Geombinatorics 23(2013), 5 – 11.

Revisiting "Small unstretchable simplicial arrangements of pseudolines"

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The aim of this note is to present new information about small unstretchable simplicial arrangements of pseudolines, that became available since 2009. Several open problems concerning such arrangements are proposed.

An *arrangement of pseudolines* is the partition of the real projective plane by a family of simple curves such that each curve differs from a straight line in a finite part only, and every two have a single point in common at which they cross transversally. Throughout, we model (or interpret) the real projective plane as the *extended Euclidean plane*, with added points "at infinity" and the line "at infinity" consisting of all the points at infinity. An arrangement is *simplicial* if every region of the partition is a triangle. An arrangement of pseudolines is *stretchable* provided it is isomorphic to an arrangement with straight lines (a *linear* arrangement). Examples of *unstretchable* simplicial arrangements of pseudolines are shown in Figures 1 and 2.

Many of the new developments are contained in the paper [1] by M. Cuntz. Applying the method of "wiring diagrams" introduced by Goodman and Pollack [2], [4], together with innovative arguments to reduce the computational effort, Cuntz is able to enumerate all simplicial arrangements of at most n = 27 pseudolines. His methodology would work for larger values of n as well, but the computational effort would be very large. Among the many results in [1] are the following:

(1) Confirmation of Conjecture 1 from [6] to the effect that all simplicial arrangements of at most 14 pseudolines are stretchable.

(2) Disproof of Conjecture 2 from [6] that there is a single unstretchable simplicial arrangement of 15 pseudolines, and precisely four such arrangements with 16 pseudolines. We discuss details below.

(3) Disproof of Conjecture 3 of [6] that all simplicial arrangements of lines are described in [5]. This will be discussed in detail in another publication.

Regarding (1) it should be stressed that without assuming that the arrangement is simplicial this is not valid. It is well known that there are unstretchable arrangements of nine pseudolines but that there are no such arrangements of at most 8 pseudolines (see, for example, [3]).

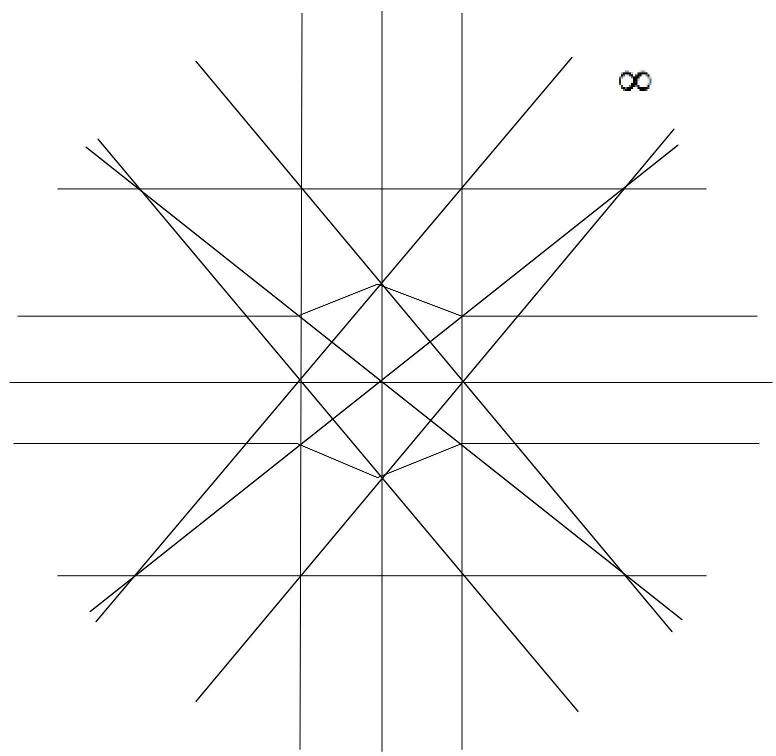

Figure 1. An unstretchable simplicial arrangement with 15 pseudolines. Adapted from [6], where it was denoted $B_1(15)$.

Concerning (2) we observe first that the supposedly unique simplicial arrangement of 15 pseudolines is $B_1(15)$ shown in Figure 1. The new such arrangement $B_2(15)$ found by Cuntz is presented in Figures 3 and 4. In [1] only a wiring diagram $B_2(15)$ is shown; our Figure 3 is an adaptation of this. Figure 4 shows a pseudoline arrangement representing $B_2(15)$. It has 3-fold rotational symmetry. I could not verify the statement in [1] that $B_2(15)$ has 6-fold rotational symmetry.

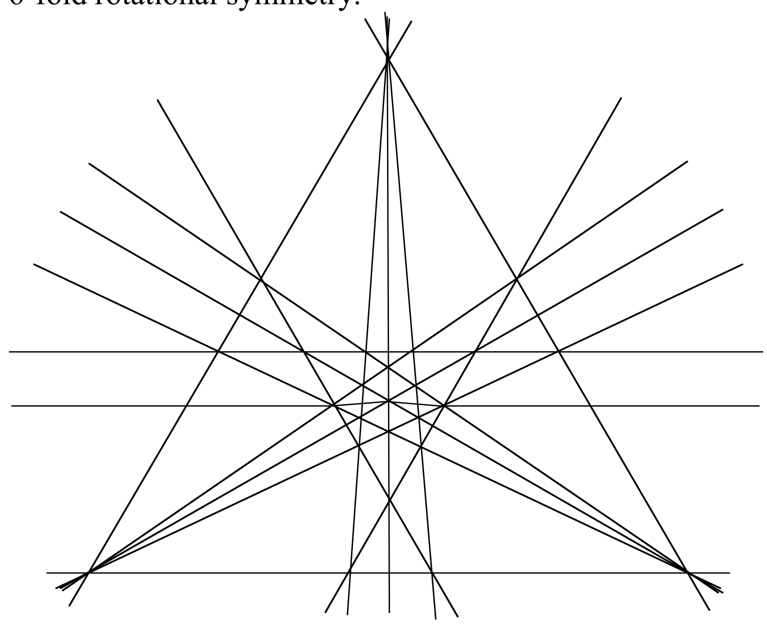

Figure 2. An unstretchable simplicial arrangement of 16 pseudolines. It is taken from [6], where it was designated $B_3(16)$. Note that the departure from straight lines is very small.

Next, we note that according to [1], besides the four unstretchable arrangements of 16 pseudolines described in [6], there are three additional arrangements of this kind. No details are given on any of these. A wiring diagram of an arrangement of 16 pseudolines is shown in [1], reproduced here as Figure 5. In Figure 6 we show a realization of this arrangement with just three non-straight pseudolines. Due to my misunderstanding of the

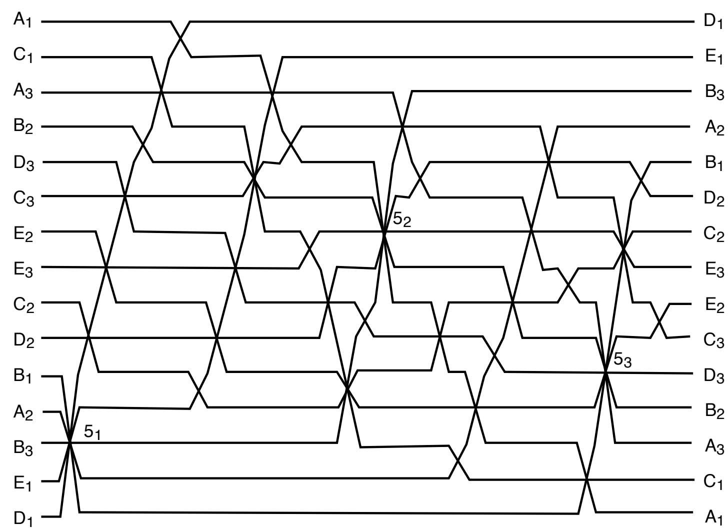

Figure 3. A wiring diagram of the new unstretchable simplicial arrangement $B_2(15)$ of 15 pseudolines found by Cuntz. Adapted from Figure 2 of [1].

corresponding part of [1], I thought that this wiring diagram is one of the three new ones. However, after the realization on Figure 6 was drawn, I realized that this arrangement is isomorphic to $B_3(16)$. As a somewhat non-trivial exercise the reader is invited to discover an isomorphism between arrangements in Figures 5 (or 6) and $B_3(16)$ shown in Figure 3.

An interesting question is how many non-straight pseudolines are required in the construction of each simplicial arrangement A of n pseudolines. We denote the smallest such number (over all arrangements isomorphic to A) by m(A). The examples in Figures 1, 2, 4 show that $m(B_1(15)) \leq 2$, $m(B_3(16)) \leq 3$, $m(B_2(15)) \leq 6$. According to a personal communication by Prof. Cuntz, the true values are $m(B_3(16)) = 1$ and $m(B_1(16)) = 2$. It would seem that the upper bounds for $B_1(15)$ and $B_2(15)$ are too high as well. However, in all cases I did not have the patience to try to find realizations with fewer non-straight pseudolines.

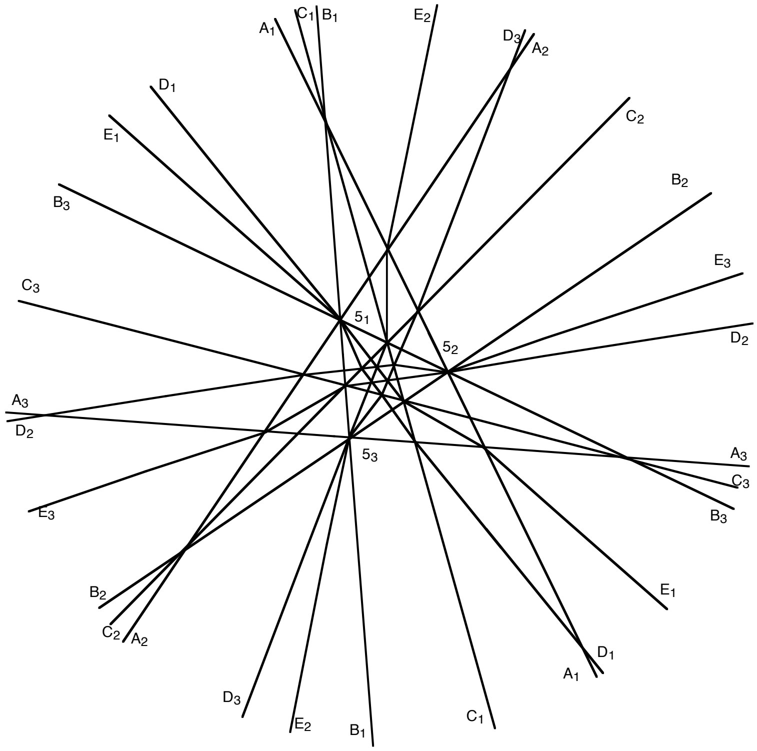

Figure 4. A presentation of the simplicial arrangement $B_2(15)$ of 15 pseudolines in the extended Euclidean model of the real projective plane. The labels of the pseudolines correspond to those in Figure 3.

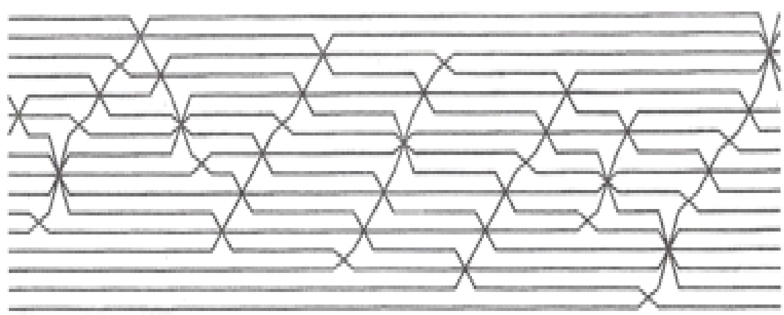

Figure 5. A wiring diagram for an unstretchable simplicial arrangements of 16 pseudolines found by Cuntz [1].

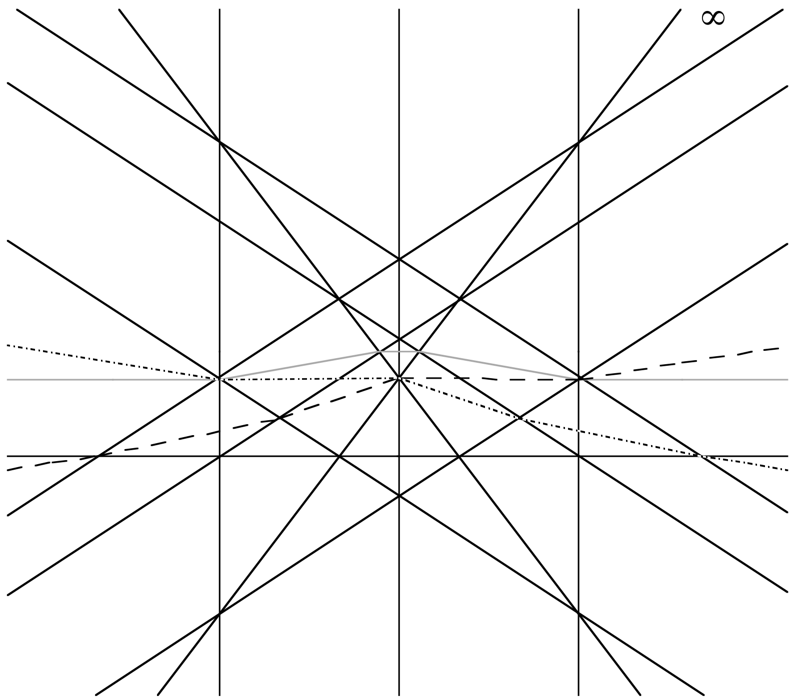

Figure 6. A realization of an unstretchable simplicial arrangement of 16 pseudolines that corresponds to the wiring diagram in Figure 5. In this presentation there are three non-straight pseudolines, shown dashed.

This leads to a related problem: Can the number of non-straight pseudolines be reduced if one of the straight lines is deleted? In all the above cases this is a question not covered by results on simplicial arrangements, since each such deletion leads to an arrangement that is not simplicial.

Another open question is whether the number m(A) can be arbitrarily large for sufficiently large arrangements A? One may conjecture that the answer is affirmative — but no examples with large m(A) are known.

Acknowledgements. The author is grateful for the helpful comments and suggestions by Professors L. W. Berman, M. Cuntz and J. Malkevitch. He also acknowledges the stay at the Helen Riaboff Whiteley Center at the Friday Harbor Laboratories of the University of Washington, which provided the atmosphere and conditions that contributed to the work reported here.

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