Geombinatorics 22(2012), 5 – 13

# Incenters and incircles

by Branko Grünbaum
University of Washington, Box 354350, Seattle, WA 98195
e-mail: grunbaum@math.washington.edu

Abstract. We find a common extensions to general polygons of the classical results concerning in- and excircles of triangles, and incircles of convex polygons.

Incircles (inscribed circles) and their centers (incenters) of triangles are well-known parts of traditional elementary geometry. The properties of incenters as point-valued functions of triangles are in some ways similar to the properties of circumcenters, although in other ways they are quite different. The similarity starts with the well-known result that for a triangle T the incenter I = I(T), the area centroid A = A(T) and the perimeter centroid P = P(T) are collinear, and that the ratio of IA to AP is 2, or, in vector notation, that I = 3A – 2P. (See [6, p. 225].) This is analogous to the more familiar relation between the orthocenter, area centroid and circumcenter of the triangle on the Euler line.

Since the area A(Q) and perimeter of P(Q) of a quite general polygon Q depend only on Q, one can define an "incenter point" I(Q) for Q by taking I(Q) = 3 A(Q) – 2 P(Q) (again as vectors). An indication that this definition may be appropriate can be found in the following result of Brassine [2] (published in 1843; the paper was forgotten until Shephard's paper [7] in 1990):

**Theorem 1.** Let Q be a **convex** n-gon all edges of which are tangent to a circle with center I. Then the area centroid A and the perimeter centroid P of Q are collinear with I, and I = 3A – 2P.

**Proof.** (This is Brassine's proof, as reproduced by Shephard.) Let Q be divided into n triangles, each of which has I as a vertex and one edge of Q as its base; see Figure 1. To each vertex of each of these triangles a mass proportional to the area of the triangle is assigned. Since the polygon has an incircle, all the triangles have the same height, thus these masses are also proportional to the lengths of the bases. Then the centroid of all $3n$ masses is the point $A$, the centroid of the $n$ masses at $I$ is $I$ itself, and the centroid of the remaining $2n$ masses is the point P. Hence $A$ divides the segment $[I, P]$ in ratio $2 : 1$, as claimed.

As an immediate consequence of Theorem 1 we see that the above definition of incenter point I(Q) satisfies two conditions: (i) It is defined for all convex polygons Q; and (ii) I(Q) coincides with the incenter of Q if Q has an inscribed circle. Moreover, Theorem 1 holds even in the more general case of "monotone" polygons; these are polygons for which the deflection (change of direction of edges) at each vertex is positive. An example is shown in Figure 2. It may be worth mentioning that such polygons were considered "convex" by some 19th century writers.

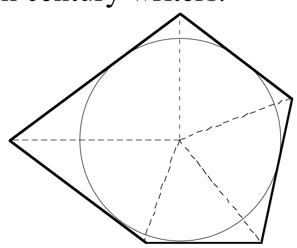

Figure 1. The division of a polygon with incircle into triangles, as used in the proof of Theorem 1.

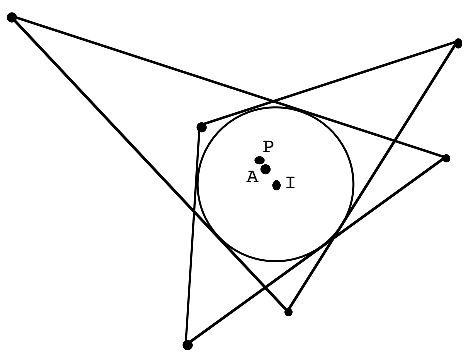


Figure 2. A monotone non-simple hexagon with an incircle. Theorem 1 is valid for such polygons.

In order to extend the above to general polygons, we need some definitions. A **polygon** Q is a cyclically ordered (oriented) sequence of points $V_1$, $V_2$, ..., $V_n$ together with the closed segments (**edges** of Q) $[V_i,V_{i+1}]$ for all i = 1, ..., n (here and throughout subscripts are understood mod n). In order to avoid lengthy explanations we shall assume that the **vertices** $V_i$ are completely arbitrary, **except** for stipulating that no two of the edges are collinear. Each such polygon Q defines also an **arrangement** of lines in the plane. By this we understand the partition of the plane into open convex regions formed by the complement of the union of the **sides** of Q, that is, the lines determined by the edges of Q. This is illustrated in Figure 3. (Concerning arrangements of lines see [4]. See also remark (ii) below.)

Many nonconvex polygons have a **touched circle** C generalizing the incircle; a circle C is said to be "touched" by Q if each **side** of Q is tangent to C. This is a generalization of the excircles of a triangle. Several examples are shown in Figure 4. In fact, a polygon Q may have more than one touched circle, see Figure 5. We call the center of a touched circle C a **t-center** of Q. For a triangle Q the set of t-centers of Q consists of the traditional incenter together with the three "excenters".

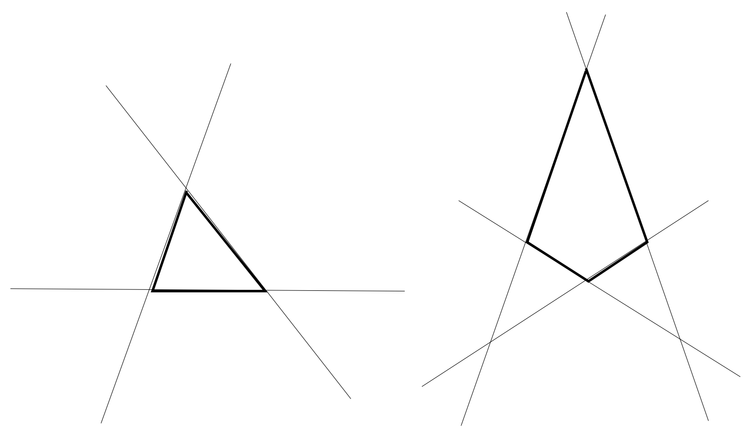

Figure 3. The arrangements generated by a triangle and a quadrilateral.

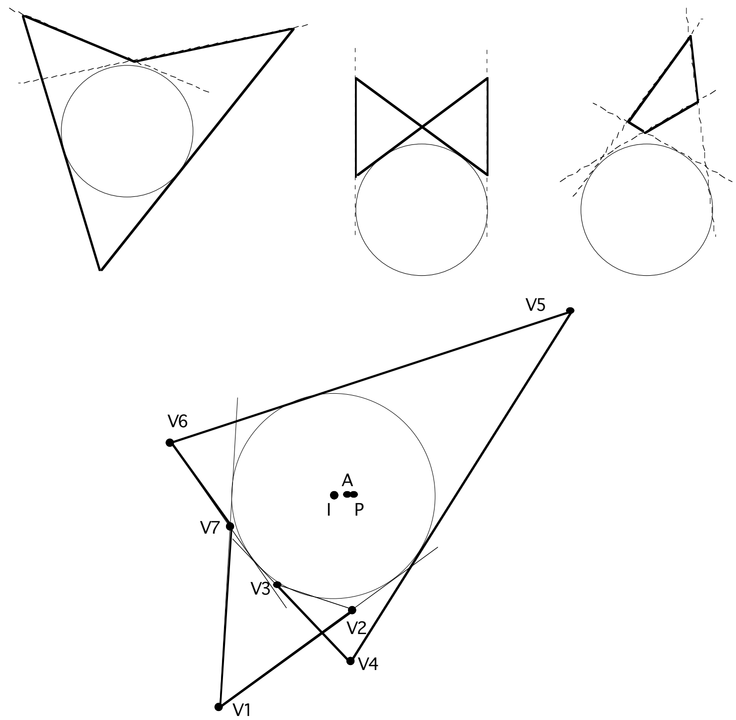


Figure 4. Examples of quadrangles and a (non-monotone) heptagon that have a touched circle, but no inscribed circle.

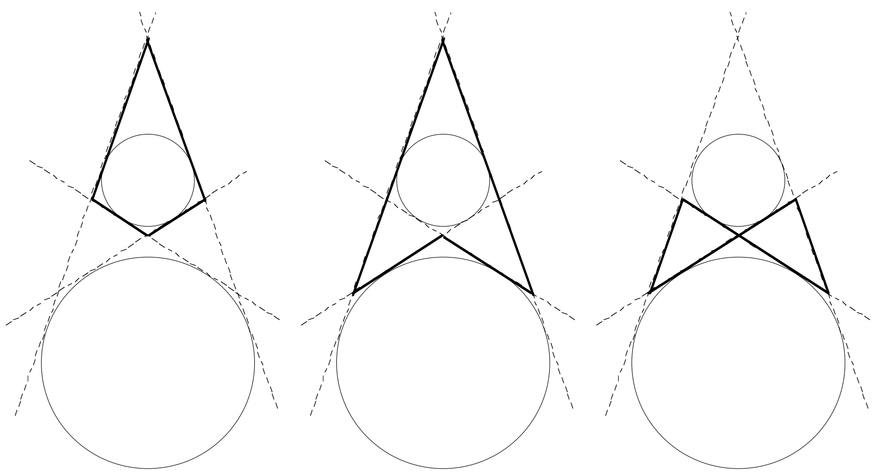

Figure 5. Examples of convex and nonconvex quadrangles with two touched circles.

**Theorem 2.** If Q is a polygon with a touched circle C and t-center T of Q, and if Q has a **suitably** defined centroid of area A and a centroid of perimeter P, then T = 3A – 2P.

**Proof.** We are dealing with oriented polygons Q. If the direction of an edge E of Q, as viewed from T, is positive (counterclockwise) – then the area of the triangle determined by E and T is taken as positive, and so is the length of T. If the direction of E (seen from T) is negative, then both the area and the length are taken as negative. This explains the meaning of "suitably" in the formulation of the theorem. Then the remaining part of proof coincides with that of Brassine.

**Remarks.**

(i) The pericenter is also known as the Nagel point ([6, p. 225]) and as the "verbicenter" [e.g. [3]).

(ii) The concept of "sides" of a triangle (or n-gon) permeates much of elementary geometry, altough without being given a proper name or recognition. This is probably due to Euclid's not considering (infinite) lines, but only arbitrarily extendable segments. Among the oldest examples of the use of "sides" is in the theorem of Menelaus, and in connection with excircles. From a different point of view the sets of sides of polygons may be considered as a generalization of Hamiltonian multilaterals in configurations, see Section 5.2 of [5].

(iii) The possibility of several t-circles of a polygon Q is explained by the fact that A and P depend on orientation of the sides (or edges) of Q with respect to the t-center.

(iv) It is easy to see that for every proper (not collapsed) triangle there are **precisely** four t-circles – the incircle and the three excircles. However, for any quadrangle there are **at most** two t-circles, and for any polygon with five of more sides there is **at most** one.

(v) The requirement that A and P exist is obviously necessary for the proof. However, it is possible for a touched circle to exists even if the area and perimeter are 0, hence the centroids A and P are not defined. Examples of this situation are the middle quadrangle in Figure 4 and the right-most one in Figure 5.

(vi) The equation T = 3A – 2P may be used to construct a point we call k-center (quasicenter K = K(Q,O) of Q w.r.t. O) starting from an arbitrary point O not on any side of Q. Since the signs of the lengths of edges depends of the position of the point O with respect to the sides of Q, we find that each of the regions of the arrangement generated by sides of Q will yield (in general) a different value of the perimeter p(Q,O) of Q, hence lead to a different pericenter P = P(Q,O) and a different k-center K = 3A – 2P. The area of Q and its centroid A do not depend on O. It follows from the proof that as long as O stays within a given region of the arrangement, the point P and therefore the k-center K will not change. Hence if the region in which we consider K(Q,O) contains a t-center T, then T will coincide with K(Q,O) for every O in that region. It should also be mentioned that if two choices for the point O are separated by every side of Q, then they lead to the same k-center, since both the area and the perimeter change sign.

(vii) As an additional consequence of the Brassine proof we see that it is reasonable to define an **inradius** i(Q,O) corresponding to the k-center K(Q,O) by i(Q,O) = 2 area(Q)/p(Q,O). If Q is a triangle, this gives the well-known formulas for the inradius and the exradii. In general, if O is in a region that corresponds to a touched circle (such as $O_1$ in Figure 6), this expression for i(Q,O) gives the radius of this circle; otherwise it gives a radius for **a circle** centered at K(Q,O). In either case, we may call a circle centered at K(Q,O) and with radius i(Q,O) a **k-circle** of Q. Probably, each k-circle possesses some extremal property. However, I do not know what property – if any – this is.

The above is illustrated in Figure 6; to avoid clutter, only a few points O and the corresponding k-centers and k-circles are shown. It should be noted that the points $O_3$ and $O_4$ yield the same

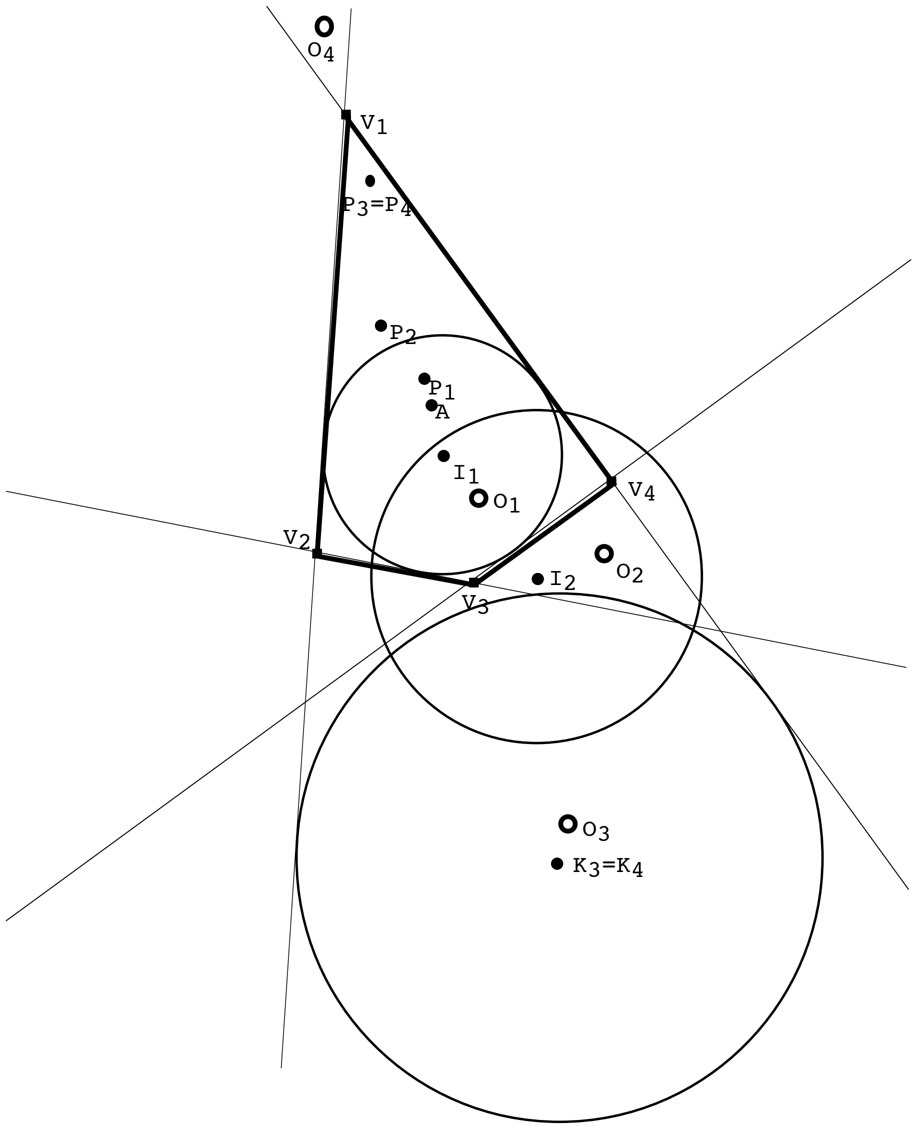


Figure 6. Three of the incircles of the quadrangle $Q = [V_1, V_2, V_3, V_4]$. Shown are the incenters and the pericenters that correspond to the points $O_1$, $O_2$, $O_3$, $O_4$ chosen. The last two of these points are separated by every side of $Q$; hence $I(Q,O_3) = IQ,O_4)$ but $i(Q,O_3) = –i(Q,O_4)$. By convention, we usually interpret the radius as the absolute value of the expression obtained.

k-center, $K(Q,O_3) = K(Q,O_4)$. However, $i(Q,O_3) = - i(Q,O_4)$. As mentioned earlier this is a general phenomenon. Hence it may be appropriate to use the absolute value for the radius – or else deal with oriented circles.

(viii) Incircles and excircles of triangles are well known and results on them are widely available, in print and on the Web. Most of the analogous literature on polygons with more than three sides deals with quadrangles only, quite frequently restricted to convex ones. A few writers discuss touched circles of general quadrangles. Bogomolny [1] calls them **exscriptible**, while in [8] **ex-tangential** is used in conection with convex quadrangles. However, none of the literature I have seen mentions the possibility of a quadrangle having two touched circles.

**References.**

[1] A. Bogomolny, Inscriptible and Exscriptible Quadrilaterals. http://www.cut-the-knot.org/Curriculum/Geometry/Pitot.shtml (accessed 5/24/2012)

[2] E. Brassine, Sur quelques propriétés des centres de gravité. J. de Math. Pures et Appl. 8(1843), pp. 46 – 48.

[3] K. W. Crain, Solution of Problem 172. Nat. Math. Magazine 12(1937/38), 194 – 196

[4] B. Grünbaum, Arrangements and Spreads. Regional Conference Series in Mathematics No. 10, Amer. Math. Soc., Providence, RI 1972.

[5] B. Grünbaum, Configurations of Points and Lines. Graduate Studies in Mathematics vol. 103, Amer. Math. Soc., Providence, RI 2009.

[6] R. A. Johnson, Advanced Euclidean Geometry. Dover, NY.

[7] G. C. Shephard, Centroids of polygons and polyhedra. Math. Gazette 74(1990), pp. 42 – 43.

[8] Ex-tangential quadrilateral. http://en.wikipedia.org/wiki/Ex-tangential_quadrilateral (accessed 5/24/2012)