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EQUIPARTITE POLYTOPES*

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ABSTRACT

A polytope P with $2n$ vertices is called **equipartite** if for any partition of its vertex set into two equal-size sets V_1 and V_2, there is an isometry of the polytope P that maps V_1 onto V_2. We prove that an equipartite polytope in $\mathbb{R}^d$ can have at most $2d+2$ vertices. We show that this bound is sharp and identify all known equipartite polytopes in $\mathbb{R}^d$. We conjecture that the list is complete.

1. Introduction

Classification of polytopes possessing a variety of symmetries has been extensively studied: centrally symmetric polytopes, vertex transitive polytopes, self-dual polytopes are few such examples. In this paper, we study a new kind of symmetry: **equipartiteness**. Its definition is included in the abstract. Examples of equipartite d-polytopes include rectangles, squares and hexagons (two types) in $\mathbb{R}^2$, various tetrahedra, regular octahedra, regular three-sided prisms, rectangular boxes, in $\mathbb{R}^3$, regular simplices in $\mathbb{R}^{2k+1}$ and more. A complete list of all equipartite polytopes in $\mathbb{R}^2$ and $\mathbb{R}^3$ is described in Section 4.

In Section 3, we prove that the number of vertices of an equipartite d-polytope is at most $2d+2$ and, later in Section 5, we show that the bound is tight by constructing equipartite d-polytopes with $2d+2$ vertices for every $d \geq 2$. When restricting the definition of equipartiteness of polytopes to their 1-skeletons, we are naturally led to the notion of equipartite graphs:

Definition 1: A graph G of order $2n$ is **equipartite** if for every n-element subset A of its vertices, there is an automorphism of G mapping A to its complementary set of vertices.

All equipartite graphs of order six and eight are depicted in Figures 1 and 2.

A full characterization of equipartite graphs was obtained in [5]. For every $n \neq 4$ there are 8 equipartite graphs of order $2n$: K_{2n}, $2K_n$, $2K_n + nK_2$, $K_{2n} \setminus nK_2$ and their complements. There are 10 equipartite graphs of order 8. These graphs play a pivotal role in limiting the number of vertices in equipartite d-polytopes and their structures. Curiously, all equipartite graphs appear as orbits in equipartite polytopes.

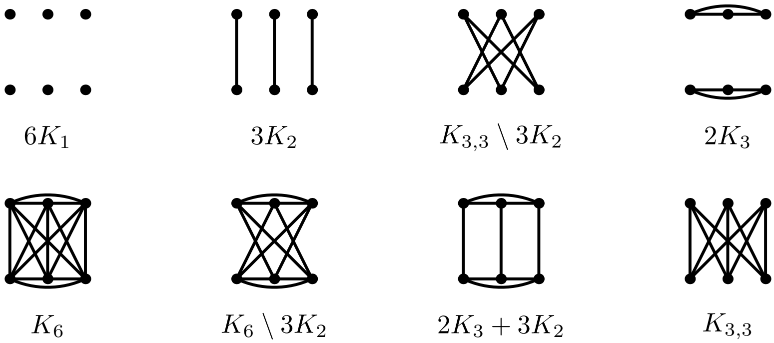

Figure 1. Equipartite graphs of order six.

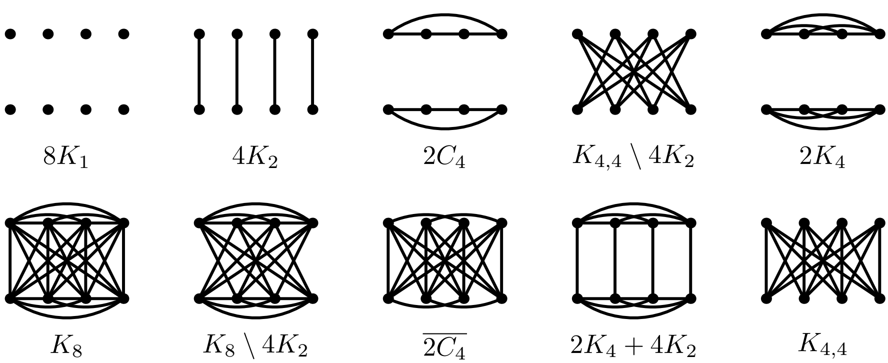

Figure 2. Equipartite graphs of order eight.

2. Definitions and notation

We use standard graph theory terminology which can be found, e.g., in [10]. A union of k vertex-disjoint copies of a graph G is denoted by $k\,G$. We write $G+H$ for an edge-disjoint union of two graphs G and H on the same vertex set; the graph $G+H$ will always be well-defined by the graphs G and H. Similarly, $G \setminus H$ stands for a graph obtained from G by removing a subgraph isomorphic to H. Again, the graph $G \setminus H$ will always be well-defined by the graphs G and H. This notation is used in Figures 1 and 2.

The d-dimensional Euclidean space is denoted by $\mathbb{R}^d$. A **d-polytope** is a convex hull of some $n \geq d+1$ points of $\mathbb{R}^d$ which are not contained in a $(d-1)$-dimensional affine flat. A **symmetry** of a d-polytope $P \subseteq \mathbb{R}^d$ is an isometry τ of $\mathbb{R}^d$ that maps P onto P. In a suitable coordinate system, each symmetry can be represented by an orthogonal transformation. If P is an equipartite polytope, then the 1-skeleton of P is an equipartite graph. Clearly, the converse is false. Two d-polytopes P and Q have the same **symmetry type** if they are combinatorially equivalent and their symmetry groups are isomorphic under an isomorphism compatible with their combinatorial equivalence [9].

A permutation group Γ acting on a set A_0 of size $2n$ has the **interchange property** [1] if for every n-element subset $A \subseteq A_0$, there is a group element $g \in \Gamma$ which interchanges A with its complement. Note that a polytope P is equipartite if and only if its symmetry group, acting as a permutation group on the vertices of P, has the interchange property.

We denote by Γ_P the group of symmetries of the polytope P. Γ_P acts as a permutation group on the vertices and on the $\binom{2n}{2}$ pairs of vertices of P.

Definition 2: For a given pair of vertices $\{u, v\} \subset V(P)$ the **orbit** of $\{u, v\}$ is is the graph $G_{u,v}$ where $V(G_{u,v}) = V(P)$ and $E(G_{uv}) = \{\gamma(u)\gamma(v) | \; \gamma \in \Gamma_P\}$.

Observation 1: When we consider the vertices of an equipartite polytope P as the vertices of a complete graph K_{2n} any orbit of Γ_P is an equipartite spanning subgraphs of K_{2n}. Furthermore, on each orbit, Γ_P acts 2-homogeneously, that is, it acts transitively on the (unordered) pairs of vertices (u, v) that belong to the orbit. This also implies that all segments belonging to the same orbit have the same Euclidean length and the union of orbits is an equipartite graph.

3. Equipartite polytopes

We first note that equipartite polytopes have a very high degree of symmetry.

Definition 3: A polytope P is **isogonal** if for any two vertices of P, there is a symmetry of P that maps one onto the other.

Proposition 4: *If a polytope P with $2n$ vertices is equipartite, then P is isogonal.*

Proof. This is a direct consequence of the interchange property [1]. For the reader's convenience we include the following simple proof.

Consider a graph G whose vertices are the vertices of P, two of them are adjacent if there is a symmetry of P that maps one of them onto the other. Clearly, the graph G is well-defined. If G contains a vertex v of degree at most $n-1$, choose a subset $A \subseteq V(G)$ such that $v \notin A$ and $|A| = n$. Note that v is adjacent to no vertex of A. Since the polytope P is equipartite, there is a symmetry that maps the vertices of A onto the complementary set of vertices of P. But this means that v must have a neighbor in A contradicting the definition of A. Hence, the minimum degree of G is at least n. Therefore, the graph G is connected (its order is $2n$). Since a composition of two symmetries of P is a symmetry of P, it follows that G is the complete graph and the polytope P is isogonal. ∎

We can now show that every equipartite d-polytope has at most $2(d+1)$ vertices. Since the only equipartite 2-polytopes can be cycles and no cycle of order > 6 is an equipartite graph we can conclude:

Proposition 5: *An equipartite 2-polytope P has at most six vertices.*

Theorem 6: *If P is an equipartite d-polytope with $2n$ vertices, then $n \leq d+1$.*

Proof. We may assume that $d \geq 3$. Let G_i be the graphs determined by the orbits that partition the complete graph on $V(P)$. The graphs G_i partition K_{2n} into edge-disjoint spanning subgraphs. As noted before, the graphs $\bigcup_{i \in I} G_i$ are equipartite for each set $I \subseteq \{1, \ldots, k\}$. In particular, each G_i is a nonempty equipartite graph of order $2n$.

Assume that $n \geq d+2$. Since $\mathbb{R}^d$ does not contain $d+2$ distinct equidistant points, no graph G_i contains a clique of order n. Note that the only nonempty equipartite graphs of order $2n$ without a clique of order n are nK_2, $K_{n,n}$ and $K_{n,n} \setminus nK_2$. Hence, each G_i is isomorphic to nK_2, $K_{n,n}$ or $K_{n,n} \setminus nK_2$. Therefore, $k \geq 3$.

At most one of the graphs G_i can be isomorphic to nK_2. Indeed, if two of the graphs G_j and $G_{j'}$ were isomorphic to nK_2, then the graph $G_j \cup G_{j'}$ would be an equipartite 2-regular graph of order $2n \geq 2(d+2) \geq 10$. But there is no such equipartite graph by Theorem 14 [5].

Since at most one of the graphs G_i is isomorphic to nK_2 and $k \geq 3$, two of the graphs G_i, say G_1 and G_2, are isomorphic to $K_{n,n}$ or $K_{n,n} \setminus nK_2$. Both

G_1 and G_2 contain $K_{n,n} \setminus nK_2$ as a subgraph. Since the graphs G_i partition the complete graph K_{2n}, the graph G_2 is a subgraph of the complement of the graph G_1. This immediately yields that $K_{n,n} \setminus nK_2$ is a subgraph of its complement, i.e., $K_{n,n} \setminus nK_2 \subseteq 2K_n + nK_2$. However, this is not true for $n \geq 5$ – a contradiction. ∎

4. Equipartite 2 and 3-polytopes

We first characterize equipartite 2-polytopes:

Theorem 7: *Equipartite polygons are precisely isogonal quadrangles and hexagons, i.e., they are rectangles, regular hexagons and hexagons whose interior angles are all equal to* $120°$ *and whose sides alternate between two lengths.*

Proof. It is easy to see that isogonal quadrangles and hexagons, i.e., squares, non-square rectangles, regular hexagons and non-regular isogonal hexagons, are equipartite. These are all isogonal polygons with at most six vertices. Since there is no equipartite polygon with eight or more vertices by Theorem 6, the statement of the theorem now follows. ∎

Equipartite 3-polytopes are more interesting. In order to describe all of them, we start by recalling that the symmetry types of all isogonal 3-polytopes have been determined [6]. It is well-known that each isogonal 3-polytope is combinatorially equivalent to one of the Platonic or Archimedean solids (we include prisms and anti-prisms among Archimedean solids). We can now show the following:

Theorem 8: *Equipartite* 3*-polytopes are precisely tetrahedra,* 3*-sided prisms,* 4*-sided prisms and* 3*-sided antiprisms.*

Proof. Each equipartite 3-polytope has at most 8 vertices by Theorem 6. It is easy to check that the only isogonal polytope on at most 8 vertices not listed in the statement of the theorem is a 4-sided antiprism. However, no 4-sided antiprism is equipartite (consider two successive vertices of one base and two non-adjacent vertices of the other). ∎

An inspection of symmetry types of isogonal tetrahedra, 3-sided prisms, 4-sided prisms and 3-sided antiprisms leads to the following exhaustive list of possible symmetry types of equipartite 3-polytopes (representatives of the symmetry types are depicted in Figure 3). The symmetry groups are denoted as in [2]:

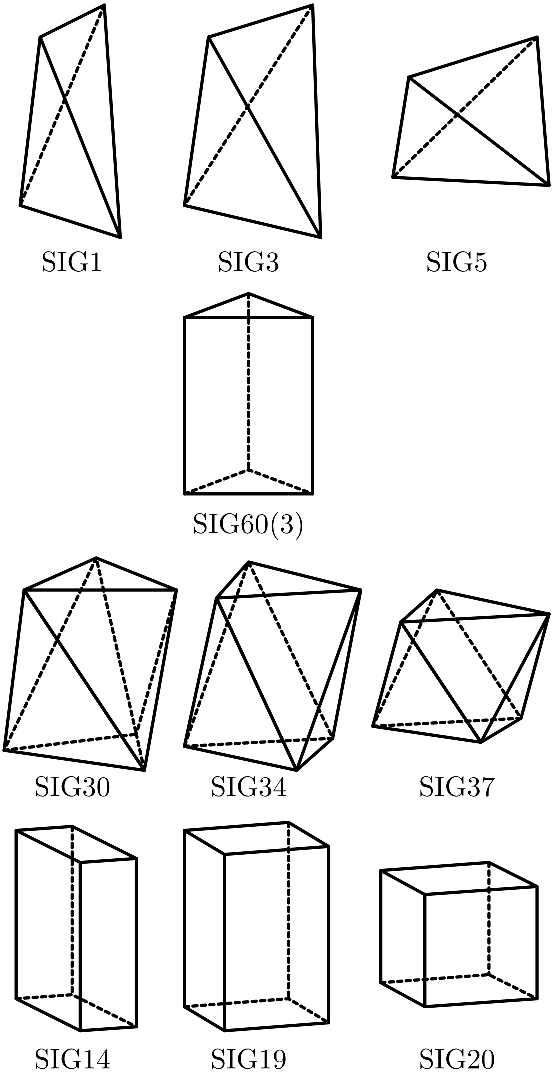

Figure 3. Examples of equipartite 3-polytopes for each possible symmetry type (the notation used is as in [6])

- **Tetrahedra**
  - The 2-parameter type SIG1 with the symmetry group $[2,2]^+$, i.e., a convex hull of congruent non-parallel segments, perpendicular to the line connecting their midpoints but not to each other.
  - The 1-parameter type SIG3 with the symmetry group $[2^+,4]$, i.e., the limit case of the above in which the segments are perpendicular to each other, but the faces are not equilateral triangles, i.e., they are non-equilateral isosceles triangles.
  - The regular tetrahedron SIG5 with the symmetry group $[3,3]$.
- **3-sided prisms**
  - The 1-parameter type SIG60(3) with the symmetry group $[2,3]$, i.e., a straight prism with an equilateral triangle as a base.
- **4-sided prisms**
  - The 2-parameter type SIG14 with the symmetry group $[2,2]$, i.e., a rectangular box with three distinct dimensions.
  - The 1-parameter type SIG19 with the symmetry group $[2,4]$, i.e., a straight prism with a square as a base and with non-square mantle faces.
  - The cube SIG19 with the symmetry group $[3,4]$.
- **3-sided antiprisms**
  - The 2-parameter type SIG30 with the symmetry group $[2,3]^+$, i.e.. a convex hull of two congruent equilateral triangles in horizontal planes, perpendicular to the line connecting their centers, with sides not parallel. The side faces are either congruent scalene triangles, or congruent non-horizontal isosceles triangles with non-horizontal bases.
  - The 1-parameter type SIG34 with the symmetry group $[2^+,6]$, i.e., the limit case of the above in which the side faces are non-equilateral isosceles triangles with horizontal bases.
  - The regular octahedron SIG37 with the symmetry group $[3,4]$.

5. Constructions of equipartite polytopes

Structure of equipartite polytopes is governed by equipartite graphs. Since the orbits of an equipartite polytope are equipartite graphs on which their automorphism group acts 2-homogeneously, there are very few possibilities. The only possible orbits are:

$$K_{2n}, \quad K_{2n} \setminus nK_2, \quad 2K_n, \quad nK_2, \quad K_{n,n}, \quad K_{n,n} \setminus nK_2, \quad 2C_4$$

The other equipartite graphs are not edge transitive and therefore cannot be orbits. Since the orbits are edge disjoint spanning subgraphs of the complete graph on $V(P)$ and the union of orbits is an equipartite graph, it is easily seen that if P is an equipartite polytope in $\mathbb{R}^d$, $d \geq 4$ then if it does not contain $2C_4$ as an orbit it must contain one of the following 3 orbits:

- K_{2n} (this is the regular simplex)
- $K_{2n} \setminus nK_2$
- $2K_n$

For instance, if P has an orbit isomorphic to $K_{n,n}$ then the only way the complete graph on $V(P)$ can be decomposed into the union of edge disjoint orbits is for it to contain $2K_n$ as an orbit.

We start with the exceptional orbit $2C_4$. We construct equipartite d-polytopes ($d = 4, 5$) with 8 vertices having the orbit $2C_4$. They are "unique" in the sense that their construction does not generalize to higher dimensions.

Take two congruent rectangles lying in a pair of orthogonal 2-dimensional subspaces of $\mathbb{R}^4$. For example: $\{(\pm a, \pm b, 0, 0), (0, 0, \pm a, \pm b)\}$

Note that for each two triples of vertices of a rectangle in $\mathbb{R}^2$ there is an isometry of the rectangle that maps one triple onto the other triple. Also any two vertices of a rectangle can be isometrically mapped onto the other two vertices.

To show that this polytope is equipartite, note, for example, that the 4 vertices $\{(a, b, 0, 0), (-a, -b, 0, 0), (-a, b, 0, 0), (0, 0, a, b)\}$ will be interchanged with the vertices $\{(0, 0, -a, -b), (0, 0, -a, b), (0, 0, a, -b), (a, -b, 0, 0)\}$ by the orthogonal matrix

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{1}$$

All remaining cases can be treated similarly by similar combinations of 2×2 unit matrices and the 2×2 sub-matrices of M. Thus, the polytope obtained by the convex hull of these 8 points in $\mathbb{R}^4$ is equipartite. In order to identify its orbits we note that the matrices

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \tag{2}$$

fix the vertices $(\pm a, \pm b, 0, 0)$ and act transitively on the vertices $(0, 0, \pm a, \pm b)$. This implies that the orthogonal matrices

$$T = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \tag{3}$$

generate the orbit $K_{4,4}$.

Hence the orbits of this polytope are: $\{4K_2,\ 4K_2,\ 4K_2,\ K_{4,4}\}$. The equipartite graphs generated by unions of these orbits are:

- $2C_4$: two matchings
- $2K_4$: the 3 matchings
- $\overline{2C_4}$: $K_{4,4}$ plus one matching
- $K_8 \setminus 4K_2$: two matchings and $K_{4,4}$
- K_8: the union of all orbits.

Thus, except for the graphs $K_{4,4} \setminus 4K_2$ and its complement $2K_4 + 4K_2$, 8 of the 10 equipartite graphs of order 8 are orbits or unions of orbits in this 4-polytope. When $a = b = 1$ we get a different 4-polytope of order 8 in which one of the orbits is $2C_4$.

Similarly, the convex-hull of the 8 points $\{(\pm 1, \pm 1, 1, 0, 0), (0, 0, -1, \pm 1, \pm 1)\} \in \mathbb{R}^5$ also yield an equipartite 5-polytope having $2C_4$ as an orbit.

In the next two propositions we construct the only possible equipartite polytope having $K_{2n} \setminus nK_2$ as an orbit.

Proposition 9: *The cross-polytope $P = conv(\pm e_1, \ldots, \pm e_d)$ where e_i are the unit vectors in $\mathbb{R}^d$ is an equipartite polytope.*

Proof. Let $F = \{f_1, \ldots, f_d\}$ be any subset of d points from $\{\pm e_1, \ldots, \pm e_d\}$ and let G be its complement. We shall construct an orthogonal matrix P that maps F onto G. Let:

- $F_1 = \{i \mid e_i \in F,\ -e_i \notin F\}$

- $F_2 = \{i \mid -e_i \in F,\ e_i \notin F\}$
- $F_3 = \{i \mid e_i \in F,\ -e_i \in F\}$

Define G_1, G_2, G_3 similarly. Clearly, $F_1 = G_2$, $F_2 = G_1$, $F_3 \cap G_3 = \emptyset$ and $|F_3| = |G_3|$. Let ψ be a bijection from the vectors whose indices are in $F_3 \cup G_3$ onto itself such that $\psi(e_i) = -\psi(-e_i)$. Define the matrix P by:

$$P_{i,j} = \begin{cases} -1 & \text{if } i = j,\ e_i \in F_1 \cup F_2, \\ 1 & \text{if } e_i \in F_3,\ e_j = \psi(e_i), \\ 1 & \text{if } e_j \in G_3,\ e_j = \psi(e_i), \\ 0 & \text{otherwise.} \end{cases}$$

Note that P is a signed-permutation matrix in which non-zero diagonal entries are -1, and the non-zero off-diagonal entries are 1. The diagonal entries interchange the vectors with indices $\in F_1$ with the vectors with indices $\in G_2$ and the vectors with indices $\in F_2$ with the vectors with indices $\in G_1$ while the off diagonal entries interchange the vectors whose indices $\in F_3$ and the vectors with indices $\in G_3$. Since P is an orthogonal matrix the cross-polytope is equipartite. ∎

Proposition 10: *If one of the orbits of an equipartite polytope P is isomorphic to $K_{2n} - nK_2$, then P is the cross-polytope.*

Proof. By the assumption, P is a polytope with $2n$ vertices $v_1, \dots, v_{2n} \in \mathbb{R}^d$ (for some d). Since P is equipartite, for every subset of n vertices $\{v_1, \dots, v_n\}$ there is an isometry of P that interchanges $\{v_1, \dots, v_n\}$ with $\{v_{n+1}, \dots, v_{2n}\}$. This implies that each such isometry fixes the point: $\sum_{i=1}^{2n} v_i$. Furthermore, since P is isogonal, the Euclidean distance of every vertex of P from $\sum_{i=1}^{2n} v_i$ is the same. Thus without loss of generality we may assume that $\sum_{i=1}^{2n} v_i = 0$ and $\|v_i\| = 1$.

Let M be the $2n \times 2n$ matrix defined by:

$$m_{ij} = \langle v_i, v_j \rangle .$$

Observe that $m_{ij} = m_{i'j'}$ whenever the pairs (i, j) and (i', j') belong to the same orbit.

Since one orbit of P, say Ω_1, is isomorphic to $K_{2n} - nK_2$, there must be exactly two orbits and the other orbit Ω_2 is isomorphic to nK_2. It follows that there are only three choices for m_{ij}:

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \alpha & \text{if } i \neq j \text{ and } ij \in \Omega_1, \\ \beta & \text{if } i \neq j \text{ and } ij \in \Omega_2, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$. It is easy to see that up to a rearrangement of rows and columns,

$$M = \begin{pmatrix} 1 & \beta & & & & & \\ \beta & 1 & & & & \alpha & \\ & & 1 & \beta & & & \\ & & \beta & 1 & & & \\ & & & & \ddots & & \\ & \alpha & & & & 1 & \beta \\ & & & & & \beta & 1 \end{pmatrix} \tag{4}$$

Since $\sum_{i=1}^{2n} v_i = 0$, we have

$$\sum_{i=1}^{2n} m_{ij} = \sum_{i=1}^{2n} \langle v_i, v_j \rangle = \left\langle \left(\sum_{i=1}^{2n} v_i \right), v_j \right\rangle = 0$$

so $\mathrm{rank}(M) \leq 2n - 1$ and

$$(2n-2)\alpha + \beta + 1 = 0. \tag{5}$$

So 0 is an eigenvalue corresponding to the eigenvector $\mathbf{1}^T = (1, 1, \ldots, 1)$. Let $u \neq \mathbf{1}^T$ be an eigenvector of M with corresponding eigenvalue λ. Since u is orthogonal to $\mathbf{1}^T$, we have $Ju = 0$, where J is the all-1 matrix. Consequently, λ is also an eigenvalue of

$$M - \alpha J = \begin{pmatrix} 1-\alpha & \beta-\alpha & & & & & \\ \beta-\alpha & 1-\alpha & & & & 0 & \\ & & 1-\alpha & \beta-\alpha & & & \\ & & \beta-\alpha & 1-\alpha & & & \\ & & & & \ddots & & \\ & 0 & & & & 1-\alpha & \beta-\alpha \\ & & & & & \beta-\alpha & 1-\alpha \end{pmatrix}$$

with equal multiplicity.

However, the eigenvalues of $M - \alpha J$ are easily identified as $\lambda_2 = 1 - \beta$ and $\lambda_3 = 1 + \beta - 2\alpha$, both of multiplicity n. Hence the sum of the multiplicities of the eigenvalues 0, λ_2 and λ_3 of M would exceed $2n$, unless two of them coincide.

If $\lambda_2 = 0$, we get $\beta = 1$ and consequently $\langle v_1, v_2\rangle = \langle v_3, v_4\rangle = \cdots = \langle v_{2n-1}, v_{2n}\rangle = 1$. But v_i are unit vectors which would imply $v_1 = v_2$, $v_3 = v_4, \ldots, v_{2n-1} = v_{2n}$ a contradiction as the vertices of P are assumed distinct. If $\lambda_2 = \lambda_3$, we obtain $\alpha = \beta$ and P is a regular simplex whose sole orbit is K_{2n}.

The remaining case is $\lambda_3 = 0$. Together with (5), this implies that $\alpha = 0$ and $\beta = -1$. The vertex set of P thus consists of n antipodal pairs of points, and distinct pairs determine orthogonal lines. We conclude that P is the equipartite cross-polytope. ∎

The remaining equipartite polytopes have the orbit $2K_n$.

Lemma 12 provides a simple tool to prove equipartiteness. It generalizes the proof above. It is based on the following well-known basic property of orthogonal real matrices:

Lemma 11: *If $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ are two sets of unit vectors in R^d such that $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle\ \forall\{i, j\}$, then there is an orthogonal matrix T such that $T(u_i) = v_i$.*

Lemma 12: *Let $U = \{u_1, \ldots, u_k\}$ and $V = \{v_1, \ldots, v_k\}$ be two disjoint sets of unit vectors in $\mathbb{R}^d$. Suppose that $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = \alpha$ for all $1 \le i < j \le k$, $\langle u_i, v_j\rangle = \langle u_j, v_i\rangle = \beta$ for all $1 \le i < j \le k$ and $\langle u_i, v_i\rangle = \gamma$ for all $1 \le i \le k$. Then $P = \operatorname{conv}(U \cup V)$ is an equipartite polytope.*

Proof. We first note that $k \le d+1$ and that there is essentially one configuration of $d + 1$ equiangular unit vectors in R^d; the unit vectors connecting the center of the regular d-simplex to its vertices. The angle between any two of these vectors is $\arccos -\frac{1}{d}$.

Let $A = \{a_1, \ldots a_k\} \subset \{u_1, \ldots, u_k, v_1, \ldots, v_k\}$ be an arbitrary k-element subset of vectors and let $B = \{b_1, \ldots, b_k\}$ be the remaining k vectors. By Lemma 11 it is enough to show that the vectors in A and B can be sequenced so that $\langle a_i, a_j\rangle = \langle b_i, b_j\rangle\ \forall\{i, j\}$.

The proof follows the same approach used in the proof of Proposition 9. Let $A = \{a_1, \ldots a_k\} = \{u_{i_1}, \ldots, u_{i_r}\} \cup \{v_{j_1}, \ldots, v_{j_s}\} \cup \{u_{h_1}, v_{h_1} \ldots, u_{h_t}, v_{h_t}\}$. Where:

(1) $r + s + 2t = k$.

(2) The sets $\{i_1, \ldots, i_r\}$, $\{j_1, \ldots j_s\}$, $\{h_1, \ldots, h_t\}$ are pairwise disjoint.

We sequence: $A = \{u_{i_1}, \ldots, u_{i_r}, u_{h_1}, u_{h_2}, \ldots, u_{h_t}, v_{j_1}, \ldots, v_{j_s}, v_{h_1}, \ldots, v_{h_t}\}$.

In this sequence, the first $r + t$ vectors belong to U and the remaining $s + t$ vectors belong to V.

We define $B = \{v_{i_1}, \ldots, v_{i_r}\} \cup \{u_{j_1}, \ldots, u_{j_s}\} \cup \{u_{p_1}, v_{p_1} \ldots, u_{p_t}, v_{p_t}\}$ where $\{h_1, \ldots, h_t\} \cap \{p_1, \ldots, p_t\} = \emptyset$ (note also that $r + s + 2t = k$ implies that these two sets have the same size).

B contains $r + t$ vectors from V and $s + t$ vectors from U. If we sequence $B = \{b_1, \ldots b_k\}$ so that the first $r+t$ vectors are from V and the last $s+t$ vectors from U the assumptions of the lemma imply that $\langle a_i, a_j\rangle = \langle b_i, b_j\rangle$. ∎

We now present three constructions of equipartite polytopes with $2K_n$ orbits.

THEOREM 13: *For every $d \geq 2$, there is an equipartite d-polytope with $2d + 2$ vertices.*

Proof. Let $u_1, \ldots, u_{d+1}$ be a set of $d + 1$ equiangular unit vectors in $\mathbb{R}^d$. Note that the convex hull of the end-points of the vectors $u_1, \ldots, u_{d+1}$ is a regular d-simplex. Set $v_i = -u_i$ for each $i = 1, \ldots, d+1$ and let P be the convex hull of the end-points of the $2d + 2$ vectors $u_1, \ldots, u_{d+1}, v_1, \ldots, v_{d+1}$. P is a centrally symmetric convex d-polytope with $2d + 2$ vertices. By Lemma 12, the polytope P is equipartite. ∎

Remark: The orbits of this polytope are $2K_n$, nK_2 and $K_{n,n} \setminus nK_2$. In $\mathbb{R}^3$ this polytope is the regular 3-cube.

THEOREM 14: *The prism and anti-prism over a regular $(d-1)$-simplex is an equipartite d-polytope with $2d$ vertices for every $d \geq 3$.*

Proof. Let $w_1, \ldots, w_d$ be a set of d equiangular unit vectors contained in a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. For $0 < \alpha < 1$, define a d-polytope P_α to be the convex hull of the end-points of the $2d$ vectors

$$U = \{\alpha w_1 + \sqrt{1-\alpha^2}\, e, \ldots, \alpha w_d + \sqrt{1-\alpha^2}\, e\}$$
$$V = \{\alpha w_1 - \sqrt{1-\alpha^2}\, e, \ldots, \alpha w_d - \sqrt{1-\alpha^2}\, e\}$$

where e is the unit vector orthogonal to the subspace of $\mathbb{R}^d$ spanned by the vectors $w_1, \ldots, w_d$. P_α is equipartite by Lemma 12.

Similarly, the convex hull of $U' \cup V'$ where

$$U' = \{\alpha w_1 + \sqrt{1-\alpha^2}\, e, \ldots, \alpha w_d + \sqrt{1-\alpha^2}\, e\}$$
$$V' = \{-\alpha w_1 - \sqrt{1-\alpha^2}\, e, \ldots, -\alpha w_d - \sqrt{1-\alpha^2}\, e\}$$

is an equipartite polytope and the theorem now readily follows. ∎

The orbits of these polytopes are $2K_d$, dK_2 and $K_{d,d} \setminus dK_2$.

Theorem 15: *The convex hull of two regular isomorphic d-simplices centered at the origin which lie in orthogonal d-dimensional subspaces of R^{2d} form an equipartite $(2d)$-polytope with $2d+2$ vertices.*

Proof. Let $u_1, \ldots, u_d$ and $v_1, \ldots, v_d$ be vectors from the origin to the vertices of each of the two regular d-simplices. We can assume without loss of generality that all the vectors $u_1, \ldots, u_d$ and $v_1, \ldots, v_d$ are unit vectors. If $i \neq j$ $\langle u_i, u_j \rangle = \langle v_i, v_j \rangle = arccos\frac{-1}{d}$ and $\langle u_i, v_j \rangle = 0$ by Lemma 12 the polytope described in the statement of the theorem is equipartite. ∎

This equipartite $2d$-polytope has $2d+2$ vertices and two orbits: $2K_{d+1}$ and $K_{d+1,d+1}$.

We now focus on the number of distinct symmetry types of equipartite $(2d+1)$-simplices:

Theorem 16: *For each $d \geq 2$, there are three distinct symmetry types of equipartite $(2d+1)$-simplices.*

Proof. The regular $(2d+1)$-simplex is clearly equipartite. Another $(2d+1)$-simplex, of a different symmetry type, can be obtained as follows:

Let $\{X_1, \ldots, X_{d+1}\}$ and $\{Y_1, \ldots, Y_{d+1}\}$ be the vertices of two regular d-simplices centered at the origin. Let $A_i = (X_{i,1}, \ldots, X_{i,d}, -1, 0, \ldots, 0) \in \mathbb{R}^{2d+1}$ and $B_i = (0, \ldots, 0, +1, Y_{i,1}, \ldots, Y_{i,d}) \in \mathbb{R}^{2d+1}$ for $1 \leq i \leq d+1$. Let a_i and b_i be vectors from the origin to the point A_i and B_i, respectively. In order to show that the convex hull C of the points $A_1, \ldots, A_{d+1}, B_1, \ldots, B_{d+1}$ is a

$(2d+1)$-simplex, we verify that they are affinely independent. Indeed:

$$\det\begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,d} & -1 & 0 & \cdots & 0 \\ 1 & x_{2,1} & \cdots & x_{2,d} & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{d+1,1} & \cdots & x_{d+1,d} & -1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & y_{1,1} & \cdots & y_{1,d} \\ 1 & 0 & \cdots & 0 & 1 & y_{2,1} & \cdots & y_{2,d} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 & 1 & y_{d+1,1} & \cdots & y_{d+1,d} \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,d} & 0 & 0 & \cdots & 0 \\ 1 & x_{2,1} & \cdots & x_{2,d} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{d+1,1} & \cdots & x_{d+1,d} & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 2 & y_{1,1} & \cdots & y_{1,d} \\ 1 & 0 & \cdots & 0 & 2 & y_{2,1} & \cdots & y_{2,d} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 & 2 & y_{d+1,1} & \cdots & y_{d+1,d} \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,d} \\ 1 & x_{2,1} & \cdots & x_{2,d} \\ \vdots & \vdots & & \vdots \\ 1 & x_{d+1,1} & \cdots & x_{d+1,d} \end{pmatrix} \cdot \det\begin{pmatrix} 2 & y_{1,1} & \cdots & y_{1,d} \\ 2 & y_{2,1} & \cdots & y_{2,d} \\ \vdots & \vdots & & \vdots \\ 2 & y_{d+1,1} & \cdots & y_{d+1,d} \end{pmatrix} \neq 0$$

Observe now that the vectors $a_1, \ldots, a_{d+1}$ and $b_1, \ldots, b_{d+1}$ satisfy the assumptions of Lemma 12. Hence, C is an equipartite $(2d+1)$-simplex. Note that the simplex C is not regular. The symmetry group of this simplex acts imprimitively with two classes of imprimitivity, namely A and B, and no other types of imprimitivity. In particular, for $d \geq 3$, there are no imprimitivity classes of size 2. The symmetry group of this simplex is the wreath product of the groups S_{d+1} and S_2.

Finally, there is a third symmetry type of equipartite $(2d+1)$-simplices. Let $a_1, \ldots, a_{d+1}$ be $d+1$ equiangular unit vectors in $\mathbb{R}^d$ and let $e_1, \ldots, e_{d+1}$ be $d+1$ vectors of an orthonormal basis of $\mathbb{R}^{d+1}$. Note that the end-points of the vectors $a_1, \ldots, a_{d+1}$ form a regular d-simplex. We now define vectors $b_1, \ldots, b_{2d+2}$.

The vector b_{2i-1} is equal to $(a_i|e_i)$ and b_{2i} to $(a_i|-e_i)$ where $(a|e)$ is a $(2d+1)$-dimensional vector obtained by concatenation of the vectors a and e. It is easy to see that the vectors $b_1,\ldots,b_{2d+2}$ are affinely independent vectors and hence the convex hull P of their end-points is a $(2d+1)$-simplex.

We first note that $\langle b_{2i-1},b_{2i}\rangle = 0$ for each $i=1,\ldots,d+1$ and $\langle b_{2i-1},b_{2k}\rangle = \langle b_{2i},b_{2k}\rangle = \arccos\frac{-1}{d}$ when $i\neq k$. So if $U=\{b_2,b_4,\ldots,b_{2d+2}\}$ and $V=\{b_1,b_3,\ldots,b_{2d+1}\}$ by Lemma 12 the $(2d+1)$-simplex P is equipartite.

Note that the symmetry group of P is imprimitive with $d+1$ classes of imprimitivity, corresponding to the pairs of vertices b_{2i-1} and b_{2i} for $i=1,\ldots,d+1$. Therefore, the symmetry group of P is the wreath product of the groups S_2 and S_{d+1}. ∎

We believe that the constructions in Sections 4 and 5 cover all possible equipartite polytopes. It is interesting to note that in $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{R}^5$ there are equipartite polytopes whose construction is not extendable to higher dimensions as they contain the orbit $2C_4$.

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