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Topological configurations (n_4) exist for all $n \geq 17$

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ABSTRACT

We show that topological (n_4) point–line configurations exist for all $n \geq 17$. It has been proved earlier that they do not exist for $n \leq 16$.

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1. Introduction

Point–line configurations have played an essential part in combinatorial geometry in the past (see [1,2]). However, many problems are still open and they can be used to test various geometric methods.

An (abstract) (n_k) configuration is a set of n (abstract) points and n (abstract) lines when each (abstract) point is incident with k of the (abstract) lines and vice versa each (abstract) line is incident with k of the (abstract) points. Of course, two lines may only intersect in one point and two points lie at most on one common line. We always assume that the configurations are *connected*, that is the configuration is connected as a hypergraph.

We distinguish three types of configurations, listed in increasing generality:

Geometric: Points and lines are ordinary points and lines in the real projective plane.
Topological: Points are ordinary points in the real projective plane, however the lines are pseudolines.
Combinatorial: Just an abstract set system as described above.

The concept of pseudolines was first used by Levi [3], for an introduction we refer the reader to [4,5]. We note in particular that each pseudoline is isotopic to a straight line, and that every two pseudolines intersect transversely.

Since each line is a pseudoline, each geometric configuration is a topological configuration and each topological configuration is in turn a combinatorial configuration. In all these cases the reverse statements are false.

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For instance, the Fano configuration (7_3) is combinatorial, but not topological and there is a topological configuration (10_3) that is not geometric.

In the earliest writings on configurations there was some confusion between combinatorial configurations and geometric configurations; it seems that it was expected that these concepts coincide. There was also a lack of clarity whether the geometric configurations are considered in the real plane or in the complex plane. Over time, these problems have been clarified. Levi [3] introduced the topological view in 1926.

The topological model we chose has a distinct advantage; it admits a concise combinatorial description using the theory of oriented matroids [4,5]. Additionally, important tools for the study of geometric configurations like Euler's formula also hold in this more general setting. So, effectively topological configurations can also be treated as purely combinatorial objects.

For given k it is a natural question to ask for which values of n do combinatorial, topological, and geometric (n_k) configurations exist. The question is trivial for $k = 2$ as an (n_2) configuration can be constructed by any n points ($n \geq 2$) in general position. A complete answer is also known in the case $k = 3$. In this case, combinatorial configurations exist for all $n \geq 7$. The combinatorial configurations for $n = 7, 8$ cannot be made into topological configurations and for $n \geq 9$ geometric configurations always exist.

In the past the case $k = 4$ has received less attention; most authors have treated exclusively the case of combinatorial or geometric configurations. The first systematic results in the case $k = 4$ are by Brunel [6] and later independently by Merlin [7]. Brunel showed that for $n \leq 12$ no combinatorial (n_4) configuration could exist. It is easy to see that for all $n \geq 13$ there exists at least one combinatorial configuration. For $13 \leq n \leq 16$ there exist only combinatorial configurations [7,8]. The second author has shown that geometric configurations exist for all $n \geq 21$ with the possible exception of the values 22, 23, 26, 29, 31, 32, 34, 37, 38, and 43 [9]. For the topological case the first and last author have solved the cases $n = 15$ and $n = 16$ [8].

The main result of the present paper is the complete solution of the corresponding problem for topological configurations (n_4):

Theorem 1. *Topological configurations (n_4) exist if and only if $n \geq 17$.*

The available information about geometric (n_4) configurations has led to the following conjecture, which modifies the conjecture formulated in [2] and other publications:

Conjecture 2. *Geometric configurations (n_4) exist for all $n \geq 20$.*

2. The cases up to $n = 20$

It is known that combinatorial (n_4) configurations exist if and only if $n \geq 13$. In the cases (13_4) and (14_4) Merlin [7] showed in 1913 that no geometric configuration exist. He also claimed that no geometric (15_4) configuration existed. His unpublished proof relied on checking this for all possible combinatorial (15_4) configurations. However, he only knew of three combinatorial types of these configurations, whereas there are four distinct types as shown by Betten and Betten [10]. Recently, it was shown [8] that no topological, and hence no geometric, (15_4) and (16_4) configurations exist.

However, for all values of n, such that $17 \leq n \leq 20$, the question remained open. The classification of combinatorial configurations by Betten and Betten [10] is complete up to $n = 18$. However, some progress has been made since then. There exist 1 972 combinatorially distinct (17_4) configurations and 971171(18_4) configurations (this is sequence A023994 in [11]). We independently verified these results using tools from Brendan McKays program nauty [12].

For most of our pictures (with the exception of Figs. 7 and 9) we use a special model of the projective plane. One common way of constructing the real projective plane is to take the 2-sphere in $\mathbb{R}^3$ and identify antipodal points. In our model we cut the sphere in half and only show the northern hemisphere in projection to the equatorial plane. Antipodal points on the equator have to be identified in our model. The points of the configurations are marked with solid dots.

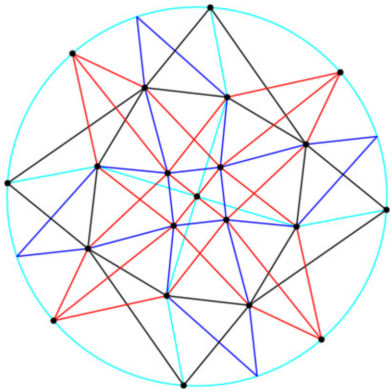


Fig. 1. The topological (17_4) configuration $\mathscr{P}$.

Table 1
(17_4) configuration $\mathscr{P}$.

1	2	3	4	1	5	6	7	1	8	9	10				
1	11	12	13	2	5	8	14	2	6	9	15				
2	7	11	16	3	5	12	15	3	6	8	16	3	7	9	17
4	5	11	17	4	6	10	13	4	7	12	14				
8	13	15	17	9	13	14	16	10	11	14	15	10	12	16	17

The case $n = 17$ is special in that there is only one combinatorial configuration that allows a realization as topological configuration. To prove this, we first have to generate all possible combinatorial configurations and then check whether they admit a topological representation.

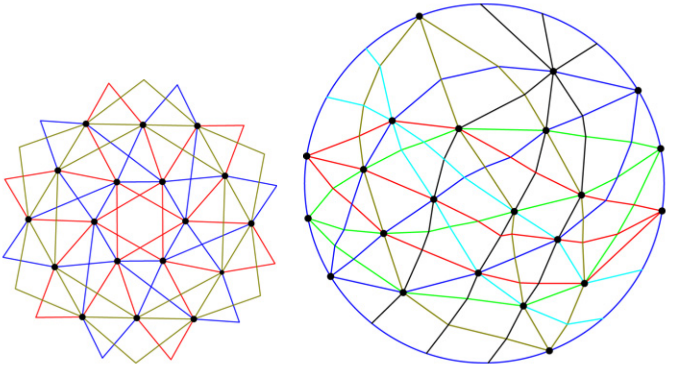
Proposition 3. *Exactly one combinatorial* (17_4) *configuration can be realized with pseudolines.*

As already mentioned, the number of combinatorial configurations (17_4) is known [10] to be 1972. We generated independently a list of these configurations using a different method. We found all possible Levi graphs, in an approach also used in [13]. For the generation of these graphs we used the program of Meringer [14].

For the second step we make the following observation: Assume we have a topological realization of a configuration $\mathcal{C}$. Then we may perturb the pseudolines so that only vertices in $\mathcal{C}$ have degree larger than 2. Hence, to check whether $\mathcal{C}$ admits a topological realization we only need to check whether the matroid corresponding to $\mathcal{C}$ is orientable. A fast method for checking orientability was developed by the third author [15]. We transform the problem of checking orientability into a satisfiability problem (SAT). The resulting satisfiability problems can then be tackled with standard SAT-solvers like ZChaff [16] or MiniSat [17]. The result was that only one (17_4) configuration (from now on called configuration $\mathscr{P}$) admits an oriented matroid. The lines of this configuration can be found in Table 1. We have depicted a topological realization of this configuration in Fig. 1.

For the other cases ($18 \le n \le 20$) the number of combinatorial configurations is by far too large, so that we cannot hope to check topological realizability for all of them. However, we managed to test many different combinatorial configurations and provide realizations with pseudolines for all these values of n. In the case $n = 18$ it was also possible to construct an example by hand (see Fig. 2(a)). This example has the additional property of admitting a rotational symmetry of order 6. The example we found with the help of a computer (see Fig. 2(b)) is not symmetric. In the case $n = 19$, we were also able to generate two distinct examples (Fig. 3).

The case $n = 20$ was especially difficult. To construct the example given in Fig. 4 we reduced the search space by considering only those (20_4) configurations that could be decomposed in a certain way in two Desargues configurations. In Fig. 4 we have hinted at the decomposition by coloring certain triangles.



(a) With rotational symmetry. (b) Without rotational symmetry.

Fig. 2. Two topological (18_4) configurations.

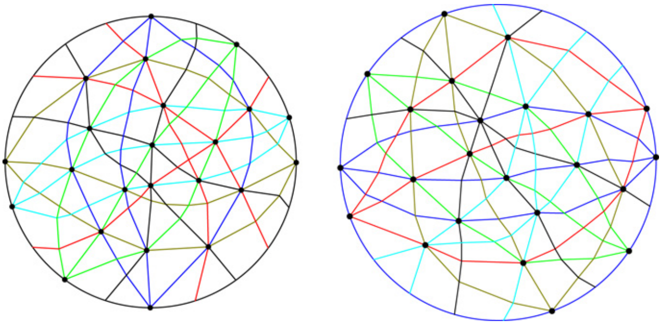


Fig. 3. Two topological (19_4) configurations.

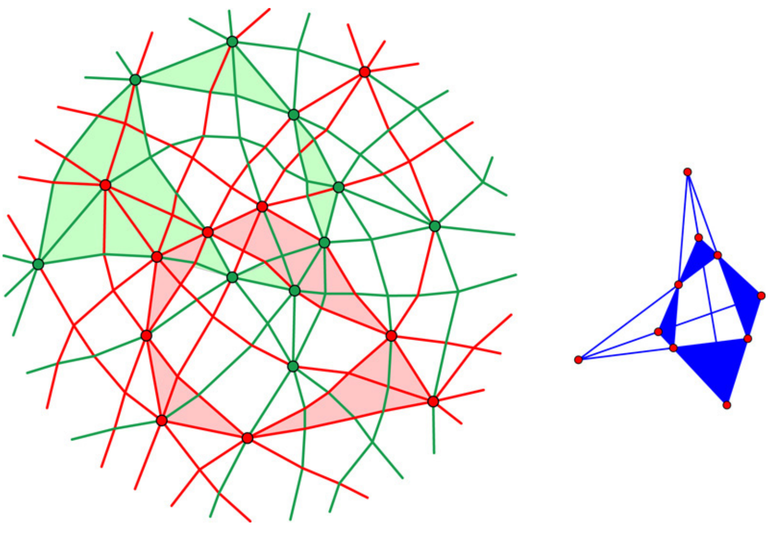


Fig. 4. A topological configuration (20_4).

3. The cases $22 \le n \le 38$, n even

The cases $22 \le n \le 38$ with n even (see Fig. 5) can be obtained with the following construction: We take the points of two regular k-gons and connect them in such a way that we always bridge the same number of points in the outer k-gon; in this way we obtain topological $(2k_4)$ configuration with a cyclic symmetry.

For the values $n \in \{24, 28, 30, 36\}$ geometric configurations were already known (see [9]).

4. Extensions of (n_4) configurations

In the following we consider extensions of (n_4) configurations on the pseudoline level. Consider a regular k-gon, $k \ge 3$, and a second one with the same center, however rotated around the center by $\frac{180}{k}$ degrees. We consider the $2k$ vertices and the $2k$ intersection points of the sides of the two k-gons. Each of the $2k$ lines that contain the sides of the k-gons is incident with precisely four of these points. Each such point is incident with two lines. We aim to add $2k$ pseudolines with this incidence structure to an existing (n_4) pseudoline configuration, in order to obtain an $((n + 2k)_4)$ configuration.

To reach this goal, we split in an (n_4) pseudoline configuration $2k$ of its points locally into pairs of points in which only two pseudolines (out of the former four pseudolines that have determined the original point) intersect. When these $2k$ points can be used as our points of the above $2k$ pseudolines and when in addition the extension by these $2k$ pseudolines fits to form a new pseudoline arrangement, we have via our construction an $(n + 2k)_4$ pseudoline configuration. Each point is again incident with four pseudolines and each pseudoline is incident with four points.

We show that this idea can indeed lead to new topological configurations. We have constructed a (23_4) pseudoline configuration in this way from our (17_4) example by adding 6 elements, see Fig. 6(a). The colored patches in Fig. 6(a) indicate where the local perturbation has taken place. In a similar way a (31_4) pseudoline configuration was obtained from a new (25_4) geometric configuration by adding 6 elements, see Fig. 4. The case (29_4) in Fig. 6(b) was found directly by hand under a certain symmetry assumption. This example than has led to the case (37_4) by adding 8 elements. The symmetry has been kept, see Fig. 8.

The final case (43_4) is a little bit more involved. We first insert a single pseudoline in the (37_4) pseudoline arrangement as depicted in Fig. 8. This pseudoline should represent a small strip of six uniform pseudolines with 12 points that have the same incidence properties between its elements and the 12 intersection points as described above for a pair of regular triangles.

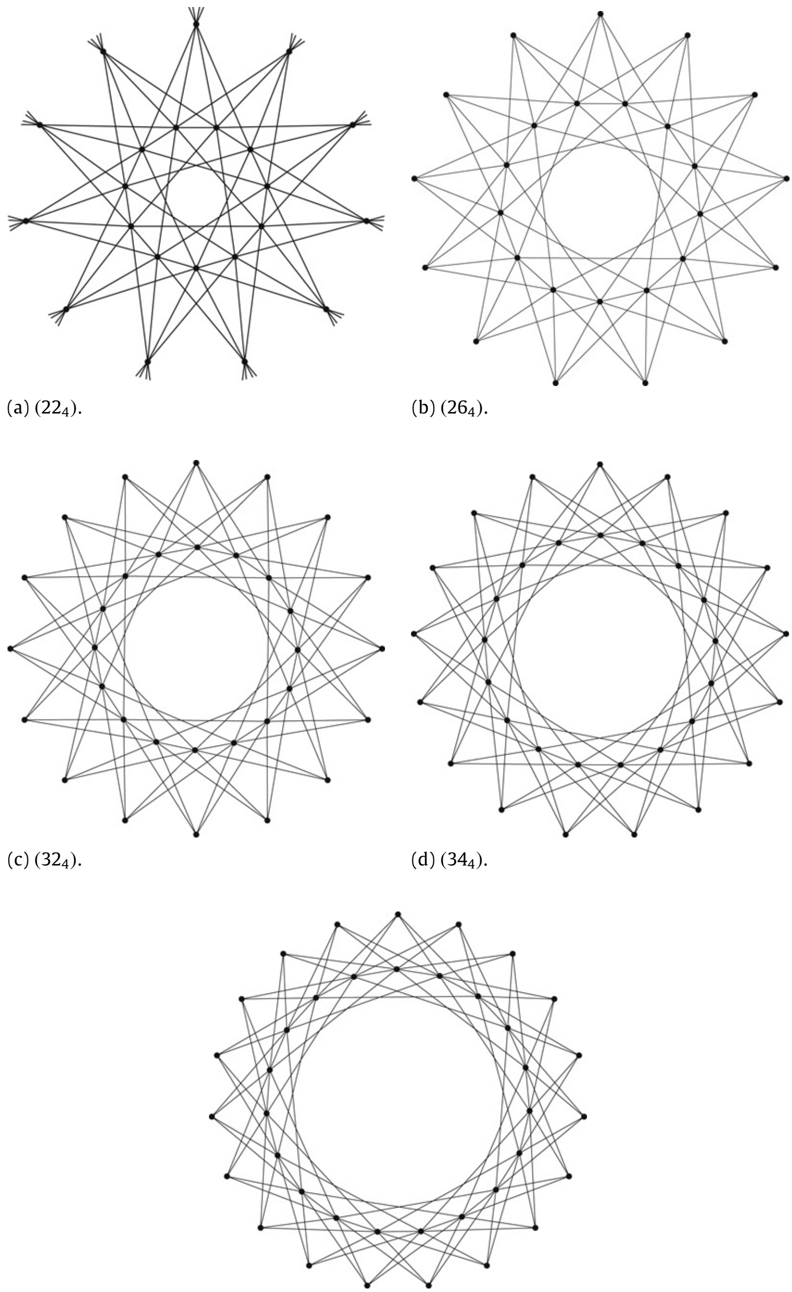
We fix a natural order of these 12 intersection points along the strip of 6 pseudolines and we partition these 12 points into 6 consecutive pairs. We can form the strip so that for each point of the original (37_4) pseudoline arrangement that is incident with the defining pseudoline of the strip, we have a pair of points of the strip in the vicinity of that point. Now it is clear that we can split the 6 points on the strip defining the pseudoline into pairs as in the previous cases and such that these pairs are incident with the pairs of the strip. We arrive at a (43_4) pseudoline configuration.

Another way of drawing a topological (43_4) configuration is shown in Fig. 9.

5. Conclusion

We have seen that topological (n_4) configurations exist if and only if $n \ge 17$. Still the question remains, is there a similar result for geometric (n_4) configurations? However, not only the (n_4) configurations are of interest, also in the cases $k \neq 4$ questions are open.

An important open question for (n_3) configurations is whether all configurations that have a geometric realization in the real projective plane have geometric realizations in the rational projective plane. In the case of (n_4) configurations 'irrational' examples are known. However, the constructions that yield such configurations need quite a number of elements. It is not known, how many elements are needed to get such a 'smallest' irrational (n_4) configuration; the smallest n known is 44. Even given a configuration it is difficult to decide whether it is 'irrational'. The most interesting case is the (21_4) configuration from [18]. The known realization is irrational. However, it is unknown whether a rational realization exists.



(a) (22_4).

(b) (26_4).

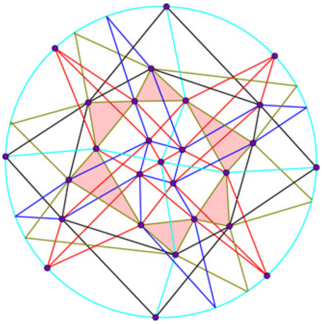
(c) (32_4).

(d) (34_4).

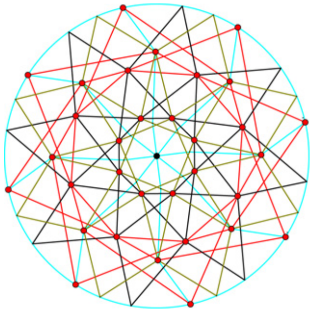
(e) (38_4).

Fig. 5. Topological configurations (n_4), $n \in \{22, 26, 32, 34, 38\}$.

Not much is known about (n_k) configurations for $k \geq 5$. For combinatorial configurations to exist it is necessary that $n \geq k^2 - k + 1$, but this is known not to be sufficient for certain k, such as $k = 7$. For topological configurations the first and third author [8] showed that $n \geq k^2 + k - 4$ has to hold;



(a) (23_4).



(b) (29_4).

Fig. 6. Topological configurations (23_4) and (29_4).

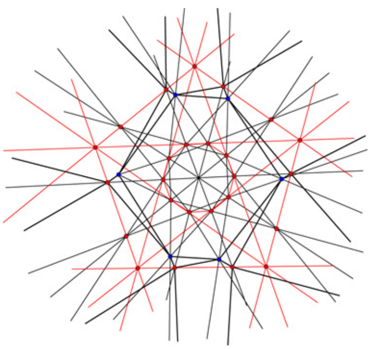


Fig. 7. A topological configuration (31_4).

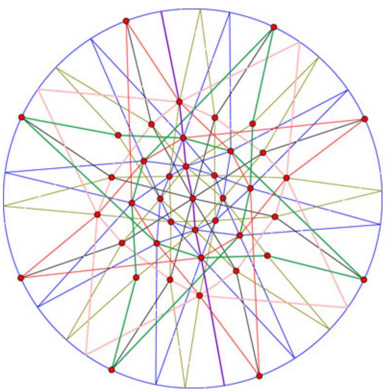


Fig. 8. A topological configuration (37_4).

this follows directly from Euler's formula. However, this bound is not sharp even in the case $k = 3$. For geometric configurations no analogous bounds are known.

Note added in proof: New results [19] about geometric configurations (n_4) have been obtained recently. These include the non-existence of geometric configurations (17_4), as well as examples of geometric configurations (18_4), (20_4), (29_4), (31_4), and (32_4), together with new methods of generating various classes of (n_4) geometric configurations.

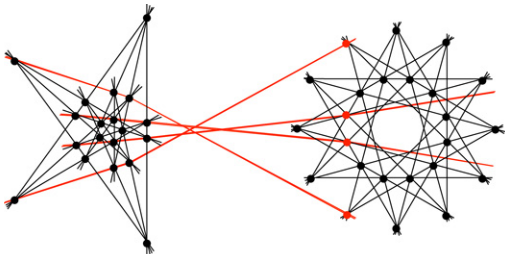


Fig. 9. A topological configuration (43_4).

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