ON A MEASURE OF ASYMMETRY OF CONVEX BODIES†

By E. ASPLUND, E. GROSSWALD and B. GRÜNBAUM

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In the present note we discuss some properties of a 'measure of asymmetry' of convex bodies in n-dimensional Euclidean space. Various measures of asymmetry have been treated in the literature (see, for example, (1), (6); references to most of the relevant results may be found in (4)). The measure introduced here has the somewhat surprising property that for $n \geqslant 3$ the n-simplex is *not* the most asymmetric convex body in E^n. It seems to be the only measure of asymmetry for which this fact is known.

Let K be a convex body in E^n, G its centroid and let $V(K)$ denote the n-dimensional volume of K. Let H be a half-space such that K does not meet $\operatorname{int} H$. We denote the 'mirror image' of H in the point G by $H^* = 2G + (-H)$. Furthermore, we define

$$K_H = K \cap H^*$$

as the intersection of K with the half-space H^*. The measure of asymmetry $A(K)$ of K is now defined by

$$A(K) = \max_H V(K_H)/V(K),$$

the maximum being taken over all half-spaces such that $K \cap \operatorname{int} H = \phi$. We also put

$$A_n = \sup\{A(K) | K \text{ a convex body in } E^n\}.$$

It is obvious that $A(K)$ is an affine invariant of K, that $A(K) = 0$ if and only if K is centrally symmetric and that $A_n \leqslant \frac{1}{2}$ for each n. Using the affine invariance and continuity of $A(K)$ and the compactness of the space of classes of affinely equivalent convex bodies in E^n (see (5)), it is not difficult to see that for each n there exists a K with $A_n = A(K)$. (The same result will follow from the proof of Lemma 1.)

Let $C_n = C_n(0)$ denote a straight cone in E^n whose base is an $(n-1)$-dimensional ball. For any x with $0 \leqslant x < 1$ let $C_n(x)$ denote the truncated cone obtained from C_n by cutting off (by means of a hyperplane parallel to the base of C_n) the upper xth part of C_n.

We now have

Lemma 1. *For each $K \subset E^n$ there exists an x such that $A(K) \leqslant A(C_n(x))$.*

Proof. Let H be a half-space such that $V(K_H)$ is maximal and let L be a straight line passing through G and perpendicular to the hyperplane $\operatorname{Bd} H$. Let $\hat{K}$ be the convex body obtained from K by spherical symmetrization ('Schwarzsche Abrundung' ((2), p. 71); Schwarz rotation process ((2), p. 100)) with respect to the line L as axis. By

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the properties of spherical symmetrization, obviously $A(K) \leqslant A(\hat{K})$. Let $P^* = \text{Bd}\, H^*$ denote the hyperplane obtained by mirroring $P = \text{Bd}\, H$ in G. It is possible to construct a truncated cone S with one base on P and with the line L for axis, which intersects P^* in $P^* \cap \hat{K}$ and satisfies

$$V(S) = V(K), \quad V(S \cap H^*) = V(K_H).$$

The truncated cone S is narrower than the supporting cone of $\hat{K}$ at $P^* \cap \hat{K}$ in the region between P^* and P, and broader in H^*. Hence, the centroid of the part of S that lies between P^* and P will be nearer to P than the centroid of the corresponding part of $\hat{K}$, and the same holds true for the parts in H^*. Altogether, the centroid of S will be nearer to P than will G, unless of course $\hat{K}$ itself was a truncated cone. But this implies that

$$A(K) \leqslant A(S) = A(C_n(x)) \quad \text{for some } x.$$

Equality can only hold if $\hat{K}$ is a truncated cone. Lemma 1 is thus proved.

LEMMA 2. $$A(C_n(x)) = \left[\left(\frac{2n}{n+1}\frac{1-x^{n+1}}{1-x^n} - 1\right)^n - x^n\right] \Big/ (1-x^n).$$

We omit the elementary computations that prove Lemma 2.

REMARK 1. It is easily seen that, for each fixed n, the limit $\lim_{x\to 1} A(C_n(x))$ exists and equals 0. Therefore Lemma 1 implies that $A_n = \max_{0 \leqslant x < 1} A(C_n(x))$.

THEOREM 1. $A_2 = \frac{1}{9}$. *The only convex sets* $K \subset E^2$ *with* $A(K) = A_2$ *are triangles.*

Proof. By the above remark and by Lemma 2,

$$A_2 = \max_{0\leqslant x<1} A(C_2(x)) = \max_{0\leqslant x<1}\left(\frac{1}{9} - \frac{8x^3}{9(1+x)^3}\right) = A(C_2(0)) = \tfrac{1}{9}.$$

On the other hand, it follows from the proof of Lemma 1 that (in the notations used there) $A(K) < A(C_2(x))$ for some x, unless $\hat{K} = C_2(x)$. Thus, if $A(K) = A_2$ it follows that $\hat{K} = C_2(0)$, which is a triangle; but then K is also a triangle. This ends the proof of Theorem 1.

It is, obviously, possible to determine A_n by finding the maximum of $A(C_n(x))$ given by Lemma 2. The resulting algebraic equations do not seem to be solvable in closed form for $n > 2$. For $n = 3$ the approximate value is $A_3 = 0{\cdot}1254\ldots$, which is attained as $A(C_3(x))$ for $x = 0{\cdot}2\ldots$.

The behaviour of A_n and of $A(C_n(x))$ for $n > 2$ is quite complicated. By routine computations the following statements may be verified:

(1) $\lim_{n\to\infty} A(C_n(x)) = e^{-2}$ for each x in $0 \leqslant x < 1$.

(2) If $n > 3$ then

$$\frac{1}{2n}\left(\frac{n+1}{n-1}\right)^n < 1,$$

and for each x satisfying $$0 < x < 1 - \frac{1}{2n}\left(\frac{n+1}{n-1}\right)^n$$

we have

$$\begin{aligned}A(C_n(x)) &= \left[\left(\frac{n-1}{n+1}+\frac{2nx^n(1-x)}{(n+1)(1-x^n)}\right)^n - x^n\right]\Big/(1-x^n)\\ &> \left(\frac{n-1}{n+1}+\frac{2nx^n(1-x)}{n+1}\right)^n - x^n\\ &> \left(\frac{n-1}{n+1}\right)^n + \left[\frac{2n^2}{n+1}\left(\frac{n-1}{n+1}\right)^{n-1}(1-x)-1\right]x^n\\ &> \left(\frac{n-1}{n+1}\right)^n\\ &= A(C_n(0)).\end{aligned}$$

Therefore, not only is $A(C_n(0)) < A_n$ but the minimal $x > 0$ such that

$$A(C_n(0)) \geqslant A(C_n(x))$$

tends to 1 with increasing n.

THEOREM 2. *The sequence A_n is strictly increasing and*

$$\lim_{n\to\infty} A_n = \max_{0\leqslant y<1}\left[-y+\exp\left(-2-\frac{2y\log y}{1-y}\right)\right]\Big/(1-y) = 0{\cdot}143\ldots$$
$$(> \lim_{n\to\infty} A(C_n(0)) = e^{-2} = 0{\cdot}135\ldots).$$

Proof. In order to establish that $A_n < A_{n+1}$, let S be a truncated n-dimensional cone such that $A_n = A(S)$ and let H_S be the corresponding 'maximal half space', containing one of the bases of S. In $E^{n+1} = E^n + E^1$, let K be the Cartesian product of $S \subset E^n \subset E^{n+1}$ and a segment $I \subset E^1$ and let H be the product of H_S and E^1. Clearly, $A(S) \leqslant A(K)$. But the symmetrization $\hat{K}$ of K cannot be a truncated cone in E^{n+1}, since its cross-sectional area, taken perpendicular to its axis, grows only as that of a truncated cone in E^n, i.e. as a polynomial of one degree too low. Hence by the proof of Lemma 1, A_{n+1} is strictly greater than A_n.

The determination of $\lim A_n$ is slightly more complicated, especially in view of the above remark (1). We define

$$A(n,y) = A(C_n(y^{1/n})) = \left[\left(1-\frac{2}{n+1}\,\frac{1-(n+1)y+ny^{1+1/n}}{1-y}\right)^n - y\right]\Big/(1-y).$$

For each y, $0 \leqslant y < 1$, the limit $A(y) = \lim_{n\to\infty} A(n,y)$ is easily seen to exist and to be given by the expression

$$A(y) = \left[-y+\exp\left(-2-\frac{2y\log y}{1-y}\right)\right]\Big/(1-y).$$

For $y \to 1$, both $A(n,y)$ and $A(y)$ tend to the limit 0. Now, we define in the square $D = \{(y,z)\,|\,0\leqslant y\leqslant 1,\ 0\leqslant z\leqslant 1\}$ a function $f(y,z)$ by means of the conditions

$$f(y,0) = A(y),$$
$$f(y,1/n) = A(n,y) \quad \text{for} \quad (n = 1,2,\ldots)$$

and for

$$z = \lambda/(n+1)+(1-\lambda)(1/n) \quad \text{with} \quad 0 \leqslant \lambda \leqslant 1,$$
$$f(y,z) = \lambda A(n+1,y)+(1-\lambda)A(n,y).$$

Then $f(y,z)$ is continuous in D, hence it achieves its maximum somewhere in D. The maximum will certainly not be achieved in the interior of D, since given $(y,z) \in \operatorname{int} D$ we can find an integer n such that $z > 1/n$ and then the inequality

$$f(y,z) < \max_{0 \leqslant t \leqslant 1} f(t, 1/n) = A_n$$

holds, because of the first part of Theorem 2 and the definition of $f(y,z)$. Hence $f(y,z)$ achieves its maximum on the boundary of D. However, only on the side $z = 0$ of D does f possess values exceeding e^{-2}. Therefore

$$\lim A_n = \max_{0 \leqslant y \leqslant 1} f(y,0) = \max_{0 \leqslant y < 1} A(y),$$

as claimed. In order to determine $\lim A_n$ numerically, we observed that $A(0) = 0$ and $A(1) = e^{-2}$. In the open interval $0 < y < 1$ the function $A(y)$ is differentiable and its derivative has only one zero, which is assumed for y_0 satisfying

$$\left(-1-\frac{2\log y_0}{1-y_0}\right)\exp\left(-2-\frac{2y_0\log y_0}{1-y_0}\right) = 1.$$

i.e. $y_0 = 0{\cdot}0435\ldots$, which yields $\lim A_n = 0{\cdot}143\ldots$. This ends the proof of Theorem 2.

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