Canad. Math. Bull. Vol. **52** (3), 2009 pp. 327–341

Geometric "Floral" Configurations

Dedicated to Ted Bisztriczky, on his sixtieth birthday.

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Abstract. With an increase in size, configurations of points and lines in the plane usually become complicated and hard to analyze. The "floral" configurations we are introducing here represent a new type that makes accessible and visually intelligible even configurations of considerable size. This is achieved by combining a large degree of symmetry with a hierarchical construction. Depending on the details of the interdependence of these aspects, there are several subtypes that are described and investigated.

1 Introduction and Definitions

A geometric (n_4) configuration is a collection of n points and n lines in the Euclidean or projective plane, so that each point lies on four lines and each line passes through four points. This paper introduces a new method of constructing (n_4) configurations and provides a method of constructing more highly-incident configurations.

The last decade has produced a number of new construction methods for (n_4) configurations. This development continues, and the primary aim of the present paper is to describe a family of related construction methods that produce five classes of (n_4) configurations, four of which have at least one degree of freedom (that is, they are *movable*). Throughout, we restrict attention to connected configurations (n_4).

Studying (n_4) configurations became accessible only when the idea of making them symmetric was introduced. That led to various kinds of classifications of (n_4) configurations, such as astral [2, 4, 5], polycyclic [3] and celestial [1, 3, 6] configurations. These configurations, and many others, are discussed in considerably more detail in the forthcoming book on configurations by Branko Grünbaum [7]. Without symmetry, (n_4) configurations would be where they were a century ago, precisely nowhere. Now the "floral" configurations introduced in this paper form another class of highly symmetric configurations, and we propose studying their properties.

2 Floral Configurations

In October 2006, Jürgen Bokowski discovered the first example of a new type of (n_4) configuration, dubbed a *floral* configuration. It is shown in Figure 1. The construction of such floral configurations has since been generalized in several directions. Here we give a brief description of five different varieties of floral configurations and

Received by the editors December 18, 2007; revised May 5, 2008.
AMS subject classification: Primary: 52C30; secondary: 52C99.

describe a few ways to construct generalized floral configurations. A more thorough exposition of floral configurations and their properties will be provided in a separate paper.

A *floral* configuration is a connected (n_4) configuration with dihedral symmetry group d_k, of order $2k$, provided that the points of the configuration are organized into disjoint *florets*. The florets are mapped onto each other by the k mirrors of the group, and each floret consists of $n = hk$ points that form, by themselves, a single transitivity class under a dihedral symmetry group not identical with the symmetry group of the configuration. However, the mirrors of the symmetry group of the configuration must be parallel to the mirrors of the symmetry groups of the florets. This definition is illustrated by the figures that follow.

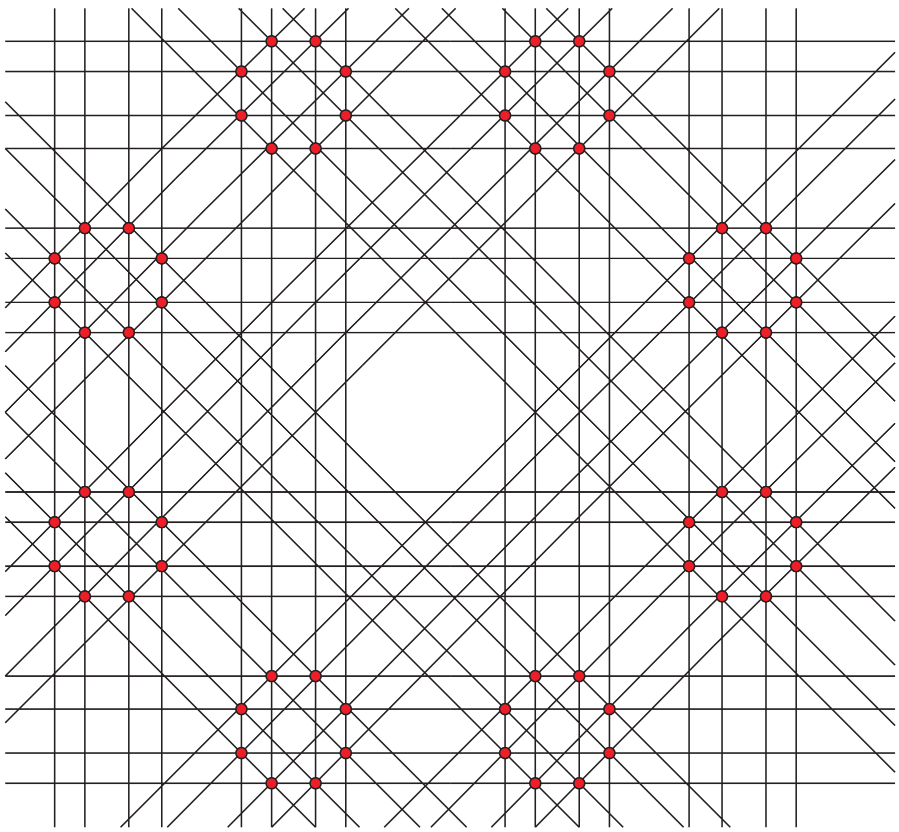

Figure 1: A (64_4) floral configuration. The configuration, each of the florets, and the incidence structure formed by the floret and its incident lines all have symmetry group d_8.

We provide here details and illustrations of five varieties of floral configurations. In all five cases, the configuration has symmetry group d_k, but the number of florets and the symmetry group of the floret depend on the variety under consideration. In two of the varieties, the florets consist of isogonal but non-regular polygons, while in the other three varieties, they consist of regular polygons. The group of symmetries of the floret considered as a set of points may differ from the group of symmetries of the floret considered as an induced irregular configuration (alias incidence structure) that is composed of the point set and all lines incident with it; in this work the word floret is reserved for the former definition. Moreover, the symmetry group of the floret is often quite different from the symmetry group of the incidence structure formed by the floret and the lines of the configuration incident with the floret. For example, in the configuration shown in Figure 1, both the floret and the lines incident with the floret have the same symmetry group, while in Figure 6, the symmetry group of the floret is d_7, and there is no line of mirror symmetry of the lines incident with the floret.

In each of the first four varieties, for a given k the configuration depends *continuously* on either one or two parameters, while the fifth variety has no degrees of freedom; in addition, in the varieties whose florets are isogonal but non-regular polygons, there is an additional degree of freedom caused by varying the spacing of the points in the florets. We only consider continuous motions of the floral configurations that preserve the floral structure of configurations (except at some discrete points of the phase space, where we may get degeneracies with point or line overlaps).

All floral configurations may be interpreted as polycyclic configurations [3]. We do not give explicit definitions of the lines that appear in the configuration, since these are easily determined (usually in several ways) in every case—but the precise specification in general would lead beyond the limits of the present paper. It should be noted that for certain choices of a parameter there may be unintended coincidences or other singularities that make the resulting incidence structure not satisfy the definition of a configuration. Such coincidences are very hard to specify in general, and we are satisfied with the warning just given.

Variety (A) consists of configurations in which each floret is an isogonal but non-regular polygon, and no floret is invariant under a symmetry of the configuration. Figure 2 shows a (64_4) configuration of variety (A). Note that although this configuration is isomorphic to the configuration shown in Figure 1, that configuration is not of variety (A), since in Figure 1 each floret is regular and is invariant under a symmetry of the configuration.

Variety (B) consists of configurations in which each floret is an isogonal but non-regular polygon, and each floret is invariant under some symmetry of the configuration. Figure 4 shows a (72_4) configuration with symmetry group d_6; each floret is mapped to itself using one of the mirrors of the configuration.

Variety (C) consists of configurations in which each floret is a regular polygon, and no floret is invariant under a symmetry of the configuration. Figure 5 shows two examples, with symmetry groups d_5 and d_6, respectively. In these examples the number of points in a floret is equal to k.

Figure 6 shows a (98_4) configuration with symmetry group d_7. Unlike previous examples, the lines incident with a single floret, along with the floret itself, do not form a symmetric incidence structure. Thus, the construction of configurations of

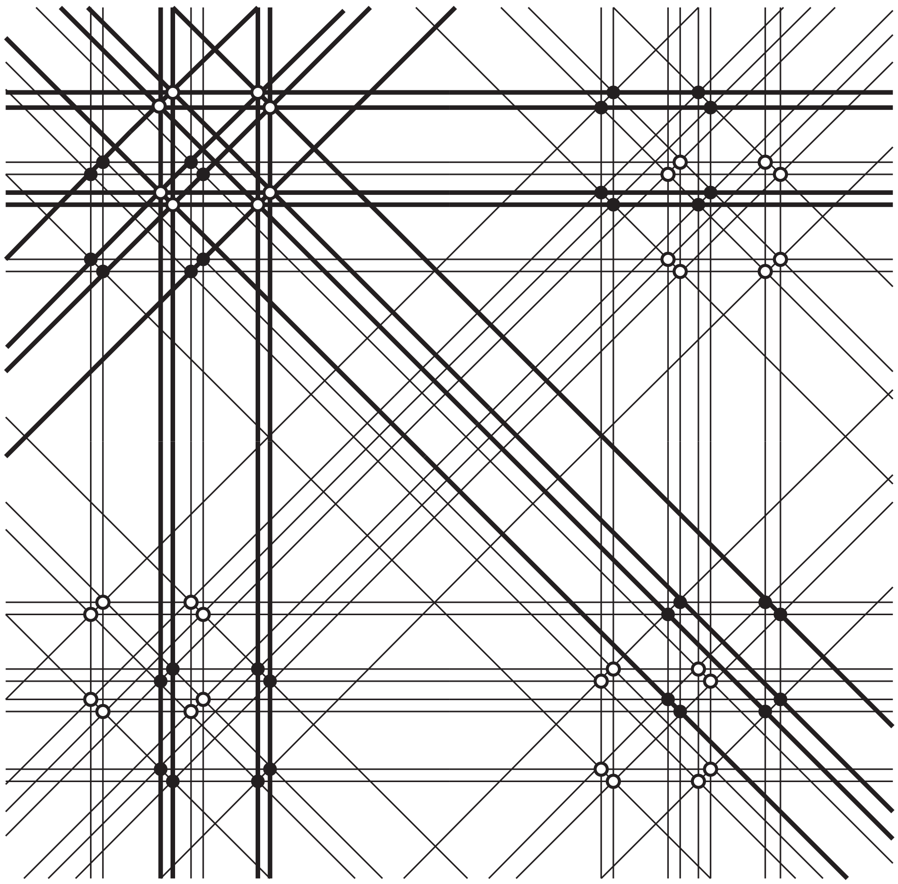

Figure 2: A (64_4) configuration of variety (A) isomorphic with the configuration in Figure 1. For clarity, one half of the florets is shown in white, the other half black. The lines of the configuration that are incident with the points of one of the white florets are shown with thick black lines. Here, $k = 4$.

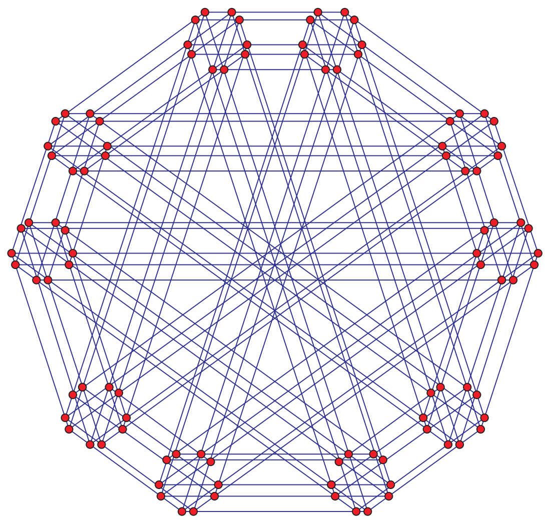

Figure 3: An example of case $k = 5$ of variety (A), forming a (100_4) configuration.

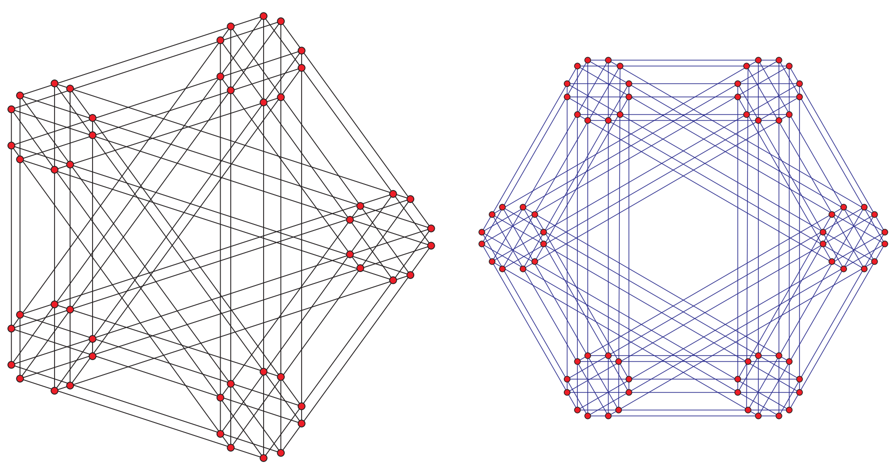

Figure 4: A (50_4) configuration with symmetry group d_5 and (72_4) configuration with symmetry group d_6, both of variety (B).

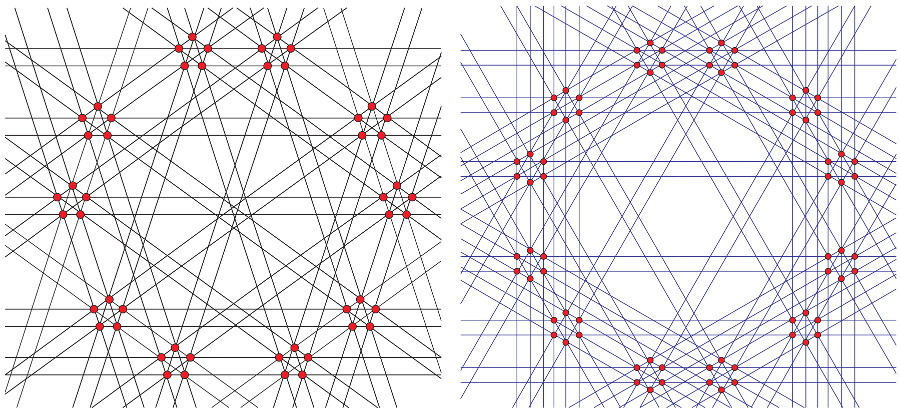

Figure 5: A (50_4) configuration and a (72_4) configuration of variety (C)

variety (C) proceeds well, even in the case that there is no mirror symmetry among lines incident with a floret.

Variety (D) consists of configurations in which each floret is a regular polygon, and each floret is invariant under some symmetry of the configuration. Figures 7 and 8 show examples of configurations of variety (D), for even and odd k. Note that the configuration shown in Figure 1 is also of variety (D), and varieties (A) through (D) all possess continuous degrees of freedom.

Variety (E) consists of configurations in which each floret is a regular polygon, and each floret is invariant under a symmetry of the configuration. In addition, the configuration is rigid (non-movable), there are lines incident with a single point in a floret, and it has lines that meet more than two florets.

Figure 9 shows a (25_4) configuration discovered in 2006 by Pisanski, that is a member of variety (E). It has no continuous placement of the florets. A similar configuration may be constructed from any odd $k \geq 5$.

We note that for some values of the continuous parameters, configurations of one variety may become configurations of another variety; for example, the configurations shown in Figures 1 and 2 are isomorphic and may be formed from each other by changing the position of the florets, but Figure 1 is a member of variety (D) and Figure 2 is a member of variety (A). The varieties themselves are disjoint.

One final note is that, depending on the placement of the florets, it can often be hard to discern that a given configuration is, indeed, a floral configuration. Figure 10 shows a floral configuration isomorphic to the configuration shown in the left-hand side of Figure 7; one floret is indicated by the shaded polygon.

3 Generalized Floral Configurations

There are a number of ways to generalize floral configurations, by relaxing in several ways the constraint that the points of the floret must be formed from some sort of

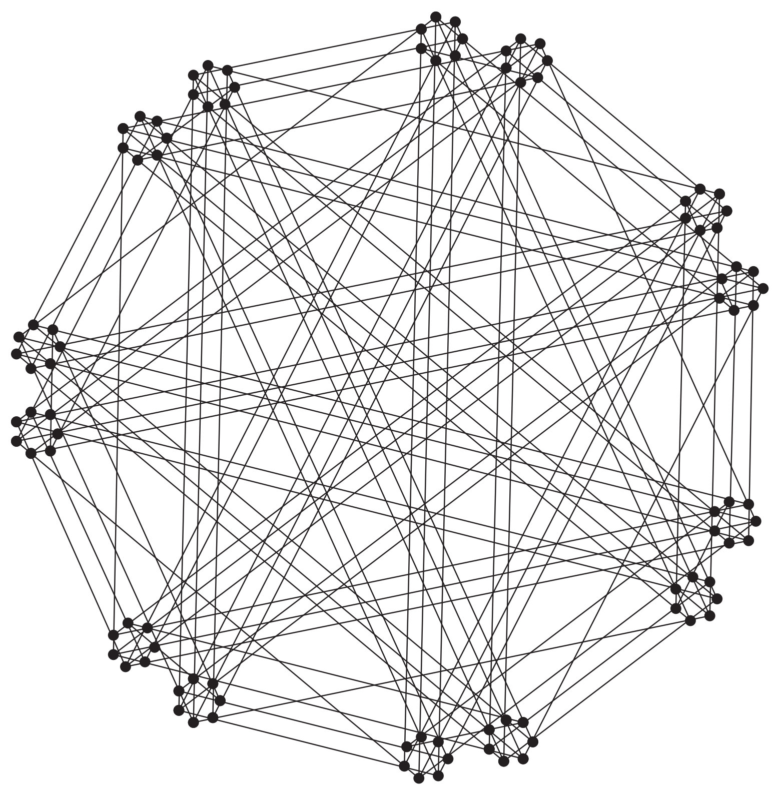

Figure 6: A (98_4) configuration of variety (C), where the lines incindent with a single floret do not exhibit symmetry.

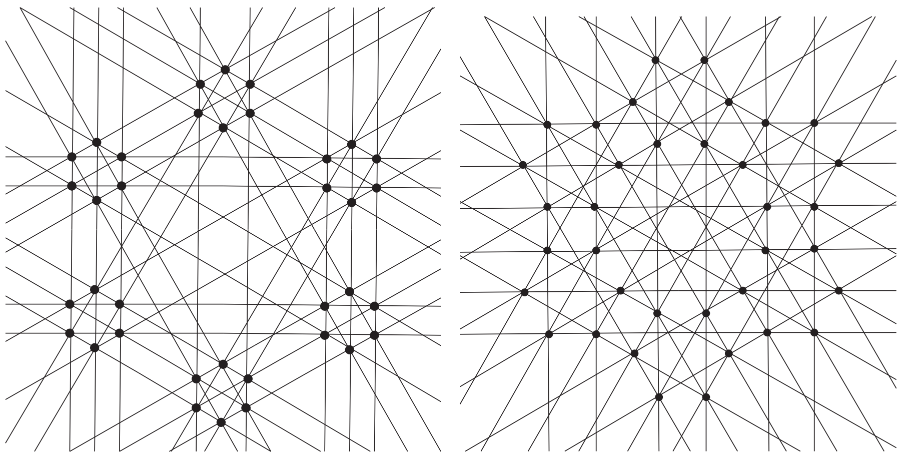

Figure 7: Two possibilities of variety (D), illustrated for the even case $k = 6$, yielding configurations (36_4).

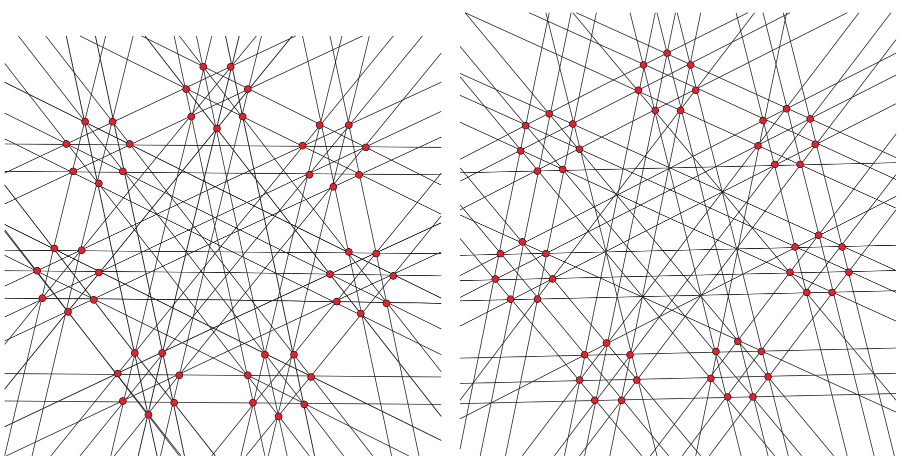

Figure 8: Two possibilities of variety (D), illustrated for the odd case $k = 7$, yielding configurations (49_4).

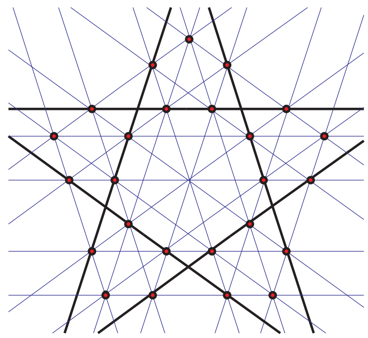

Figure 9: A variety (E) configuration (25_4) discovered in 2006 by Pisanski. Configurations in variety (E) have no continuous variability of the placement of florets. Except in the position shown, the thick black lines are not incident with collinear quadruplets of points.

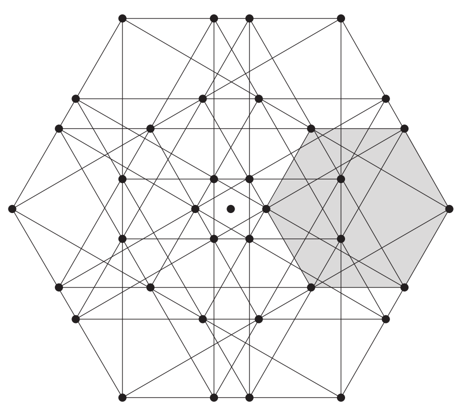

Figure 10: A configuration isomorphic to the one shown in the left-hand side of Figure 7; a single floret is indicated by the shaded polygon.

regular or isogonal polygon; in this case, we will refer to *generalized florets*. In many floral configurations (for example, all those shown as examples of varieties (A)–(D)), the incidence structure formed by a floret and its incident lines is a (p_4, n_2) configuration. It is possible to substitute other configurations or incidence structures; sometimes, an (n_4) configuration is the result, as in Figure 6, where the incidence structure is a four-valent graph, and sometimes a more highly incident configuration is formed.

Figure 11 shows an (n_4) configuration where the incidence structure formed by a generalized floret and all but one symmetry class of incident lines has two symmetry classes of three-valent points and one class of four-valent points; an additional set of lines is introduced so that each line in the class passes through two points in each of two florets. Unlike the floral configurations discussed in the previous section, the entire configuration has cyclic, rather than dihedral symmetry.

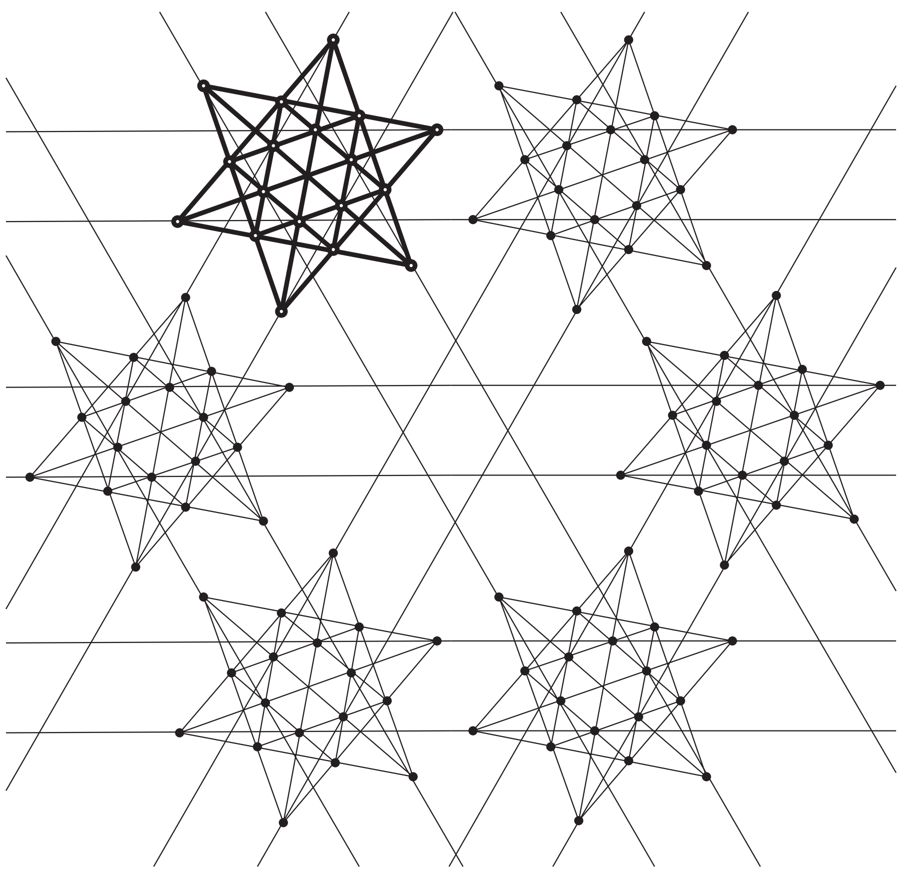

Figure 11: A cyclically symmetric generalized floral (108_4) configuration, with symmetry group c_6. One of the underlying incidence structures is shown with thick black lines.

Beginning with a dihedrally symmetric (n_4) configuration as the incidence structure associated with a generalized floret, it is possible to construct a (p_4, n_8) configuration. Figure 12 shows a $(294_4, 147_8)$ configuration with symmetry group d_7, formed analogously to variety (C) of the floral configurations, where the incidence structure formed by a floret and its incident lines is the celestial configuration $7\#(2,1;3,2;1,3)$ (see [1,6] for more information on celestial configurations, and [8] for information on the specific configuration).

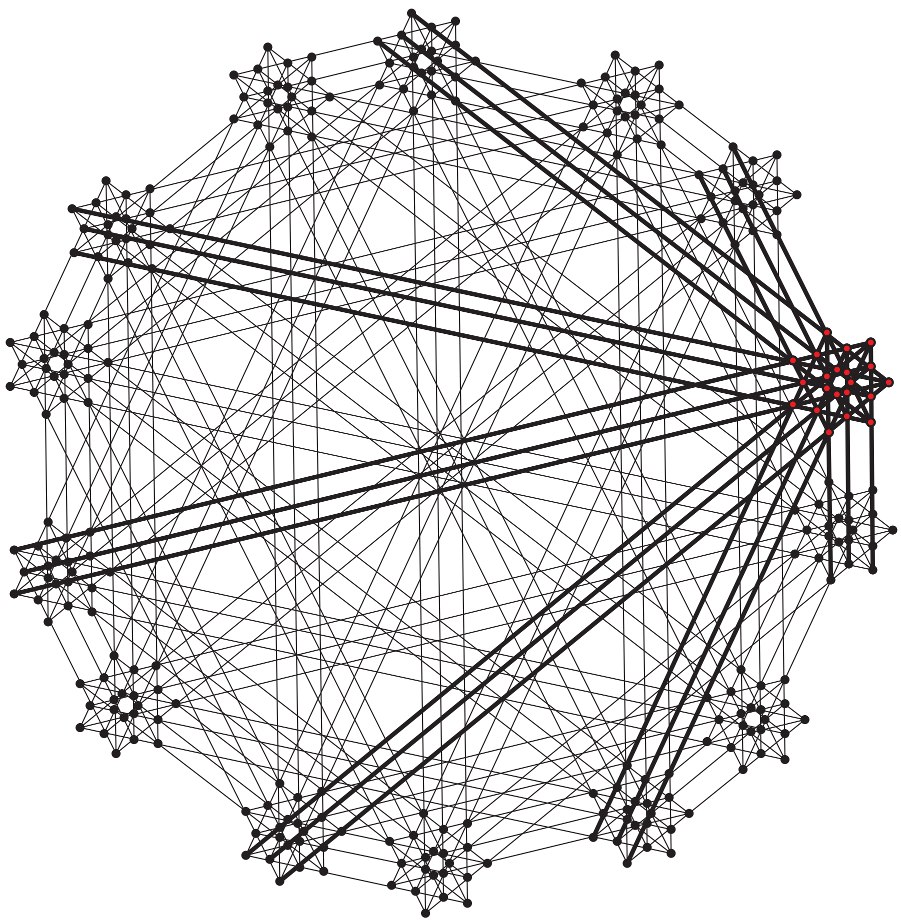

Figure 12: A variety (C) $(294_4, 147_8)$ generalized floral configuration, using a (21_4) celestial configuration as the incidence structure associated with the generalized floret. The lines incident with a single generalized floret (the celestial configuration) are shown with thick black lines.

Another way to generalize floral configurations begins with a floral configuration and then uses that entire configuration as the incidence structure associated with the generalized floret of a new configuration. Figure 13 shows such a configuration, using as its initial generalized floret a version of the (n_4) variety (A) floral configuration shown in Figure 2. Such a configuration is called a three-level floral configuration. This "nesting" process may be repeated as many times as desired. In a q-level configuration, a (p_4, n_{2^q}) configuration is formed. Unfortunately, despite Figure 13, only two-level floral configurations are very intelligible.

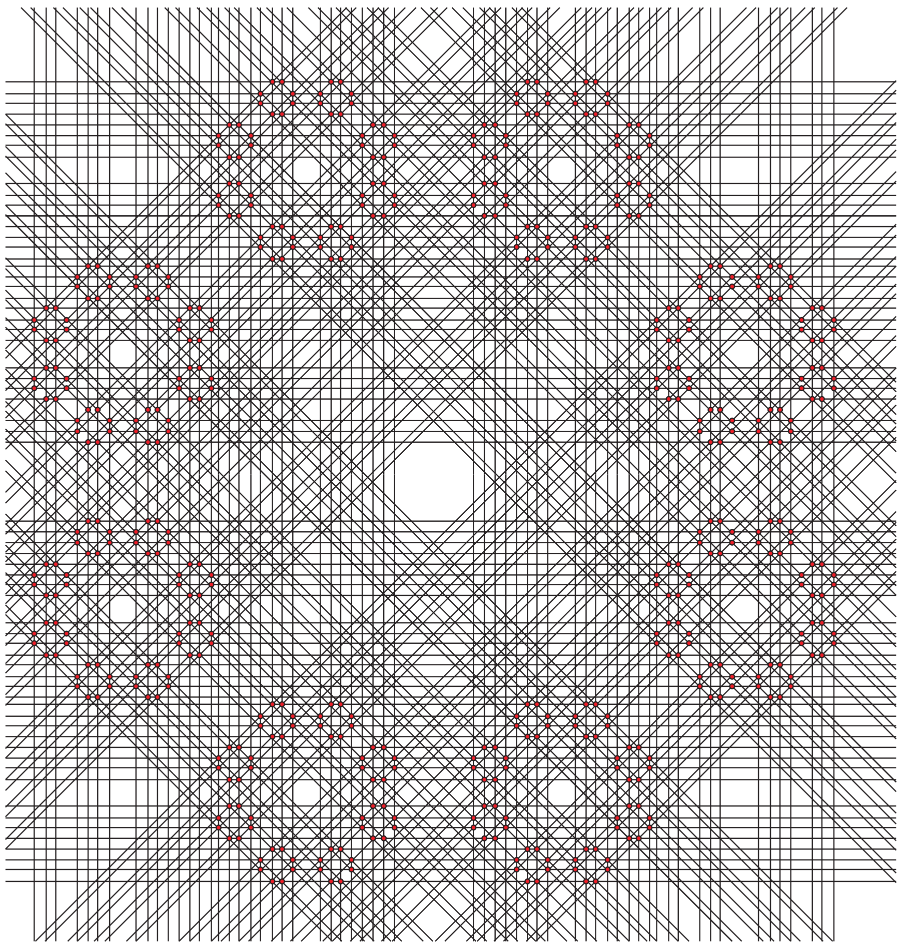

Figure 13: A $(512_4, 256_8)$ three-level floral configuration, formed using a version of the (64_4) variety (A) floral configuration shown in Figure 2.

Finally, beginning with an isogonal or regular polygon as the points of a floret but using more lines to connect the vertices of the polygon, and then constructing a configuration as in the construction of floral configurations, yields (p_q, n_4) configurations where $q > 4$. For example, if as a floret an isogonal 16-gon is used $(k = 8)$ with vertices labelled consecutively as $1, 2, \ldots, 16$, and lines are drawn connecting vertices i and $i + 1$, i and $i + 3$, i and $i + 5$ and i and $i + 7$ (so that the floret and its incident lines form a $(16_8, 64_2)$ configuration), and the construction method of variety (A) is used, then the result will be a $(256_8, 512_4)$ configuration. Constructing a 3-level configuration using the floret will yield a (4096_8) configuration. This is of interest, as very few examples of (n_k) configurations for $k > 4$ are known.

4 Conjectures and Open Questions

Floral configurations solve several interesting questions, and they also raise more questions.

4.1 Open Questions Answered by Floral and Generalized Floral Configurations

In [1], the following questions about "movable" configurations were posed.

Question Are there symmetric (n_4) configurations with a continuous parameter where the number of points in a single symmetry class is odd?

Question Does there exist a symmetric, movable (p_q, n_k) configuration, where one of q or k is more than 4? If so, are there infinite families of such configurations?

Floral configurations answer the first question affirmatively. For example, variety (D), for any odd k, provides an infinite family of isomorphic (n_4) configurations with a continuous parameter: the size of the configuration relative to the radius of the circle formed by the centers of the florets. Figure 14 shows two (49_4) configurations with the same floret and the same radius of circle connecting the centers of the florets, but different sized florets.

Generalized floral configurations answer the second question affirmatively. For example, beginning with a celestial configuration as a floret, (p_4, n_8) configurations may be produced. Beginning with a regular or isogonal polygon, using as a floret a $((2p)_{2t}, (2pt)_2)$ configuration and constructing a configuration analogously to variety (A) or (C), a $((4p^2)_{2t}, (2p^2)_4)$ configuration is produced. These configurations are movable, since, for instance, the offset between the generalized florets may be changed continuously.

4.2 Open Questions

Question 1 For every floral configuration with symmetry group d_k, the number of points in each floret is a multiple of k. More specifically, the following combinations are known to us:

- Variety (A): $2k$ florets, each with $2k$ points;

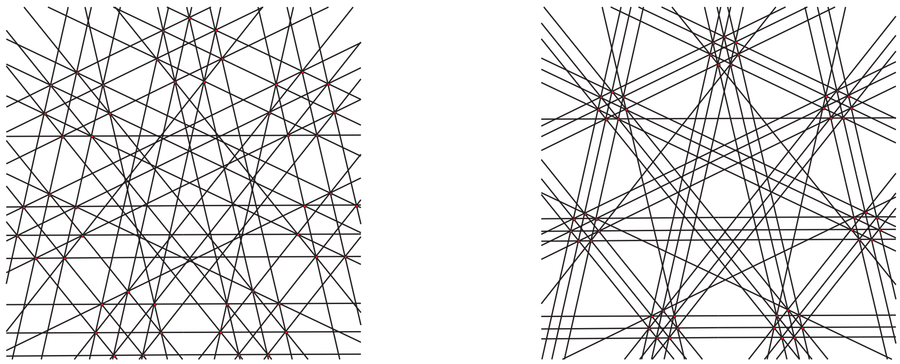

Figure 14: Two (49_4) configurations with the same floret and the same radius of the circle formed by the centers of the florets, but different sized florets; each may be continuously moved to the other by changing the radius of the floret.

- Variety (B): k florets, each with $2k$ points;
- Variety (C): $2k$ florets, each with k points;
- Varieties (D) and (E): k florets, each with k points;

Are these the only possibilities?

Question 2 Do there exist (n_6) generalized floral configurations? What about (n_k) for any $k \equiv 2 \bmod 4$? Or (p_q, n_k) for every even pair q, k?

Question 3 What is the level of connectedness of a floral configuration?

Question 4 Does every floral configuration have a Hamiltonian circuit?

Question 5 What is a useful notation for floral configurations? Generalized floral configurations?

Question 6 Is there a reasonable extension to pseudolines?

References

[1] L. W. Berman, *Movable (n_4) configurations.* Electron. J. Combin. **13**(2006) Research paper 104.
[2] ———, A characterization of astral (n_4) configurations. Discrete Comput. Geom. **26**(2001), no. 4, 603–612.
[3] M. Boben and T. Pisanski, *Polycyclic configurations.* European J. Combin. **24**(2003), No. 4, 431–457.
[4] B. Grünbaum, *Astral (n_k) configurations.* Geombinatorics **3**(1993), no. 3, 32–37.
[5] ———, *Astral (n_4) configurations.* Geombinatorics **9**(2000), no. 2, 127–134.
[6] ———, *Configurations of points and lines.* In: The Coxeter Legacy: Reflections and Projections. American Mathematical Society, Providence, RI, 2006, pp. 179–225.

[7] ———, *Configurations of points and lines*. To be published in the Graduate Studies in Mathematics Series, American Mathematical Society, Providence, RI.

[8] B. Grünbaum and J. F. Rigby, *The real configuration* (21_4). J. London Math. Soc **41**(1990), no. 5, 336–346.

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