "Some theorems on abstract graphs", *Proc. London Math. Soc.* (3), 2 (1952), 69–81.

4. Th. Motzkin, *Contributions à la théorie des graphes*, C. R. Congrés Internat. Math. (Oslo, 1936, Vol. 2, pp. 133–134 (1937)).

5. J. Petersen, "Die Theorie der regulären Graphs", *Acta Math.*, 15 (1891), 193–220.

6. E. Steinitz, *Vorlesungen über die Theorie der Polyeder* (Springer, Berlin, 1934).

† The present paper was completed in July 1960. In a revised version of [**1′**] it is stated that a graph without a simple covering path has been found by T. A. Brown, unpublished note, RAND Corporation, August 1960. Brown's paper is to appear in the *Pacific J. Math.* For non-constant valence already the rhombic dodecahedron provides a counter-example (Problem E 711, *Amer. Math. Monthly*, 53 (1946), 146 and 593).

7. P. G. Tait, "Listing's 'Topologie'," *Phil. Mag.* (5), 17 (1884), 30–46. (Sci. Papers, Vol. 2, pp. 85–98.)
8. W. T. Tutte, "On Hamiltonian circuits", *Journal London Math. Soc.*, 21 (1946), 98–101.
9. ———, "The factorization of linear graphs", *Journal London Math. Soc.*, 22 (1947), 107–111.
10. ———, "A theorem on planar graphs", *Trans. American Math. Soc.*, 82 (1956), 99–116.
11. H. Whitney, "A theorem on graphs", *Annals. of Math.*, 32 (1931), 378–390.
12. ———, "Congruent graphs and the connectivity of graphs", *American J. of Math.*, 54 (1932), 150–168.

University of California,
Los Angeles, Calif., U.S.A.