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ON COMPONENTS IN SOME FAMILIES OF SETS¹

BRANKO GRÜNBAUM AND THEODORE S. MOTZKIN

1. **Introduction.** Helly's well-known theorem [3] states that all the members of a family \mathcal{C} of compact convex subsets of the Euclidean n -space E^n have a point in common provided every $n+1$ members of \mathcal{C} have a common point. On the other hand (Motzkin, cf. Hadwiger-Debrunner [2] for further reference), there exists no (finite) number h with the following property: If \mathcal{K} is a family of subsets of E^n (even of E^1) such that each member of \mathcal{K} is the union of at most two disjoint, compact, convex sets, and such that every h members of \mathcal{K} have a common point, then all the members of \mathcal{K} have a common point.

A consideration of the examples which establish the nonexistence of h led to the idea that there might exist theorems of Helly's type for such families \mathcal{K} if an additional condition is imposed on \mathcal{K} : the intersection of any two members of \mathcal{K} should also be representable as the union of at most two disjoint, compact, convex sets. The present paper contains a theorem in this direction together with related results on families \mathcal{K} whose elements are disjoint unions of members of another family \mathcal{C} .

In §2 we give the definitions of the properties we consider, and the statements of our main results. The proofs follow in §3. Remarks, examples, and counter-examples are given in §4.

2. **Definitions and results.** We shall deal mainly with families of subsets of some set, on whose nature nothing is assumed.

For a set A or an ordinal μ we denote by $\text{card } A$ resp. $\text{card } \mu$ the corresponding cardinal. Thus, for a family of sets $\mathcal{C} = \{C_\alpha : \alpha \in A\}$ we have $\text{card } \mathcal{C} = \text{card } A$. The letter ω is used only for initial ordinals.

For a family of sets $\mathcal{C} = \{C_\alpha : \alpha \in A\}$ we put $\pi\mathcal{C} = \bigcap_{\alpha \in A} C_\alpha$ and $\sigma\mathcal{C} = \bigcup_{\alpha \in A} C_\alpha$.

We define $K = C_1 + C_2$ to be an abbreviation for the statement " $K = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$." Similarly, for $\mathcal{C} = \{C_\alpha : \alpha \in A\}$, we write $K = \sum_{\alpha \in A} C_\alpha = \Sigma\mathcal{C}$ for " $K = \sigma\mathcal{C}$ and $C_\alpha \cap C_\beta = \emptyset$ for all $\alpha, \beta \in A$ with $\alpha \neq \beta$."

If $K = \Sigma\mathcal{C}$, each member of \mathcal{C} is a *component* of K and $\Sigma\mathcal{C}$ is a *decomposition* of K .

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For any family \mathcal{C} and any cardinal γ let $[\mathcal{C}]_\gamma = \{\Sigma \mathcal{C}' : \mathcal{C}' \subset \mathcal{C}, \text{card } \mathcal{C}' < \gamma + 1\}$ and $[\mathcal{C}] = \{\Sigma \mathcal{C}' : \mathcal{C}' \subset \mathcal{C}\}$. For $K \in [\mathcal{C}]$ let $c(K) = \min \{\text{card } \mathcal{C}' : K = \Sigma \mathcal{C}', \mathcal{C}' \subset \mathcal{C}\}$.

This paper deals with some properties of families of sets which we proceed to define.

DEFINITION 1. A family \mathcal{C} is γ -*intersectional* (for a finite or infinite cardinal $\gamma \geq 1$) if for every subfamily $\mathcal{C}' \subset \mathcal{C}$ with $\text{card } \mathcal{C}' < \gamma + 1$ we have $\pi \mathcal{C}' \in \mathcal{C}$. The family \mathcal{C} is *intersectional* if it is γ -intersectional for every $\gamma \geq 1$.

Obviously, if $\gamma^* \leq \gamma$ and \mathcal{C} is γ -intersectional, it is γ^* -intersectional as well. Every family is 1-intersectional; every 2-intersectional family is \aleph_0 -intersectional.

DEFINITION 2. A family \mathcal{C} is γ -*nonadditive* (for a finite or infinite cardinal $\gamma \geq 2$) if for every subfamily $\mathcal{C}' \subset \mathcal{C}$, with $\emptyset \notin \mathcal{C}'$ and $1 < \text{card } \mathcal{C}' < \gamma + 1$, such that $\Sigma \mathcal{C}'$ is defined, we have $\Sigma \mathcal{C}' \notin \mathcal{C}$. The family \mathcal{C} is *nonadditive* if it is γ -nonadditive for every $\gamma \geq 2$.

EXAMPLES. The family of all closed [open] subsets of E^n is intersectional [\aleph_0 -intersectional]. The family of all connected and open [compact] subsets of E^n is nonadditive [\aleph_1 -nonadditive; see [4]]. In the set of ordinals $\{\alpha : \alpha < \omega, \text{card } \omega = k\}$, for any $k > \aleph_0$, all segments of the form $[\alpha, \beta]$ or $[\beta, \omega)$, where α, β are limit-ordinals, form a family \mathcal{S} which is intersectional and nonadditive. For any set S with $\text{card } S = k \geq \aleph_0$ the family of all subsets of S with complements of cardinal less than k is \aleph_0 -intersectional and nonadditive.

DEFINITION 3. A family \mathcal{C} has the *Helly property of order h with limit γ* (h, γ cardinals with $2 \leq h < \gamma$) if for each subfamily $\mathcal{C}' \subset \mathcal{C}$, with $\text{card } \mathcal{C}' < \gamma + 1$, the condition " $\pi \mathcal{C}^* \neq \emptyset$ for all $\mathcal{C}^* \subset \mathcal{C}'$, with $\text{card } \mathcal{C}^* < h + 1$ " implies $\pi \mathcal{C}' \neq \emptyset$. The family \mathcal{C} has the *unlimited Helly property of order h* if it has the Helly property of order h with limit γ for every $\gamma > h$.

EXAMPLES. The family of all compact subsets of any topological space has the unlimited Helly property of order \aleph_0 . The family of convex subsets of E^n has the Helly property of order $n + 1$ with limit \aleph_0 ; that of compact convex subsets has the unlimited Helly property of order $n + 1$ (Helly's theorem). The family of all closed segments $[\alpha, \beta]$ of a well-ordered set has the unlimited Helly property of order 2; if segments $[\alpha, \mu)$, for a limit ordinal μ , are included, the family has the Helly property of order 2 with limit card μ .

The first theorem gives a criterion for the uniqueness of the decomposition of K .

THEOREM 1. Let $\mathcal{C} = \{C_\alpha : \alpha \in A\}$ be 2-intersectional and γ -nonadditive, and $K \in [\mathcal{C}]_\gamma$. If $K = \sum_{\alpha' \in A'} C_{\alpha'}$ with $A' \subset A$, $\text{card } A' < \gamma + 1$,

and $C_{\alpha'} \neq \emptyset$ for all $\alpha' \in A'$, and if $K = \sum_{\alpha'' \in A''} C_{\alpha''}$ with $A'' \subset A$, $\text{card } A'' < \gamma + 1$, and $C_{\alpha''} \neq \emptyset$ for all $\alpha'' \in A''$, then there exists a one-to-one map ϕ from A' onto A'' such that $C_{\alpha'} = C_{\phi(\alpha')}$ for all $\alpha' \in A'$. In other words, the components of K are uniquely determined.

As an immediate corollary we have:

COROLLARY. *Let \mathcal{C} be 2-intersectional and γ -nonadditive, and let $K \in [\mathcal{C}]_n$ (i.e., $c(K) \leq n$), where n is a finite cardinal and $\gamma \geq n$. Let $K^* \in [\mathcal{C}]_\gamma$, $K \subset K^*$, and let some n different components of K^* each have a nonempty intersection with K . Then different components of K are contained in different components of K^* , and, in particular, $c(K) = n$.*

Obvious examples show that the corollary may fail for infinite n .

The next theorem shows that $[\mathcal{C}]_\gamma$ is, in a sense, weakly intersectional: if the intersections of all members of certain subfamilies of $\mathcal{K} \subset [\mathcal{C}]_\gamma$ belong to $[\mathcal{C}]_\gamma$, then for each subfamily of \mathcal{K} the intersection of its members belongs to $[\mathcal{C}]_\gamma$.

THEOREM 2. *Let \mathcal{C} be γ -intersectional and γ' -nonadditive, $\mathcal{K} \subset [\mathcal{C}]_\gamma$ and $\pi\mathcal{K} \in [\mathcal{C}]_{\gamma'}$. Then there exists a subfamily $\mathcal{K}' \subset \mathcal{K}$, with $1 + \text{card } \mathcal{K}' \leq c(\pi\mathcal{K})$, such that different components of $\pi\mathcal{K}$ are contained in different components of $\pi\mathcal{K}'$; in particular, $c(\pi\mathcal{K}') \geq c(\pi\mathcal{K})$.*

A result of Helly's type for members of $[\mathcal{C}]_2$ is given by

THEOREM 3. *Let \mathcal{C} be γ -intersectional and \aleph_0 -nonadditive, with the Helly property of order h and limit γ^* , $\gamma^* \geq \aleph_0 > h$. Let $\mathcal{K} \subset [\mathcal{C}]_2$ be such that $\text{card } \mathcal{K} < \gamma + 1$ and $K' \cap K'' \in [\mathcal{C}]_2$ for all $K', K'' \in \mathcal{K}$. Then \mathcal{K} has the Helly property of order $2h$ with limit γ^* .*

3. Proofs.

PROOF OF THEOREM 1. Obviously

$$K = \sum_{\alpha' \in A'; \alpha'' \in A''} (C_{\alpha'} \cap C_{\alpha''})$$

is a decomposition of K . If for each $\alpha' \in A'$ and each $\alpha'' \in A''$ either $C_{\alpha'} \cap C_{\alpha''} = \emptyset$ or $C_{\alpha'} \cap C_{\alpha''} = C_{\alpha'}$, the theorem is proved. Assume on the contrary that there exists an $\alpha'_0 \in A'$ and an $\alpha''_0 \in A''$ such that $C_{\alpha'_0} \cap C_{\alpha''_0}$ is neither \emptyset nor $C_{\alpha'_0}$. Let $A_0'' = \{\alpha'' \in A'' : C_{\alpha'_0} \cap C_{\alpha''} \neq \emptyset\}$. Then $2 \leq \text{card } A_0'' < \gamma + 1$ and $C_{\alpha'_0} = C_{\alpha'_0} \cap K = C_{\alpha'_0} \cap \sum_{\alpha'' \in A''} C_{\alpha''} = \sum_{\alpha'' \in A_0''} (C_{\alpha''} \cap C_{\alpha'_0})$, in contradiction to the γ -nonadditivity of \mathcal{C} .

PROOF OF THEOREM 2. (i) Let $c(\pi\mathcal{K}) \geq 2$. Then there exist points x_1 and x_2 contained in different components C_1^*, C_2^* of $K^* = \pi\mathcal{K}$. For some $K_0 \in \mathcal{K}$ the points x_1 and x_2 are contained in different com-

ponents of K_0 ; indeed, otherwise there would for each $K \in \mathcal{K}$ exist a component C' of K with $x_1, x_2 \in C'$. Now $C = \pi \{ C' : K \in \mathcal{K} \} \in \mathcal{C}$ but, on the other hand, $C = C \cap K^* \supset (C \cap C_1^*) + (C \cap C_2^*)$, and none of the components is empty (since $x_i \in C \cap C_i^*$), contradicting the γ' -non-additivity of \mathcal{C} . If $c(K^*) = 2$, it follows at once from the corollary to Theorem 1 that different components of K^* are contained in different components of K_0 .

(ii) We now assume that $c(K^*) = n$ is finite, $n > 2$, and that the theorem is proved for all n' with $n' < n$. We start as in (i) with a set $K_0 = \sum_{\nu \in N} C_\nu \in \mathcal{K}$, where $\text{card } N = c(K_0) \geq 2$, such that $C_1 \cap K^* \neq \emptyset$ and $C_2 \cap K^* \neq \emptyset$. Let $q_\nu = c(K^* \cap C_\nu) \geq 0$ for $\nu \in N$. By Theorem 1 we have

$$(*) \quad \sum_{\nu \in N} q_\nu = c(K^*) = n.$$

This implies that $N_0 = \{ \nu \in N : q_\nu > 0 \}$ is finite and contains at most n elements. Let us assume that $N_0 = \{ 1, 2, \dots, t \}$ and that the components of K_0 are labeled in such a way that $q_\nu \geq 2$ for $1 \leq \nu \leq s$, and $q_\nu = 1$ for $s < \nu \leq t$. If $s = 0$, then (*) implies $t = n$, and by the corollary to Theorem 1 the n components of K_0 contain the n components of K^* , as claimed. Thus we are left with the case $s \geq 1$; then $2 \leq t < n$, $q_1 \geq 2$ and, by the choice of K_0 , $q_2 \geq 1$; therefore, by (*), $q_\nu < n$ for all $\nu \in N_0$. This allows us to apply the inductive assumption to each of the s families $\mathcal{K}_\nu = \{ C_\nu \cap K : K \in \mathcal{K} \}$, $1 \leq \nu \leq s$. It follows that for each ν , with $1 \leq \nu \leq s$, there exists a subfamily $\mathcal{K}'_\nu \subset \mathcal{K}_\nu$, containing $p_\nu \leq q_\nu - 1$ members, such that the different components of $C_\nu \cap K^*$ are contained in different components of $\pi \mathcal{K}'_\nu$. The family $\mathcal{K}' = \{ K_0 \} \cup (\bigcup_{\nu=1}^s \mathcal{K}'_\nu)$ satisfies all the conditions of the theorem. Indeed, different components of K^* are, by the corollary to Theorem 1, contained in different components of $\pi \mathcal{K}'$; but on the other hand, \mathcal{K}' contains only $1 + \sum_{\nu=1}^s p_\nu \leq 1 - s + \sum_{\nu=1}^s q_\nu = 1 - s + n - (t - s) = n + 1 - t \leq n - 1 < c(K^*)$ members.

(iii) There remains the case in which $k = c(K^*)$ is infinite. Let ω be the initial ordinal of k and let $K^* = \pi \mathcal{K} = \sum_{\nu < \omega} C_\nu^*$. For each $\nu < \omega$ let $x_\nu \in C_\nu^*$. As in (i), for each pair $\nu, \mu < \omega$ with $\nu \neq \mu$ there exists some $K_{\nu, \mu} \in \mathcal{K}$ such that x_ν and x_μ are contained in different components of $K_{\nu, \mu}$. Let $\mathcal{K}' = \{ K_{\nu, \mu} : \nu, \mu < \omega \}$. Then $\text{card } \mathcal{K}' \leq (\text{card } \omega)^2 = k$. For the family \mathcal{K}' we have $c(\pi \mathcal{K}') \geq k$ since x_ν and x_μ belong to different components of $\pi \mathcal{K}'$. By an argument similar to that used in the proof of Theorem 1 it follows that different components of K^* are contained in different components of $\pi \mathcal{K}'$. This ends the proof of Theorem 2.

PROOF OF THEOREM 3. For some fixed h assume the theorem false; let k be the minimal cardinal for which there exists a family with card $\mathcal{K} = k$ contradicting the theorem.

(i) Assume k finite. Then for each $K^* \in \mathcal{K}$ we have $\pi\{K \in \mathcal{K}: K \neq K^*\} \neq \emptyset$. Let $\mathcal{K}_i = \{K \in \mathcal{K}: c(K) = i\}$ for $i = 1, 2$, and let $K = C_1 + C_2$ for all $K \in \mathcal{K}_2$. We assume that \mathcal{K} is chosen in such a way that card $\mathcal{K}_1 + 2$ card \mathcal{K}_2 (the total number of components of members of \mathcal{K}) is minimal. This implies that for each $K' \in \mathcal{K}_2$ and $i = 1, 2$, there exists a $K^0 = K^0(C'_i) = K^0(K', i) \in \mathcal{K}$ such that $\pi\{K \in \mathcal{K}: K \neq K^0\} \subset C'_i$.

We shall show that $C'_i \cap K \neq \emptyset$ for all $K' \in \mathcal{K}_2, K \in \mathcal{K}$, and $i = 1, 2$. Let us assume, to the contrary, that there exists $K' \in \mathcal{K}_2, K_0 \in \mathcal{K}$ and $i = 1$ or 2 such that $C'_i \cap K_0 = \emptyset$. (Without loss of generality we shall assume $i = 1$.) Since $\emptyset \neq \pi\{K \in \mathcal{K}: K \neq K^0(K', 1)\} \subset C'_i$, it follows that $K_0 = K^0 = K^0(K', 1)$. Then $C'_i \cap K \neq \emptyset$ for all $K \neq K^0$; also $C'_2 \cap K \neq \emptyset$ for all $K \in \mathcal{K}$, since otherwise $K' \cap K \cap K^0 \subset (C'_1 \cap K^0) \cup (C'_2 \cap K) = \emptyset$ would contradict the assumption that any $3 < 4 \leq 2h$ members of \mathcal{K} have a nonempty intersection. Therefore, for each $K \neq K^0, c(K' \cap K) = 2$; hence, for some component C_j of K we have $K \cap C'_2 = C_j \cap C'_2$. Now

$$\begin{aligned} \pi\{C_j: K \in \mathcal{K}, K \neq K^0\} &= C'_2 \cap \pi\{C_j: K \in \mathcal{K}, K \neq K^0\} \\ &= C'_2 \cap \pi\{K \in \mathcal{K}: K \neq K^0\} \subset C'_1 \cap C'_2 = \emptyset. \end{aligned}$$

Since \mathcal{C} has the Helly property of order h it follows that for some subset \mathcal{K}_0 of \mathcal{K} , such that $K^0 \notin \mathcal{K}_0$ and with card $\mathcal{K}_0 = h_0 \leq h$, we have $\pi\{C_j: K \in \mathcal{K}_0\} = \emptyset$. For the family $\mathcal{K}^* = \{K', K^0\} \cup \mathcal{K}_0$ we have therefore $\pi \mathcal{K}^* \subset (C'_1 \cap K^0) \cup (C'_2 \cap \pi \mathcal{K}_0) = \emptyset$, although card $\mathcal{K}^* \leq h_0 + 2 \leq h + 2 \leq 2h$. This contradiction establishes our assertion.

Next, let $K^* \in \mathcal{K}_2$ be chosen arbitrarily. For each $K \in \mathcal{K}_2$ it follows from the above and from $c(K^* \cap K) \leq 2$ that $c(K^* \cap K) = 2$ and that different components of K intersect different components of K^* . Let the components of K be re-labeled, if necessary, in such a way that $C_i^* \cap C_i \neq \emptyset$ for $i = 1, 2$. We claim that for all $K', K'' \in \mathcal{K}_2$ we have $C'_i \cap C''_i \neq \emptyset, i = 1, 2$. Indeed, otherwise we would have (since each component of one set intersects every other set), $C'_1 \cap C''_1 = C'_2 \cap C''_2 = \emptyset$, and therefore $K^* \cap K' \cap K'' = \emptyset$, which is impossible. Thus, for any $K', K'' \in \mathcal{K}_2$,

$$C'_i \cap C''_j \begin{cases} = \emptyset & \text{if } i \neq j \\ \neq \emptyset & \text{if } i = j. \end{cases}$$

Now we consider the families $\mathcal{C}_i = \mathcal{K}_1 \cup \{C_i: K \in \mathcal{K}_2\}$ for $i = 1, 2$. The assumption $\pi \mathcal{K} = \emptyset$ implies that $\pi \mathcal{C}_i = \emptyset$ for $i = 1, 2$. Since $\mathcal{C}_i \subset \mathcal{C}$,

there exist h or less members of \mathcal{C}_i whose intersection is empty, $i=1, 2$. But then the intersection of the corresponding members of \mathcal{K} is also empty, although it involves at most $2h$ members of \mathcal{K} . The contradiction reached proves the theorem for finite k .

(ii) Let k be infinite, $k < \gamma^*$, and the theorem true for all families with less than k members. Let ω be the initial ordinal of k , let A be the set of ordinals $A = \{\alpha: \alpha < \omega\}$, and let $\mathcal{K} = \{K_\alpha: \alpha < \omega\}$. By the inductive assumption we have $\bigcap_{\alpha < \mu} K_\alpha \neq \emptyset$ for each $\mu < \omega$. If for some K_α one of its components does not intersect some K_β , we omit this component and take the other component to be the new K_α . By the inductive assumption, the new K_α satisfy $\bigcap_{\alpha < \mu} K_\alpha \neq \emptyset$ for all $\mu < \omega$. From here on we proceed as in the final part of (i): we re-label (if necessary) the components of some K_α with $c(K_\alpha) = 2$, construct the families \mathcal{C}_i and derive a contradiction from the assumption that $\bigcap_{\alpha < \omega} K_\alpha = \emptyset$. This terminates the proof of Theorem 3.

4. Remarks. 1. Theorem 2 fails if $\text{card } \pi\mathcal{K}$ is infinite and \mathcal{K}' is assumed to satisfy $\text{card } \mathcal{K}' < \text{card } \pi\mathcal{K}$. E.g., starting from the family \mathcal{S} (preceding Definition 3), with $\text{card } \omega = k > \aleph_0 = \text{card } \omega_0$, let $\mathcal{K} = \{[\omega_0, \alpha] \cup [\alpha + \omega_0, \omega): \alpha \text{ limit ordinal } < \omega\}$. Then $c(\pi\mathcal{K}) = k$, but the intersection of any $k' < k$ members of \mathcal{K} has only k' components. Similar examples are easily found for $c(\pi\mathcal{K}) = \aleph_0$.

2. Probably the most interesting immediate application of Theorem 3 is to convex sets in E^n . To satisfy the condition of nonadditivity we may consider, e.g., families consisting only of closed (or only of open) convex sets. The following example shows that Theorem 3 does not hold if \mathcal{C} is, e.g., the family of all convex sets in E^2 . (Simple examples of a similar nature show the necessity of nonadditivity assumptions in Theorem 2.) Let D denote a closed disc with center 0. Let K_0 be obtained from D by deleting 0. Let $x_i, i=1, 2, \dots, 6$, be equidistant points on the boundary of D , ($x_i = x_{i+6}$). For each $i, 1 \leq i \leq 6$, let K_i be obtained from D by deleting the open small arc of $\text{Bd } D$ determined by x_{i-1} and x_{i+1} , and the open sector determined by these two points and 0. Then each $K_i, 0 \leq i \leq 6$, as well as the intersection of any two K_i , is the disjoint union of two convex sets, and any six K_i have a nonempty intersection. Nevertheless, $\bigcap_{i=0}^6 K_i = \emptyset$. As is easily verified, the same reasoning applies to the case where 7 or 8 equidistant points are chosen on $\text{Bd } D$. We conjecture that for the family of all convex sets in E^2 a result analogous to Theorem 3 holds, with 9 instead of $2h$.

3. The following statement (with obvious refinements) is conjectured: If \mathcal{C} is an intersectional and nonadditive family with un-

limited Helly property of order h and if $\mathcal{K} \subset [\mathcal{C}]_n$ is such that the intersection of any 2, 3, \dots , n members of \mathcal{K} also belongs to $[\mathcal{C}]_n$ then \mathcal{K} has the unlimited Helly property of order nh . Simple examples show that $nh-1$ may not be substituted for nh in this conjecture. If \mathcal{C} is the family of segments in E^1 , the conjecture is easily provable.

4. Let $\mathcal{C}^{(n)}$ denote the family of all compact, convex subsets of E^n ; in [1], a function $\Delta(K)$, with $0 \leq \Delta(K) \leq +\infty$, was defined for all compact sets $K \subset E^n$ in such a way that $\Delta(K) < \infty$ if and only if $K \in [\mathcal{C}^{(n)}]_{\aleph_0}$. Theorem 2 of [1] may be formulated as follows: For any finite $n \geq 1$ and real $d < \infty$ there exists a finite $h = h(n, d)$ such that the family $\{K \in [\mathcal{C}^{(n)}]_{\aleph_0} : \Delta(K) \leq d\}$ has the unlimited Helly property of order h . By applying the methods of [1] it may be shown that for each finite $n \geq 1$ and $d < \infty$ there exists a finite $k = k(n, d)$ such that $\Delta(K) \leq d$ implies $K \in [\mathcal{C}^{(n)}]_k$.

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