Dirac's conjecture concerning high-incidence elements in aggregates

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The sixty years old conjecture by G. A. Dirac [Di51] is so simple it can be explained to a grade-school child in five minutes — but it has resisted efforts to solve it for sixty years. We shall get to it in a moment, after setting up the formal definitions required by mathematicians that could be dispensed with when speaking to the child.

An *aggregate of points* $\mathcal{A}$ consists of a finite set $\mathcal{P}$ of n distinct points $\{p_j \mid 1 \le j \le n\}$ together with the family of all lines $\{L_i \mid i \in I\}$ that contain at least two points of $\mathcal{P}$. It is convenient to stipulate that not all points of $\mathcal{P}$ are collinear; otherwise lots of exceptions would arise. The dual concept is an *aggregate of lines,* that is, a finite family of lines together with all their points of intersection, assuming not all lines are concurrent; it is convenient to think of it in the projective plane, considered as the Euclidean plane augmented by the points and line "at infinity".

The concept of aggregates (of points, or of lines) has been explicitly defined only recently, but has been the topic of investigations for more than a century. Many papers that concern aggregates have been labeled as dealing with arrangements, or configurations, or various other general-use words. It is my hope that a precise terminology — that distinguishes between these several kinds of objects — will, in time, become more widely used. In what follows we shall use this terminology and other modern concepts, even though other expressions are used in the original literature.

One of the early questions concerning aggregates of points was posed by Dirac [Di51]. In an aggregate $\mathcal{A}$ of n points $\{p_j \mid 1 \le j \le n\}$, we associate with each p_j the *multiplicity* t_j of p_j defined as the number of lines L_i in $\mathcal{A}$ that contain p_j. The maximum of the values of t_j is the *multiplicity* $t(\mathcal{A})$ of the aggregate $\mathcal{A}$. Dirac's question is how small can t($\mathcal{A}$) be if $\mathcal{A}$ ranges over all aggregates of n points; we denote this minimum by t(n). The topic can easily be reformulated in terms of aggregates of lines. Then we are asking for the multiplicity of each line (the number of points of the aggregate on it, which is the same as the number of edges in the arrangement of lines generated by the lines of the aggregate in the projective plane). Dirac's problem is to minimize the maximal number of vertices (or edges) on each line in aggregates of n lines. We shall use both versions interchangeably, while noting that aggregates of lines are visually and perceptually simpler to understand.

In [Di51] Dirac proved that $t(n) > \sqrt{n}$, but commented that this *seems to be far from best possible*. (A more accessible proof of $t(n) > \sqrt{n}$ is in the book [Kl 91] by Klee and Wagon.) Dirac continued with the conjecture (written here in the usual notation, with $\lceil x \rceil$ denoting the "ceiling" of x)

... it seems likely that ... $t(n) \geq \lceil n/2 \rceil$ ... . I have checked the truth of this for $n \leq 14$. ... In the case of t(n) it is easy to see that this is best possible, since $t(n) = \lceil n/2 \rceil$ for the following configurations ...".

Dirac supports this conjecture by general examples, which are illustrated in Figure 1. An alternative notation for Dirac's conjecture is $t(n) = [(n+1)/2]$.

Various examples (that we shall mention soon) show that Dirac's conjecture is not valid in general. This does not detract from Dirac's work – after all, conjectures are made to be either confirmed or refuted, and for all that is known, by changing the conjecture just a little eliminates all but few available counterexamples. However, a serious problem with Dirac's statement is the assertion *I have checked the truth of this* [conjecture] *for $n \leq 14$* since there are at least four values of $n \leq 14$ for which the conjecture fails, and t(n) is smaller than this bound. Two examples are illustrated in Figure 2, the two other examples are listed in the Table.

At about the same time as Dirac's paper, there appeared a paper by Motzkin [Mo51], often quoted for a variety of results it contains. Motzkin mentions (on p. 452) that for each n-point aggregate

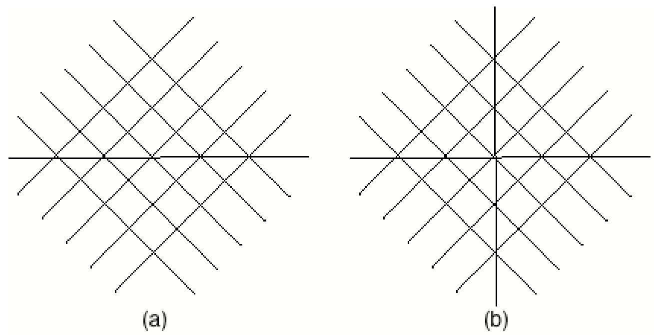

Figure 1. (a) Duals (in the projective plane) of the typical examples constructed by Dirac [Di51] to show that t(n) = ⌈n/2⌉. For even n = 2k, the example $\mathcal{A}$ consists of two pencils of n/2 – 1 concurrent lines, that intersect a different line in the same points (that is, the two pencils of lines are projectively equivalent) together with that line and the line connecting the two apexes of the pencils (here points at infinity, connected by the line at infinity). In the diagram n = 12, and as is easily verified t($\mathcal{A}$) in the illustration is 6, and in general t($\mathcal{A}$) = n/2. For odd n = 2k – 1, the same lines as before are used except that the connecting line is not included; then t($\mathcal{A}$) = 6 = ⌈11/2⌉ again, and in general t($\mathcal{A}$) = ⌈n/2⌉. (b) A slight modification of Dirac's construction, showing greater symmetry and leading to the same results.

$\mathcal{A}$ the existence of an ordinary line in $\mathcal{A}$ (that is, a line through precisely two points of $\mathcal{A}$) would follow from t($\mathcal{A}$) ≥ n/2, but that this "has not been proved or disproved." In any case, since t($\mathcal{A}$) is an integer, t($\mathcal{A}$) ≥ n/2 is the same as t($\mathcal{A}$) ≥ [(n+1)/2] = ⌈n/2⌉, hence equivalent to Dirac's conjecture. However, it should be noted that although Motzkin explicitly avoids conjecturing what is the value of t(n), in some publications the conjecture t($\mathcal{A}$) ≥ n/2 for all $\mathcal{A}$ has been called the Dirac-Motzkin conjecture.

The first new information regarding t(n) emerged in [Gr72], where several examples that contradict the "Dirac-Motzkin" conjecture were given. A longer list of counterexamples is contained in the Table at the end of this paper; it is mainly based on [Gr09].

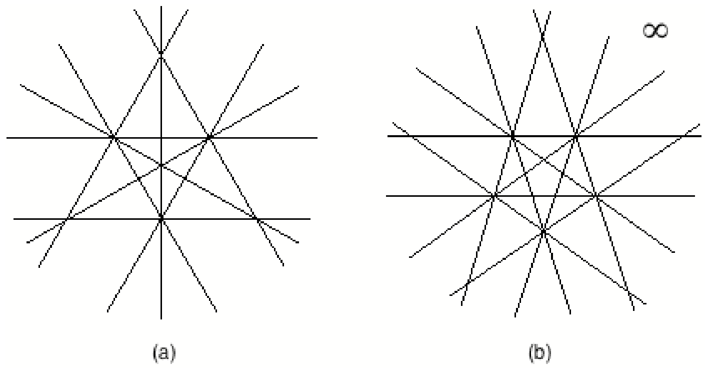

Figure 2. Two examples of aggregates with $n \leq 14$, that contradict Dirac's claim that $t(n) = \lceil n/2 \rceil$ for $n \leq 14$. The simplicial arrangement denoted A(9,1) in [Gr09], shown in (a), has $n = 9$ but $t = 4$. The aggregate shown in (b) is the dual of the example with $n = 11$ and $t = 5$ found by Akiyama *et al.* [Ak10].

Several books deal briefly with Dirac's conjecture. Klee and Wagon [Kl91] discuss the difficulties in establishing any related result, and suggest that one might try to prove $t(n) \geq n/3$ — which is not contradicted by any known examples. However, there does not seem to have been any action on this suggestion.

Two other books deal with Dirac's problem, each on half a page. The main contribution of Felsner [Fe04, p. 86] is *... a family of [aggregates of] lines showing that $t(n) \leq \lceil n/2 \rceil - 2$ for all n of the form $n = 12k + 7$ ...* The construction is not made explicit, but is illustrated by the example for $k = 2$ (that is, $n = 31$); the resulting aggregate $\mathcal{A}$ has $t(\mathcal{A}) = 14$. Figure 3 show a possible interpretation of the Felsner construction for $n = 19$ (with $t = 8$), as well as an example with $n = 31$ that is slightly different from Felsner's.

Brass, Moser and Pach [Br05, p. 313] ascribe to Dirac a conjecture which is their own extension of Dirac's: *There is a constant c such that $t(n) \geq n/2 - c$.* They add the comment: *Many small examples in [Gr72] show that the conjecture is false with $c = 0$. An infinite family of counterexamples was constructed by Felsner (personal communication): $6k + 7$ points, each of them incident to at most $3k + 2$ lines.* This comment about Felsner is reproduced in [Ak10].

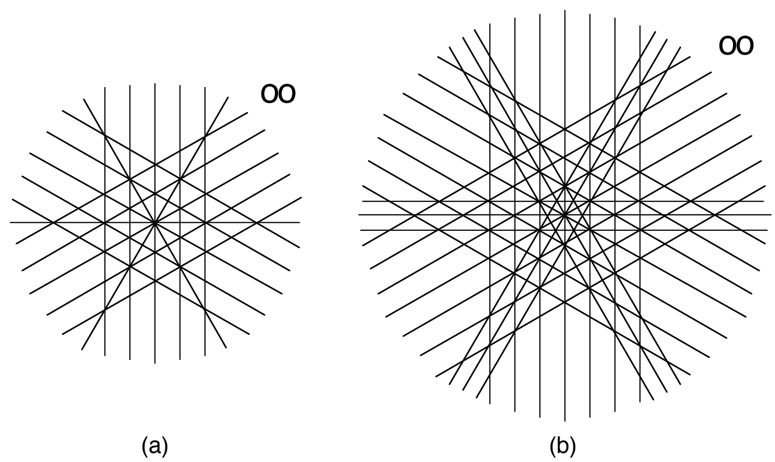

Figure 3. (a) An aggregate of 19 lines, with t = 8; it arises as the simplicial arrangement A(19,1), in the notation of [Gr09]. (b) An aggregate of 31 lines, with t = 14; it arises from the simplicial arrangement A(31,2), and is different from the examples with 31 lines and t = 14 given in [Fe04] and[Br05].

Without any additional explanation there follows in [Br05] a diagram that is a counterexample to Dirac's conjecture with c = 0. This aggregate $\mathcal{A}$ has n = 31 and t($\mathcal{A}$) = 14. It is not clear from the text whether this example is due to Felsner, or to the authors. I have been able find a generalization of this example for all even k (so that the values of n coincide with those in Felsner's examples), but I have not been successful in modifying the given example for odd values of k to confirm the authors' claim. Brass and Pach (private communications) indicated that the statement is in error for odd k, and that a corrected version will appear in the Japanese translation of [Br05].

In a recent paper [Ak10], J. Akiyama *et al.* mention examples that improve the previously known results; this will be made precise below. (I am indebted to the authors for a preprint of [Ak10].)

It would seem that a slight modification of the conjectures in [Di51] and [Br05] accommodates more closely all known results, and exhibits a certain balance:

Conjecture. Except for finitely many values of n = 12k + m we have t(n) = [n/2] if m = 0,1,4,5,8,9, and t(n) = [n/2] − 1 if m = 2,3,6,7,10,11.

Support for this conjecture in case m = 0,4,8 comes from the aggregates that correspond to the families of simplicial arrangements R(1) in the notation of [Gr09] as well as the construction of Dirac (see Figure 1), and a variety of other examples. For m = 1,5,9 the values are those arising from simplicial arrangements R(2); an alterative construction is given in [Ak10], where also a special construction for n = 49 is given, see Figure 4. For m = 7 the bound is obtained through the family of examples in [Fe05]. For m = 3 and 11, there are only a few examples known that conform to the conjecture; they are listed in the Table. In [Ak10] there is a family of examples for m = 3 that gives a value of t(n) higher by 1 than our conjecture. For m = 2,6,10 the values are implied by the monotonicity of t(n).

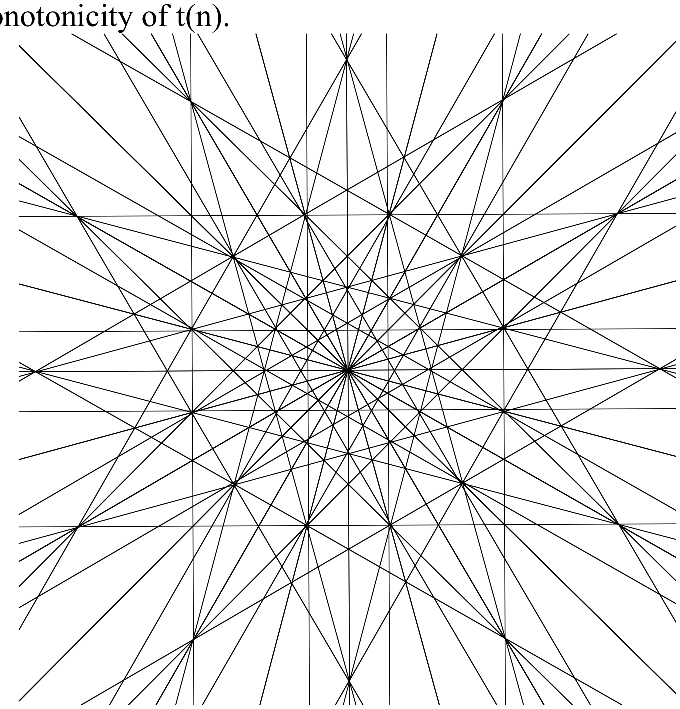

Figure 4. The (dual of an) example in [Ak10] with n = 49 and t = 22.

Dirac's theorem $t(n) > \sqrt{n}$ makes use of incidence properties only, so it is valid in appropriate combinatorial structures. It would be interesting to investigate the values of $t^*(n)$, defined in analogy to $t(n)$ but for aggregates of pseudolines.

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Table of values of t(n). (a) Parameter n. (b) Value of t(n) given by our Conjecture. (c) Smallest known value of t(n); italics indicate a value smaller than the one in (b), and bold-face marks entries higher than the conjecture. (d) Example for the value in (c); in many cases there are additional examples. Symbol A(n,j) refers to the simplicial arrangement in the notation of [Gr09]; A(n,j)\i results from A(n,j) by the deletion of (any) i lines. X(n) is an example from [Ak10], and F[n] is an example from [Fe04].

(a)	(b)	(c)	(d)
6	2	**3**	A(6,1)
7	2	**4**	A(7,1)
8	4	4	A(8,1)
9	4	4	A(9,1)
10	4	**5**	A(10,1)
11	4	**5**	X(11)
12	6	6	A(12,1)
13	6	6	A(13,1)
14	6	6	A(15,1)\1
15	6	6	A(15,1)
16	8	8	A(16,1)
17	8	8	A(17,1)
18	8	8	A(18,2)
19	8	8	A(19,1)
20	10	*9*	A(20,5)
21	10	10	A(21,1)
22	10	10	A(24,2)\2
23	10	10	A(24,2)\1
24	12	*10*	A(24,2)
25	12	*10*	A(25,5)
26	12	12	A(26,2)
27	12	12	A(27,1)
28	14	*12*	A(28,2)
29	14	*12*	A(29,2)
30	14	*12*	A(30,2)
31	14	*12*	A(31,1)
32	16	16	A(32,1)
33	16	16	A(33,1)
34	16	*16*	A(37,2)\3
35	16	16	A(37,2)\2
36	18	*16*	A(37,2)\1
37	18	*16*	A(37,2)
38	18	19	A(38,1)
39	18	**19**	X(39)
40	20	20	A(40,1)
41	20	20	A(41,1)
42	20	20	F(43)\1
43	20	20	F(43)
44	22	22	A(44,1)
45	22	22	A(45,1)
46	22	*22*	X(49)\3
47	22	22	X(49)\2
48	24	*22*	X(49)\1
49	24	*22*	X(49)
50	24	**25**	A(50,1)
51	24	**25**	X(51)
52	26	26	A(52,1)
53	26	26	A(53,1)
54	26	*26*	F(55)\1
55	26	26	F(55)