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Tilings by some nonconvex parallelohedra

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A *parallelohedron* is any polyhedron P in 3-dimensional space with the property that there is a tiling of the space consisting of translated copies of P. This concept, as well as that of the more general zonohedra, are due to Fedorov [F]. He proved that any *convex* parallelohedron is combinatorially equivalent to one of the five polyhedra in Figure 1. In the later years the theory of convex parallelohedra and zonohedra was extended to higher dimensions, see for example [Z]. However, no general studies of *nonconvex* parallelohedra have been published prior to [G]. But two particular types of such polyhedra (and their higher-dimensional analogues) have been studied in some detail by Stein [S] and his collaborators. These they called *crosses* and *semicrosses.* A "cross" is a set of congruent cubes consisting of a central cube and a stack of k ≥ 1 cubes attached (face-to-face) to each of its faces. A "semicross" is formed in the same way, but with stacks attached on only one-half of the faces, all adjacent to each other.

In this note we shall use a notation different from that of Stein but more suitable for our purposes. A (u,v,w)-*semicross* has stacks of u, v, or w, respectively, attached to three adjacent faces of a

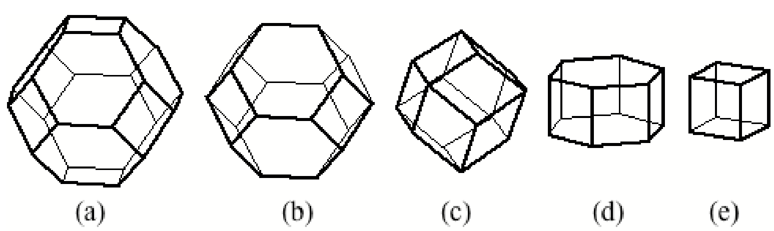

Figure 1. Representatives of the five combinatorial types of convex parallelohedra, as determined by Fedorov [F]. (a) Truncated octahedron (an Archimedean polyhedron); (b) Elongated dodecahedron (with regular faces, but not Archimedean); (c) Kepler's rhombic dodecahedron K (a Catalan polyhedron); (d) Archimedean 6-sided prism; and (e) cube.

central cube; we shall always assume that u ≥ v ≥ w. In Figure 2 we show a (1,1,1)-semicross P and a (3,2,1) semicross. In the writings by Stein and others about semicrosses, P is called a (1,3) semicross, where 1 is k in the earlier definition, and 3 denotes the dimension. Our notation does not include the dimension since we are restricting the discussion to the 3-dimensional case; however, in contrast to the earlier works, we do not require the three stacks to have the same size.

One of the results of Stein [S] is that the (k,k,k) semicross is a parallelohedron if and only if k = 1. Our results imply that every (u,v,1) semicross is a parallelohedron. To formulate our results precisely, it is convenient to introduce some additional concepts.

A tiling by a semicross is a *lattice tiling* provided any two tiles are related by a translation vector, where all such vectors form a lattice (that is, an additive group). It is called an *integer tiling* if any two tiles are related by translation vectors with integer components. *Integer lattice tilings* are both lattice and integer tilings.

Theorem 1. *Every* (u,v,1) *semicross admits an integer lattice tiling.*

Theorem 2. *The* (1,1,1) *semicross* P *admits two distinct integer lattice tilings, as well as a continuum of tilings that are*

- *lattice tilings but are not integer tilings;*
- *integer tilings but are not lattice tilings;*
- *neither lattice tilings nor integer tilings.*

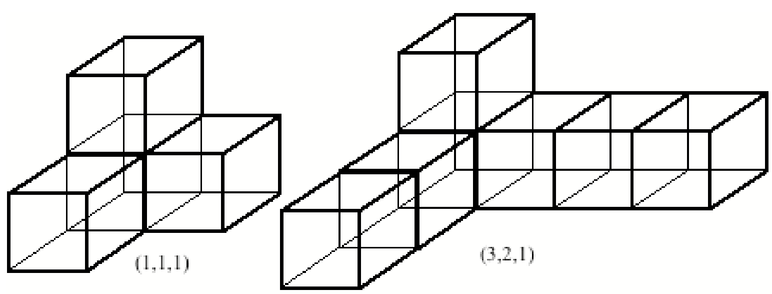

Figure 2. Examples of (1,1,1) and (3,2,1) semicrosses.

In the literature, questions about semicross tilings have been recast into algebraic garb, and treated accordingly. This may have been caused by the wish to obtain results in all dimensions, but possibly also by the difficulty to present the 3-dimensional tilings in an intelligible way. The semicrosses we are concerned allow a quite simple geometric approach, which we shall now explain and use in the proof of the theorems.

We assume that the semicrosses are composed by unit cubes, centered at the origin of the x-y-z coordinate system. A schematic representation of such semicrosses is shown in Figure 3. In this representation, we are imagining that we are looking down on a semicross from far on the z-axis. The two horizontal "arms" of the semicross are indicated by shaded squares, the vertical arm by the black square. Since we are interested in (u,v,1) semicrosses only, the black square indicates a single cube above the cubes of the two other arms. Then a section of an integer tiling, parallel to the x-y plane, may appear like the example in Figure 4, where the white squares indicate cubes that are the vertical arms of the semicrosses in the layer below.

The same Figure 4 may be interpreted as a visual proof of Theorem 1. Indeed, the set of white squares is translation equivalent to the set of black squares, and this means that the layer underneath the one shown arises from it by an integer translation. This translation can be repeated indefinitely downwards, and has an inverse that can be indefinitely extended upwards — hence leads to an integer lattice tiling. An analogous argument proved Theorem 1 in general. ♦

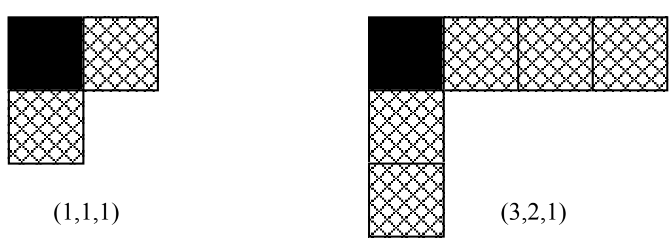

Figure 3. The schematic representation of the semicrosses in Figure 2.

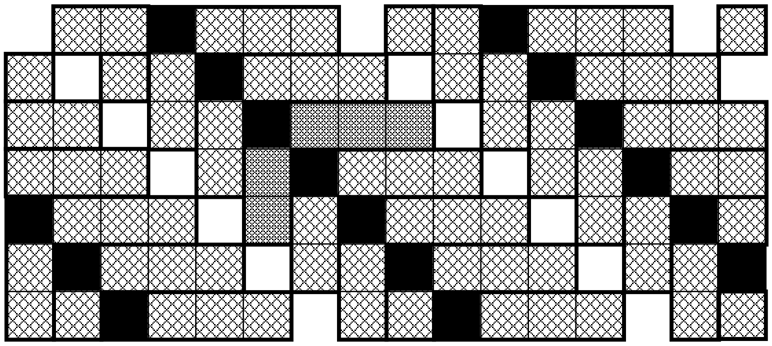

Figure 4. A schematic representation of an integer lattice tiling by (3,2,1) semicrosses, one of which is highlighted by distinct shading.

For the proof of Theorem 2 we first note that the two tilings represented in Figure 5 are the two distinct tilings mentioned in Theorem 2. The fact that they are different tilings can be seen in various ways; probably simplest is to observe that with respect to every coordinate plane the semicrosses in the "square tiling" have end-squares shared with central cubes, but this does not hold for the "diagonal tiling". We may mention that the tiling by (1,1,1) semicrosses shown in Figure 6 that appears to be different from the ones in Figure 5 is actually the same as the diagonal tiling.

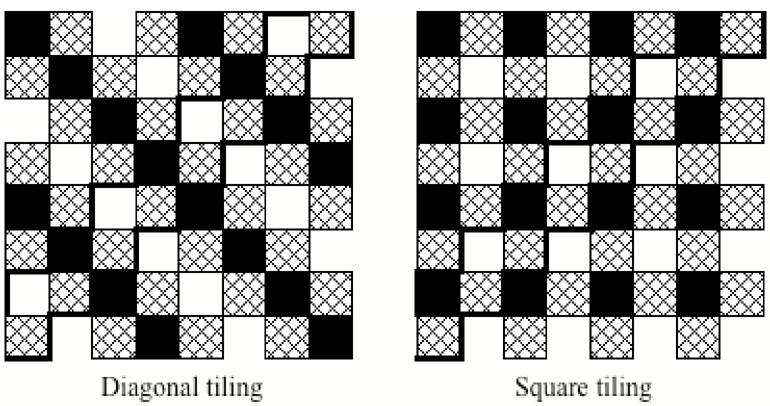

Figure 5. A schematic representation of the two integer lattice tilings by the (1,1,1) semicrosses, mentioned in Theorem 2. The heavily drawn zigzags bound the vertical slabs used in the proof of the other parts of Theorem 2.

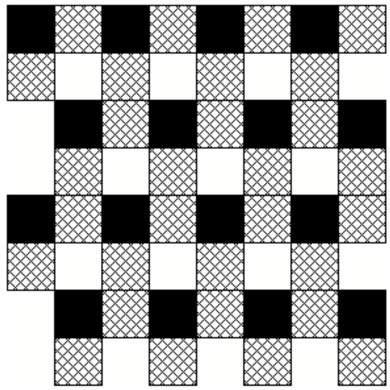

Figure 6. A schematic representation of an integer lattice tiling by the (1,1,1) semicrosses, which is in fact the same as the diagonal tiling in Figure 5.

One interesting property of both tilings in Figure 5 is that there are essentially 2-dimensional collections of tiles that can be considered as "supertiles" from which the tilings arise by translations. These supertiles are the ones indicated by the zigzags in Figure 5; two layers of the (1,1,1) semicrosses that make these supertiles by repetition in the vertical direction are shown in Figure 7. The most remarkable property of these zigzag vertical walls is the fact that

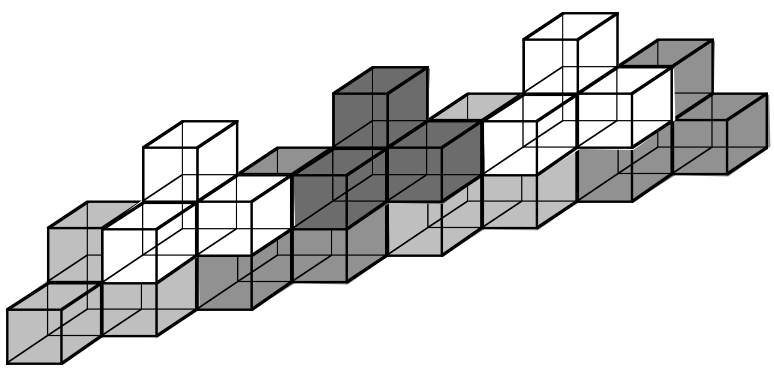

Figure 7. Two layers of the "vertical wall" indicated by the zigzag lines in Figure 5. The two layers can be translationally repeated to create an infinite wall translated copies of which can be vertically displaced by any distance.

adjacent ones can slide up and down with respect to each other. In fact, the diagonal tiling can be obtained from the square tiling by sliding every other wall one unit up (or down). But such slides through any collection of distances are also possible, and suitable choices yield all the various tilings specified in the second part of Theorem 2. ♦

Comments.

During recent investigations that led to the paper [G], I happened to notice the following sentence on page 532 of [S]:

> We should point out that there are tilings of R^3 by (1,3) semicrosses which are not a lattice and that there is a tiling in which the centers form a lattice L, such that $R^3/L = Z_2 \times Z_2$.

As I could not imagine such tilings, and could not find any details about them, my curiosity led me to experimental investigations using my grandsons set of "Duplo"s (a variety of Leggo's). Forming units that are a fair approximation of (1,1,1) semicrosses, it was easy to see that there are (at least two) integer lattice tilings. The most surprising discovery — facilitated by the fact that Duplo blocks sitting above each other stick together — was the existence of the zigzag walls. The obvious freedom of motion provided by these supertiles then led to the other parts of Theorem 2, and to the relationship between the two integer lattice tilings.

The following conjecture was made in [G]:

Conjecture. *Let P be a sphere-like polyhedron, with no pairs of coplanar faces. If the boundary of P can be partitioned into pairs of non-overlapping "patches"* $\{S_1, T_1\}$; $\{S_2, T_2\}$; ... ; $\{S_r, T_r\}$, *each patch a union of contiguous faces, such that the members in each pair* $\{S_i, T_i\}$ *are translates of each other, and the complex of "patches" is topologically equivalent as a cell complex to one of the parallelohedra in Figure 1, then P is a parallelohedron. Conversely, if no such partition is possible then P is not a parallelohedron.*

Only a small effort is needed to verify that the semicrosses (u,v,1) do not admit any such partitions even though they are parallelohedra. The reason that this does not disprove the conjecture is that no interpretation of the condition in the conjecture is applicable to semicrosses: If the squares are considered as faces — there are coplanar ones; if coplanar triplets of squares are considered as face — there are no translates.

We may generalize crosses in a way similar to what we did for semicrosses. A (u,v,w) *cross* consists of a central cube on opposite sides of which are attached stacks of u, v, w cubes, respectively. It is known that (k,k,k) crosses tile 3-dimensional space if an only if k = 1 or 2 (see, for example, [SS]). By a slight modification of the approach used in the proof of Theorems 1 and 2, we can establish:

Theorem 3. *The* (k,1,1) *cross admits integer lattice tilings for every* $k \geq 1$*, and so do the* (2,2,1) *and* (3,2,1) *crosses.*

As before, it is easy to provide graphic proofs. The difference is that now the white squares are of two kinds: solid dots indicate cubes that are bottom cubes of the crosses in the next higher layer, while hollow dots indicate top cubes of the next lower level. This is illustrated in Figure 8, by the (2,1,1) cross and the {3,2,1} cross. The other crosses mentioned in Theorem 3 admit analogous tilings.

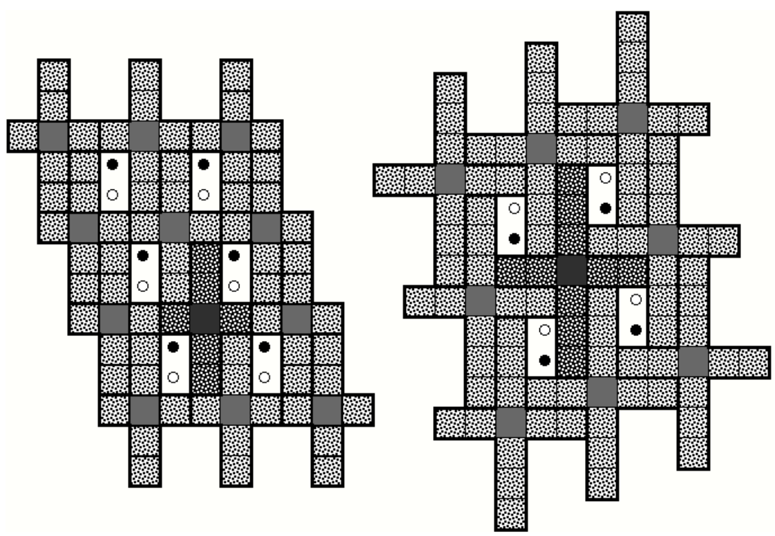

Figure 8. A visual proof of the possibility of tilings by the (2,1,1) and the (3,2,1) crosses.

It is interesting that there seems to be no mention of parallelohedra in any of the writings on crosses and semicrosses.

At present it is not known how to characterize the semicrosses and crosses (in the sense considered here) that are parallelohedra – – with or without assumptions of lattice or integer restrictions. As is evident from the results of this note, there are possibilities that go beyond the traditionally considered crosses and semicrosses.

References.

[F] E. S. Fedorov, Nachala Ucheniya o Figurah. [In Russian] (=Elements of the theory of figures) Notices Imper. Petersburg Mineralog. Soc., 2nd ser., 24(1885), 1 – 279. Republished by the Acad. Sci. USSR, Moscow 1953.

[G] B. Grünbaum, The Bilinski dodecahedron, and assorted parallelohedra, zonohedra, monohedra, isozonohedra and otherhedra. (Submitted for publication)

[S] S. K. Stein, Factoring by subsets. Pacific. J. Math. 22(1967), 523 – 541.

[SS] S. K. Stein and S. Szabó, *Algebra and Tiling: Homomorphisms in the Service of Geometry*. Math. Assoc. of America 1994.

[Z] G. M. Ziegler, Lectures on Polytopes. Springer, New York 1995.