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DID YOU REALLY PROVE THAT ?

Branko Grünbaum
Department of Mathematics, Box 354350
University of Washington Seattle, WA 98195-4350
e-mail: grunbaum@math.washington.edu

In his many publications, Paul Erdös made uncountably many conjectures; some were subsequently proved, others disproved; quite a few are still open. To the best of my knowledge, Erdös never published results that were later shown to be invalid. However, I was present in an hour-long lecture in which he was "proving" a theorem that shortly afterwards was shown to be false (see [G1] for details). This note discusses errors of a particular kind that were made by mathematicians, some of them quite famous. In most of the cases listed below the error was not noticed for a long time.

The type of errors I have in mind is the following. When trying to enumerate objects of a certain kind it is often advantageous to replace the objects in question by another (usually more general) type of objects, the enumeration of which is easier. From this enumeration one is then supposed to go back to the original quest. Obviously, there is nothing wrong with this *provided* it is carried our correctly. However, errors arise if in this last step either some possibilities are missed on the unstated assumption that to each of the derived objects corresponds precisely one of the original object, or all the possibilities of the general enumeration are assumed without justification to fit the more special original situation. The next few paragraphs will present examples illustrating these errors.

(1) Several old instances concern the enumeration of (n_3)-configurations. An (n_3)-configuration is a collection of n distinct straight lines in the plane together with a collection of n distinct points, such that each of the points is on precisely three lines and every line is on precisely three points. Well-known examples are the Pappus configuration (with n = 9) and the Desargues configuration (with n = 10). More on configurations and the errors discussed here can be found in [G2].

Late in the nineteenth century, Kantor [K], Martinetti [M] and others considered the case n = 10 and attempted to enumerate the possible (10_3) configurations by counting the *combinatorial* possibilities. By this are meant sets of n objects ("*points*") and n triplets of points ("*lines*") with the appropriate *incidences*. The easy enumeration of these objects yields ten distinct (nonisomorphic) (10_3) combinatorial configurations. The logical error arose when claiming that therefore there are ten geometric configurations. However, a detailed analysis by Schröter [S] showed that while nine of them can be represented geometrically, one combinatorial (10_3) admits no such representation. In fact, Kantor [K] compounded the error by presenting a diagram which was purported to show this particular configuration; naturally, the diagram was incorrect.

(2) Steinitz [St1] purported to show in 1894 hat each (n_3) configuration can be geometrically constructed stepwise so that only the last incidence of a line with three points is possibly unsatisfied. The idea of the construction is that the points and lines (considered combinatorially) can be ordered in such a way that each -- except for the last one -- is incident with at most two of the ones preceding it in the ordering. The argument is nontrivial but correct. It follows, again correctly, that one can always -- except for the last one -- find (geometric) points or lines that satisfy the previous incidences. However, where Steinitz goes wrong is in neglecting the possibility that additional incidences happen, that are not required or allowed by the configuration. A sketch of an easy example is shown in Figure 1, where the Pappus theorem implies that the line L must pass through the point P, although this incidence is not allowed by the configuration. Remarkably, this error remained unnoticed till 2000, when T. Pisanski presented in [P] examples like the one in Figure 1. It is a puzzle how such a gross logical error remained hidden for more than a century. Steinitz [St2] surveys the topic of configurations without noticing the error. This was about the last time Steinitz wrote about configurations; could he have noticed his mistake but in the competitive atmosphere in German universities decided to leave well alone? On the other hand, the

thesis [St1] is written in such an unreadable style that apparently nobody checked the details for more than a century. In fact, the second part of the thesis, that deals with elucidation of the cases in which the last incidence is also fullfillable, is so confusing that I have not heard of anybody claiming to have understood it. The "final polynomial" technique of Bokowski and Sturmfels [BS] provides insights in this question.

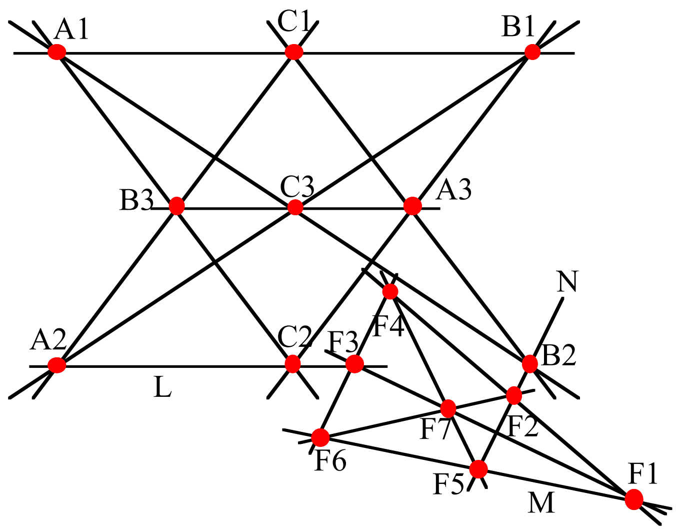

Figure 1. An example of the failure of Steinitz's theorem. By Pappus theorem, the lowest horizontal line $A_2C_2F_3$ is necessarily incident with an additional point, B_2. From [G2, Figure 2.3.3].

(3) Brückner [B] attempted to enumerate the simple convex 4-polytopes with 8 facets by enumerating the corresponding Schlegel diagrams in 3-space. Even if we disregard some errors in the actual enumeration of the diagrams, there remains the logical error of assuming that every complex that appears as if it were a Schlegel diagram of a convex polytope in the next higher dimension is actually such a Schlegel diagram. For 2-diagrams and convex polyhedra (3-polytopes) this was tacitly assumed for a very long time, but it was actually proved only by Steinitz [St3] in 1922. (For an alternative proof, and history of the result, see [G3]). On the other hand, for 3-diagrams and 4-polytopes it was established by Grünbaum-Sreedharan [GS] that this is not valid, and that therefore Brückner's enumeration is not correct in principle.

(4) Similar to (1) are the problems that arose in the enumeration of isohedral and isogonal tilings (tessellations) of the plane. Here a tiling is called *isohedral* if all tiles are images of each other under symmetries of the tiling; it is called *isogonal* if all vertices are equivalent under symmetries of the tiling. In both cases one can attempt to enumerate the possible types by considering suitably *marked* (or labeled) tilings. Details are shown in both cases in [GSh]. However, previous enumerations were wrong, through the assumption that every marked tiling can be realized by an (unmarked) tiling. (For references to the many papers see [GSh, Section 6.6]). In fact, there are 93 marked isohedral tilings, and the same number of isogonal ones. However, there are only 81 types of (unmarked) isohedral tilings, and 91 types of (unmarked) isogonal tilings. The differences of the actual numbers from the expected numbers (that were asserted by many authors) arise since the symmetries involved imply, in some cases, additional symmetries and hence lead to a type already counted. An example is shown in Figure 2.

This also illustrates the fact that the duality which exists between marked isohedral and isogonal types does not remain valid in all circumstances.

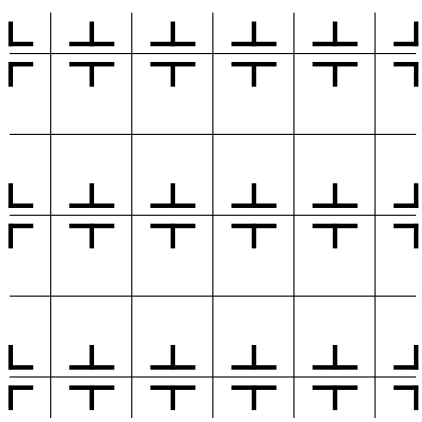

Figure 2. An example of a tiling by squares marked by T. Every tiling by unmarked quadrilateral tiles that has the same symmetries necessarily has additional symmetries, and hence is not of the same type. For details see [GSh, Chapter 6.2].

(5) Very frequently an enumeration or investigation is found to be easier if a suitable *dual* question is considered. However, in practice this often leads to errors. Besides the instance of isohedral/isogonal tilings, one should mention the case of *uniform* 3-polyhedra and their polars. A 3-dimensional polyhedron is uniform provided it is isogonal (all vertices are equivalent under symmetries of the polyhedron) and all faces are regular polygons. (There are various definition of regular polygons, but they are all equivalent to the one requiring that all pairs of mutually incident vertex and edge be equivalent under symmetries of the polygon.) An often-unstated assumption is that the vertices be distinct points. It has been known at least since Kepler in the seventeenth century, that except for the Platonic solids and the prisms and antiprisms there are precisely 13 distinct convex uniform polyhedra. Without the assumption of convexity, there are precisely 75 uniform polyhedra (other than the regular ones, and the prisms and antiprisms). Naturally, polars (which a special kinds of dual polyhedra) of uniform polyhedra are expected to have regular vertices and to be isohedral. This had been uncritically asserted to be true – although it is not. The reasons for failure are of (at least) two kinds. First off, some of the uniform polyhedra have coplanar but distinct faces; hence their polars have coinciding distinct vertices – which is inadmissible. This happens for example in both [H] and [W]. Second, some of the uniform polyhedra have faces that pass through the center of the polyhedron; this implies that no finite points is the polar of such a plane. In [W] this is avoided by locating some vertices "at infinity"; but this again would lead us out of the family of polyhedra as generally understood and accepted.

References

[B] M. Brückner, Über die Ableitung der allgemeinen Polytope und die nach Isomorphismus verschiedenen Typen der allgemeinen Achtzelle (Oktatope). Verhandel. Konink. Akad. Wetenschap. (Eerste Sectie), vol.10, no. 1 (1909).

[BS] J. Bokowski and B. Sturmfels, Computational Synthetic Geometry. Lecrture notes in Mathematics #1355, Springer 1989.

[G1] B. Grünbaum, Erdös vignettes. Geombinatorics 6(1997), 79 - 81.

[G2] B. Grünbaum, Configurations of Points and Lines. Graduate Studies in Mathematics vol. 103, Amer. Math. Soc. 2009.

[G3] B. Grünbaum, Convex Polytopes. Wiley 1967. Second ed. Edited by V. Kaibel, V. Klee and G. M. Ziegler, Springer 2003.

[GSh] B. Grünbaum and G. C. Shephard, Tilings and Patterns. W. H. Freeman, New York 1987. Reprint Dover, 2013.

[GS] B. Grünbaum and V. P. Sreedharan, An enumeration of simplicial 4-polytopes with 8 vertices. J. Combinat. Theory 2(1967), 437 – 465.

[H] Z. Har'El, Uniform solutions for uniform polyhedra. Geometriae Dedicata **47,** 57–110 (1993)

[K] S. Kantor, Die Configurationen $(3,3)_{10}$. Wien. Berichte LXXXIV(1881), 1291 – 1314 + plate.

[M] V. Martinetti, Sulle configurazioni piane μ_3. Annali di Matematica Pura ed Applicata (2) 15(1887), 1 – 26.

[P] T. Pisanski, Strong and weak realizations of configurations. Lecture Notes from the Klee-Grünbaum Festivl of Geometry, Ein Gev, Israel, April 9 – 16, 2000.

[S] H. Schröter, Über die Bildungsweise und geometrische Construction der Configurationen 10_3. Nachr. Ges. Wiss. Göttingen 1889, 239 – 253.

[St1] E. Steinitz, Über die Constructions der Configurationen n_3. PhD thesis, Breslau 1894.

[St2] E. Steinitz, Konfigurationen der projectiven Geometrie. Encyklopädie der math. Wissenschaften, Vol 3, Geometrie, Part IIIAB5a, 481 – 516 (1910). French translation (incomplete): Configurations, by E. Merlin. Encyclopédie des Sciences Mathématiques tome III, vol. 2 (1913), 144 – 160.

[St3] E.Steinitz, Polyeder und Raumeinteilungen. Enzyklop. Math. Wis., vol. 3, Geometrie, Part 3AB12, pp.1 – 139 (1922).

[W] M. J. Wenninger, Dual Models. Cambridge University Press, Cambridge (1983).