

Research Article

Geometric realization of some triangle-free combinatorial configurations 22_3

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Received 17 April 2012; Accepted 15 May 2012

Academic Editors: E. Gutkin, B. Mohar, and J. S. Snoeyink

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The main purpose of this paper is to illustrate the mutual benefit to combinatorics and geometry by considering a topic from both sides. Al-Azemi and Betten enumerate the distinct *combinatorial* (22_3) configurations that are triangle free. They find a very large number of such configurations, but when taking into account the automorphism group of each, they find two cases in which there is only a single configuration. On the heuristic assumption that an object that is unique in some sense may well have other interesting properties, the *geometric* counterparts of these configurations were studied. Several unexpected results and problems were encountered. One is that the combinatorially unique (22_3) configuration with automorphisms group of order 22 has three distinct geometric realizations by astral configurations.

In [1], Al-Azemi and Betten consider *combinatorial configurations* (22_3) . Here by (n_k) we mean an incidence structure of n distinct *points* and n distinct *blocks*, each block contains k points, each point belongs to k blocks, and each pair of distinct points belongs to at most one block. A *triangle* in a configuration is a set of three points P_1, P_2, P_3 , and three blocks B_1, B_2, B_3 , such that for $i, j = 1, 2, 3$, P_i is incident with B_j if and only if $i \neq j$. A configuration with no triangles is said to be *triangle-free*. There exists considerable literature on triangle-free combinatorial configurations; in particular, data on (n_3) configurations for $n \leq 21$ were given by Betten et al. in [2], and this has been extended in [1] to the case $n = 22$. The latter paper also determines (among other results) the classification of the 157,211 triangle-free configurations (22_3) by the size of the group of automorphisms of the configurations. Two particularly interesting results are that there is a single configuration with group of order 11, and a single configuration with group of order 22. The present note arose from the question whether these two special configurations have interesting geometric representations, in particular, with isometric symmetries group isomorphic with the automorphisms group or one of its nontrivial subgroups. Our main result is that the answers are affirmative. Some explanations of the geometric setting are required.

A *geometric configuration* (n_k) is a collection of n distinct points and n distinct *straight* lines in the Euclidean plane, each point lying on exactly k lines and each line going through exactly k points. In an obvious way, each geometric configuration is isomorphic to (can be interpreted as) a combinatorial configuration. The converse is well-known not to be valid. In fact, the determination of general conditions for a combinatorial configuration to be realizable by a geometric one is one of the deepest problems regarding configurations. More details on the concepts involved, and on the results known in this connection, may be found in [3].

A geometric configuration (n_3) is called *astral* if $n = 2m$ and its points and lines form two transitivity classes under rotational symmetries, each consisting of m points or lines, respectively. As explained in detail in [3, Chapter 2], the points of such a configuration are situated at the vertices of two concentric regular m -gons. The configuration is denoted $m\#(b, c; d)$ if the following conditions are all satisfied; Figure 1 provides an example illustrating $11\#(4, 2; 3)$ while Figure 2 shows the simplified notation we will use throughout the paper:

- (i) the points P_j in one of the transitivity classes are labeled consecutively (clockwise, or else counterclockwise) for $j = 0, 1, \dots, m - 1$;
- (ii) a line denoted L_j (for $0 \leq j < m$) passes through the points labeled P_j and P_{j+b} (subscripts mod n);
- (iii) the line L_j passes through one point Q_j of the other class;
- (iv) points Q_j and Q_{j+c} determine a line M_j ;
- (v) the line M_j passes through a point P_{j+d} .

Depending on the transitivity class selected, and on clockwise or counterclockwise orientation, up to four different symbols correspond to each astral configuration. Usually one selects the lexicographically highest among them.

Conversely, given a symbol $m\#(b, c; d)$, one may start constructing the corresponding geometric configuration. As becomes obvious at once, it is necessary to determine where on the line L_j does the point Q_j lie. This leads to a quadratic equation for the ratio r (which is the same for all j) in which Q_j divides the segment P_jP_{j+b} ; This is to ensure that condition (v) is satisfied. Then the resulting construction depends on the roots of this equation. If there are two distinct real roots there are two geometrically distinct but combinatorially isomorphic geometric configurations $m\#(b, c; d)$; if there is a single root (of multiplicity 2) there is only one geometric configuration; if the roots are conjugate complex numbers there is no geometric configuration in the real Euclidean plane. All the astral configurations are also easily seen to be selfdual. Additional details and properties of astral and other (n_3) configurations are given in Chapter 2 of [3].

By construction, all astral configurations have as geometric symmetries all rotations of the cyclic group C_m and no mirror symmetries. In the case we are interested, this is the cyclic group C_{11} . It is easy to determine what are the $11\#(b, c; d)$ configurations that are triangle-free. A consideration of all the relevant cases leads to exactly six symbols $11\#(b, c; d)$ that yield triangle-free configurations. These are:

$$\begin{aligned} &11\#(4, 2; 1) \quad 11\#(4, 3; 2) \quad 11\#(5, 3; 1) \\ &11\#(4, 2; 3) \quad 11\#(5, 1; 3) \quad 11\#(5, 3; 4). \end{aligned} \tag{0.1}$$

1

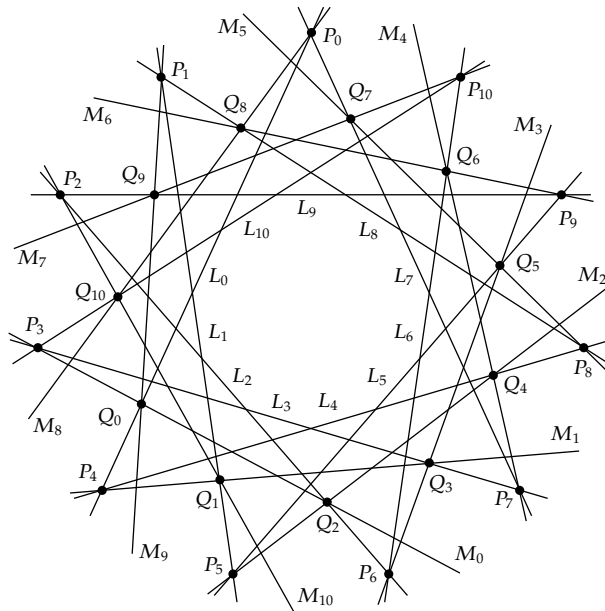


Figure 1: The configuration $11\#(4,2;3)$ labeled as explained in the text. The parameter $r = |Q_0P_b|/|P_0P_b|$ equals 0.188459 for this configuration.

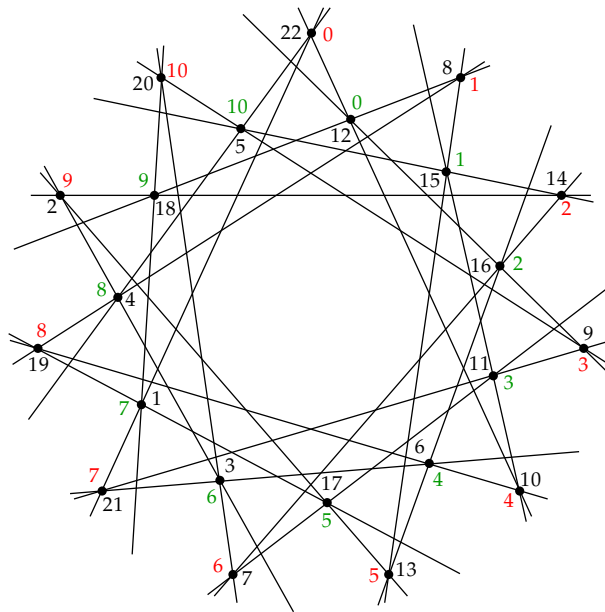


Figure 2: The configuration $11\#(4,2;3)$ labeled by red and green numerals to reduce clutter. The numerals in black are the labels of the elements in the configuration described in Section A.4 of [1]. Since the lines of the geometric configuration corresponds to the blocks in Figure 7 of [1], this establishes the isomorphism between the geometric configuration $11\#(4,2;3)$ and the unique triangle-free combinatorial configuration (22_3) in A.4 of [1].

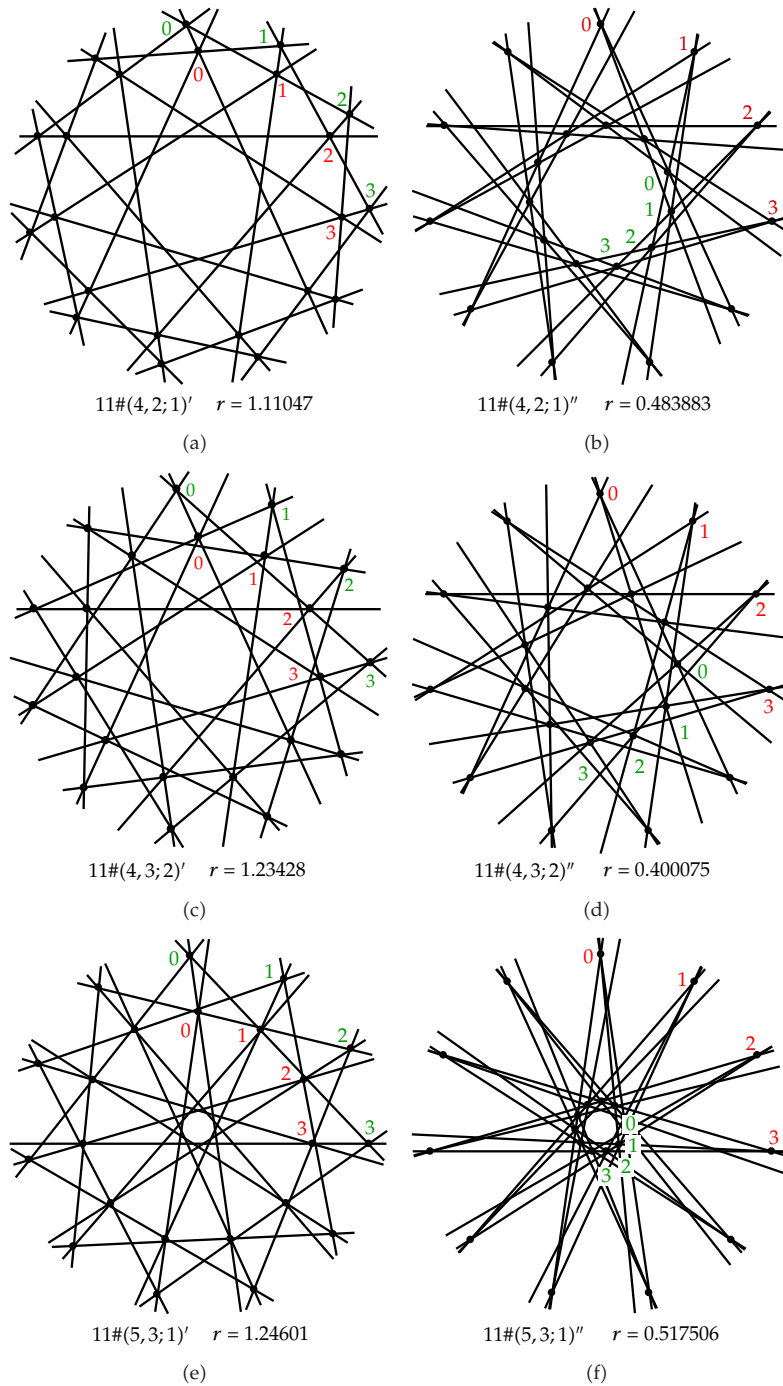


Figure 3: The three pairs of astral triangle-free configurations with symmetry and automorphisms groups C_{11} . To reduce clutter, only a few of the labels are shown.

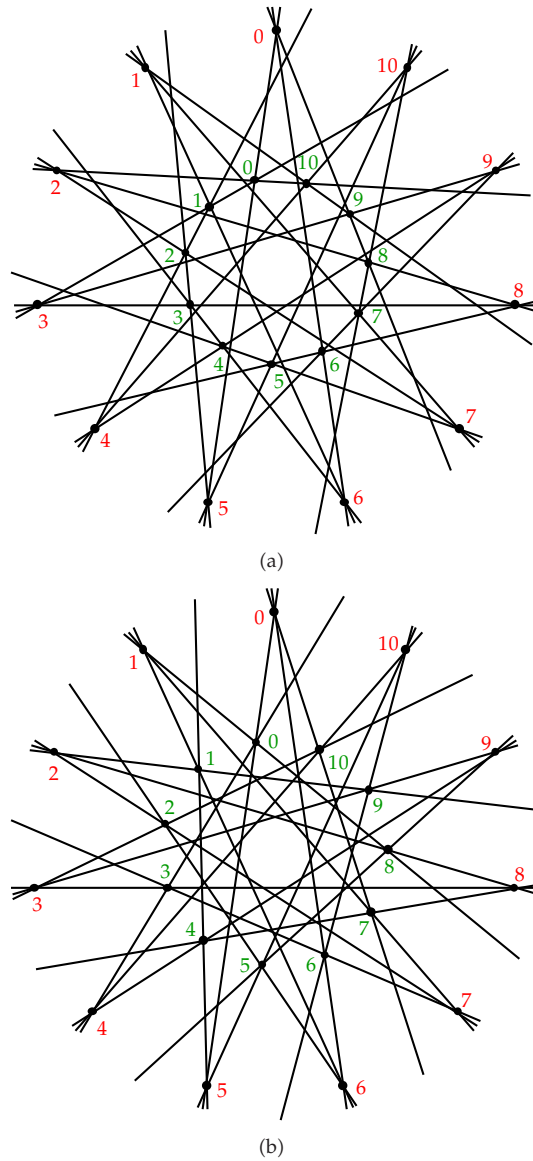


Figure 4: The astral configurations (a) $11\#(5,1;3)$ and (b) $11\#(5,3;4)$.

As visible from Figure 3, each symbol in the first row above corresponds to a pair of geometrically distinct (but combinatorially isomorphic) configurations. Figures 2 and 4 show geometric configurations that correspond to the symbols in the second row, which lead to one configuration for each symbol. As is easily verified (see Figures 2, 3, and 4), all the configurations in the both rows are triangle-free and have geometric symmetry group C_{11} , hence with automorphism group containing C_{11} .

The isomorphism of the astral configuration $11\#(4,2;3)$ with the combinatorial configuration described in Section A.4 of [1] is shown in Figure 2.

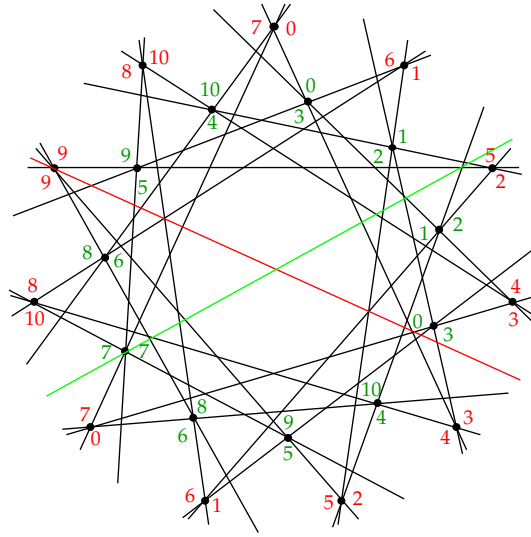


Figure 5: A combinatorial automorphism of $11\#(4,2;3)$, that is, additional to the rotational symmetries. It arises by “split-mirror” reflection. The images of the upright numerals, each reflected in the mirror of the same color, are shown in italics. In fact, the “reflected” configuration is (up to a rotation) the one that arises by using in the construction as described above, the orientation opposite the one used originally. The particular “split-mirror” reflection is chosen so as to coincide with the one in (A.13) of [1], under the correspondence of Figure 2.

Guided by the generators (in (A.13) and (A.14) of [1]) of the automorphism group of the configuration in A.4 of [1], it was possible to find the geometric basis for the larger, 22-element group, of automorphisms of $11\#(4,2;3)$. An additional generator corresponds to a “split-mirror” reflection: one transitivity class of points is reflected in one mirror, the other in an appropriate different mirror. This is illustrated in Figure 5. This automorphism was not anticipated by the previous geometric knowledge. While not an isometry, it obviously has a geometric context. 2

What about the other two configurations in the second row, shown in Figure 4? It is easy to find “split-mirror” reflections for each, hence they also have automorphism groups of order 22. This seems to be in contradiction with the uniqueness result of [1]; however, that quandary is resolved by showing that each is, in fact, combinatorially isomorphic with $11\#(4,2;3)$, as expected from the combinatorial uniqueness result. The isomorphism is documented and illustrated in Figure 6. Thus we have **the following proposition.**

Proposition 1. *The unique triangle-free combinatorial configuration (22_3) with automorphism group of order 22 found in [1] has precisely three geometrically distinct realizations by astral configurations.*

Here, and in Proposition 2, the assertion that the precise number of geometrically distinct astral configurations is 3 (resp., 6) follows from well-known facts about that kind of configurations.

It should be noted that although there has been no claim in the literature that geometric astral configurations with different symbols $m\#(b, c; d)$ are combinatorially distinct, this was the unstated belief held by at least some workers in the field (including the present writer). Hence this isomorphism is rather unexpected. On the other hand, it should be noted that for

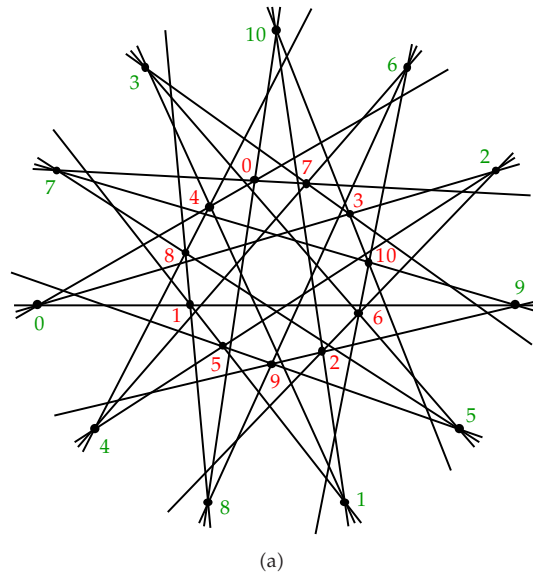


Figure 6: The astral configurations (a) $11\#(5,1;3)$ and (b) $11\#(5,3;4)$ labeled to show isomorphism with $11\#(4,2;3)$.

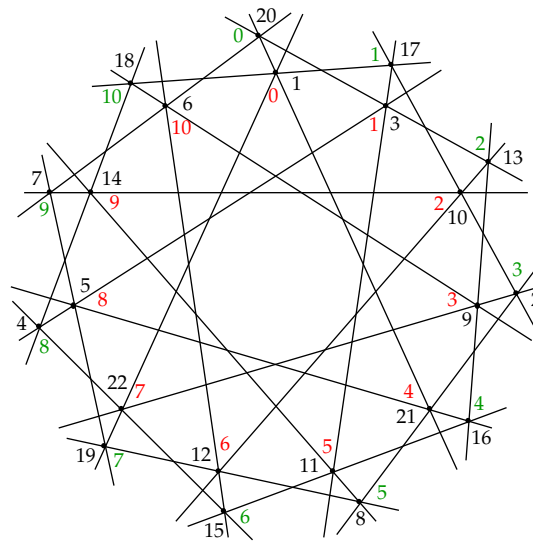


Figure 7: The astral configuration $11\#(4,2;1)$ (shown by red and green labels) is isomorphic with the combinatorial configuration given by (A.5) in Section A.2 of [1] (shown by black labels).

(n_4) astral configurations such behavior has been noted earlier. Figure 3.6.3 of [3] shows three geometrically distinct configurations that are combinatorially isomorphic; the same holds for their polars.

Turning to the astral configurations in Figure 3, we begin by observing that the first one, $11\#(4,2;1)'$, is isomorphic with the combinatorial configuration with automorphism group C_{11} described in Section A.2 of [1], and given by the blocks listed there in (A.5). This

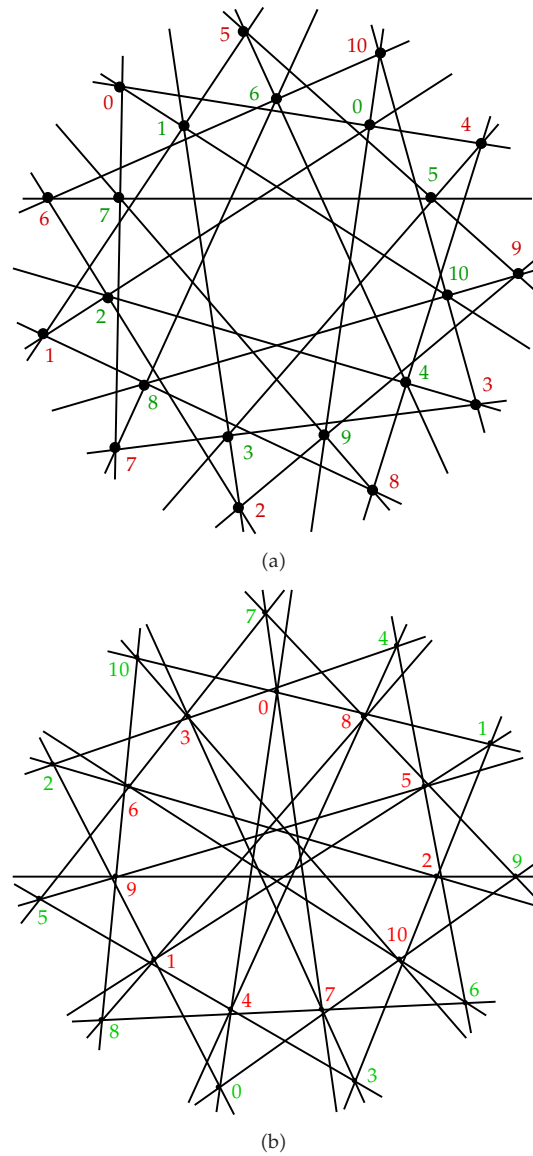


Figure 8: Astral configurations (a) $11\#(4,3;2)$ and (b) $11\#(5,3;1)$ labeled to show the isomorphism with $11\#(4,2;1)$.

is illustrated in Figure 7. The configuration $11\#(4,2;1)''$ is isomorphic to $11\#(4, 2; 1)'$, hence to the unique configuration with automorphism group C_{11} just mentioned.

Concerning the other two pairs of configurations in Figure 3, it turns out that the situation is analogous to the one discussed previously.

Proposition 2. *The unique triangle-free combinatorial configuration (22_3) with automorphism group of order 11 found in [1] has precisely six geometrically distinct realizations by astral configurations; they form three pairs of configurations with the same symbol.*

This is illustrated in Figure 8. The author's attempts to establish isomorphism between configurations from different pairs were successful for the pairs $11\#(4,2;1)$ and $11\#(4,3;2)$. However, the third pair $11\#(5,3;1)$ resisted the search for isomorphism. The author obliged to Dr. Marko Boben for the isomorphism on which Figure 8(b) is based.

Beyond this, the geometric and combinatorial results concerning cases of uniqueness deserve to be extended to other values of n , especially in cases with prime m , as does the existence of "split-mirror" reflections in other astral (n_3) configurations. Also, the behavior of (n_k) configurations (for $k \geq 4$) deserves to be studied as to the possibilities of geometrically distinct realizations of given combinatorial configurations. Similar investigations regarding topological configurations (formed by points and pseudolines) should also be rewarding.

Conjecture 3. *All (n_3) astral configurations $m\#(b,c;d)$ with $b + c = 2d$ admit "split-mirror" reflections and have automorphism groups with $n = 2m$ elements.*

Multiastal configurations are defined in section 2.9 of [3] as generalizations of astral configuration; essentially, they have as points $k \geq 3$ sets of vertices of concentric regular n -gons, with suitable restrictions. We may conjecture the following.

Conjecture 4. *The automorphism group of any multiastal configuration has at most $k.n$ elements.*

A different challenge is posed by the question whether there are geometric realizations of the three triangle-free combinatorial configurations (22_3) with automorphism group of order 24. If any is found to exist, one would wish to see how geometrically symmetric each may be. But if no realization exists—this would resolve the old conjecture that every 3-connected combinatorial (n_3) configuration with $n > 10$ is geometrically realizable with points and straight lines in the Euclidean plane.

(*Note added April 8, 2012:* Dr. Marko Boben has found geometric realizations of all three of these configurations, albeit with trivial group of isometric symmetries.)

Boben et al. [4] investigated the existence or nonexistence of geometric triangle-free configurations (v_3) for all $v \leq 21$; many details of the history of the topic may also be found in this paper. Related more general results are in [3] and in Boben et al.'s paper [5]. Although it was easy to determine directly that there are no triangles in any of the (22_3) configurations listed above, I am grateful to Dr. Marko Boben for a list of general criteria for the presence of a triangle in an astral configuration. These are analogous to the criteria in [5], and they confirmed my checking by hand that the configurations in question are triangle-free.

References

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Composition Comments

1. There is a difference between the manuscript and the electronic version in Figure 3 and we followed the manuscript. Please check.
2. We made the highlighted changes as per journal style. Please check.



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