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GEOMETRIC REALIZATIONS OF SPECIAL TOROIDAL COMPLEXES

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Abstract. For which positive integers n is it possible to find geometric complexes that topologically are tori such that all n faces are k-gons, where $k = 3$, or else 4, or 5, or 6? The answers we are able to provide do not solve the problem completely, but leave certain undecided cases. The main aim of the paper is to show how varied are the geometric shapes that represent the various cases. As will be seen, several of the polyhedra are quite hard to imagine or visualize, and we provide detailed insights into their construction, including coordinates of their vertices in some of the cases.

1. Introduction

If nothing else is expressly stated, we shall assume that we are dealing with polyhedral complexes, that is, families of planar polygons, not necessarily convex; these are the *faces* of the complexes. The polygons are assumed to be *simple*; that is, there are *no collinear adjacent* edges, all vertices are distinct, and any two edges have in common at most an endpoint of both. We also require that the faces incident with a common vertex form a circuit in which adjacent faces share an edge that is incident with the vertex in question; hence there are at least three faces incident with each vertex. Adjacent faces cannot be coplanar. To simplify the language, we shall use the term "polyhedra" for such complexes. Except in the last section, we shall assume that the polyhedra are *acoptic*, that is, the intersection of any two faces is either empty, or else every connected component is a vertex or an edge of both. If such an intersection of two faces has more than one component, we shall say that the faces are *overarching*. An example is shown in Figure 1. Note that acoptic polyhedra with convex faces have no overarching faces; it follows that triangulations have no overarching faces. By the analogue of the Jordan curve theorem in the plane, each acoptic polyhedron is the boundary of a solid.

The torus is an orientable manifold, its Euler characteristic is 0, that is $V - E + F = 0$; here and throughout we shall use V, E, and F to denote

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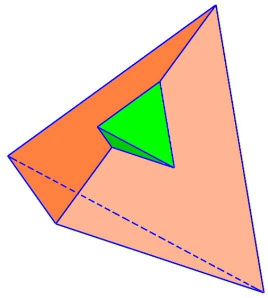


Figure 1: A "notched tetrahedron" has a pair of overarching faces.

the number of vertices, edges and faces of the polyhedron. The polyhedra we are interested in are those tori in which all faces have the same number k of sides. It is a simple consequence of the Euler characteristic of the torus that k must be one of 3, 4, 5, 6. While the terms "triangulation" and "quadrangulation" are commonly used for the first two of these cases, for the other two we shall use "quintangulation" and "hexangulation"; these terms are possibly not without reproach, but will do for present purposes. To stress the distinction between the polyhedra considered here and more general constructs, we shall use the term "geometric." Unless the contrary is explicitly stated, by "polyhedron" we mean "torus."

2. Triangulations

As each edge belongs to two faces, the number F of faces must be even. It is well known that combinatorial (or topological) triangulations of the torus exist for all $F = 2n$, with $n \geq 1$.

Proposition 2.1. *For every even $F \geq 14$ there are geometric triangulations of the torus with F faces.*

Proof. For $F = 14$ there are the well-known Császár tori, see [5, 8, 20, 21]. Attaching a tetrahedron along a face of a given toroidal polyhedron eliminates one face from each, hence leads to a net increase of two faces; the existence for every even $F \geq 16$ follows. It is a consequence of the Euler relation that no triangulated acoptic tori exist with $F \leq 12$. □

Remark 2.2: For every $F = 4m$, $m \geq 5$, there exist geometric triangulations that are isogonal (that is, vertex-transitive); see [12].

Conjecture 2.3. *There are no geometric isogonal triangulations of the torus with $F = 2m$, where m is odd.*

3. Quadrangulations

We first note:

Proposition 3.1. *There exist geometric non-overarching quadrangulations of the torus with F convex faces for every $F \geq 9$ except possibly for $F = 10, 11$.*

Proof. There are two basic constructions we use. First, there are generalized "picture frames." For all integers $p \geq 3$ and $q \geq 3$ we can construct "picture frames" for p-sided "pictures," with q-sided cross-sections. These give quadrangulations with $F = pq$, thus yielding the values $F = 9, 12, 15, 16, 18, 20, \ldots$. □

Examples of two "picture frame" polyhedra with $F = 15$ are shown in Figure 2.

The second construction consists of attaching to a given quadrangulation a suitable image of a cube. This eliminates one face of the given quadrangulation and one face of the cube, hence there is a net gain of four faces. Applying this construction to the smallest quadrangulation just described ($F = 9$), we first obtain $F = 13$ (see Figure 3), and then $F = 17$. Since now the consecutive values $F = 15, 16, 17, 18$ are available, adding multiples of 4 completes the proof of Proposition 3.1.

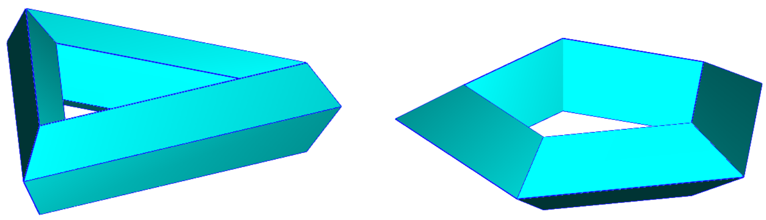


Figure 2: A picture frame for a triangular picture, with pentagonal cross section, and one for a pentagonal picture with a triangular cross section.

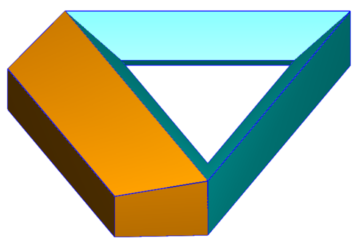


Figure 3: An acoptic quadrangulation with F = 13 convex faces, obtained by attaching a suitable image of a cube to a quadrangulation with $F = 9$.

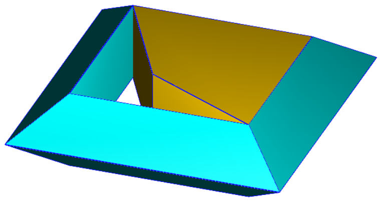


Figure 4: An acoptic quadrangulation with $F = 14$ convex faces.

A quadrangulation with $F = 14$ is shown in Figure 4; it is obtained by a different construction. There are various alternative constructions of quadrangulations for almost all values of F.

Conjecture 3.2. *There exist no quadrangulations, of the kind described in Proposition 3.1, with* $F = 10$ *or 11.*

Proposition 3.3. *If overarching faces are admitted, then acoptic quadrangulations with* F *faces exist for all* $F \geq 9$*.*

Proof. Examples with $F = 10, 11$ (the exceptional values in Proposition 3.1) are shown in Figures 5 and 6. □

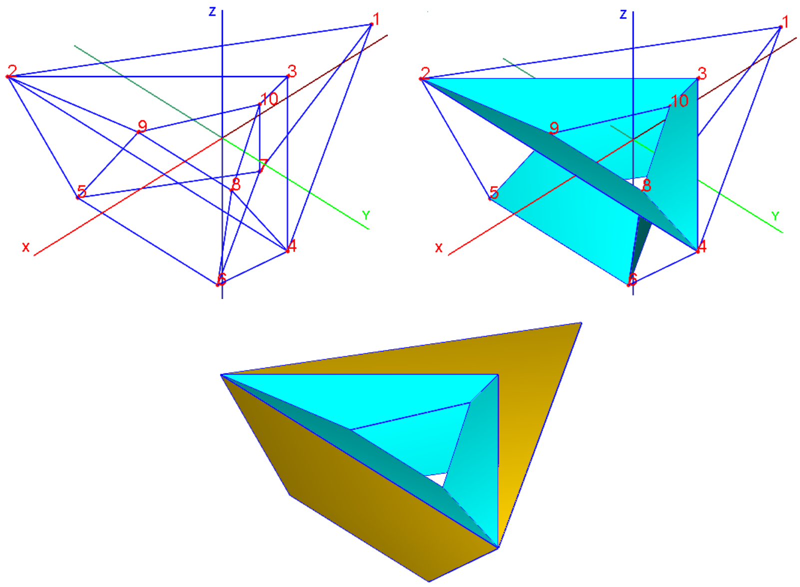


Figure 5: The steps in the construction of a quadrangulation with $F = 10$ faces, some of which are overarching. The coordinates of the vertices, and the lists of faces, are given in Table 1 in the Appendix.

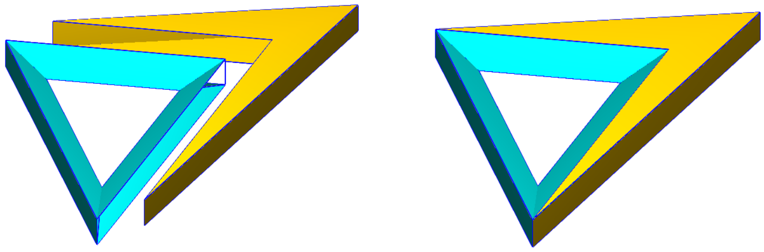


Figure 6: The construction of a quadrangulation with $F = 11$ faces, from a picture frame for a triangular picture and with a triangular cross section, and a prism over a dart-shaped quadrangle.

Conjecture 3.4. *There exist no quadrangulations, of the kind described in Propositions 3.1 and 3.3, with $F \leq 8$.*

Remark 3.5: All picture-frame quadrangulations obtained by the first construction in the proof of Proposition 3.1 are *combinatorially* both *isogonal* (vertex-transitive) and *isohedral* (face-transitive).

Remark 3.6: Schwörbel proves (see [16, 17, 23]) that all triangulations, quadrangulations and hexangulations in which the *flags* form one transitivity class under combinatorial automorphisms are geometrically realizable as acoptic polyhedra with possibly non-convex faces. (A flag is a triplet consisting of mutually incident vertex, edge and face.)

Remark 3.7: A question deeper than the one considered in this paper is whether all combinatorial quadrangulations of the torus that have no overarching faces can be geometrically realized provided non-convex faces are admitted. An affirmative answer was conjectured in [9]. An indication that admitting non-convex faces expands the family of acoptically realizable quadrangulated tori is given by the Ljubic' torus (see [9]); Simutis [18] proved that this torus is not realizable with convex faces. Another confirming example (though not a quadrangulation) is the toroidal map shown in Figure 7. It was shown in [6] that it is not geometrically realizable with convex faces. However, in Figure 8 we show how to realize it with just one non-convex face.

4. Quintangulations

In this case F must again be even.

Proposition 4.1. *For every even $F \geq 12$ there are convex-faced quintangulations of the torus, except possibly for $F = 14$.*

Proof. We start with trapezohedra, that is, Catalan polyhedra that are polar to the antiprisms. For each $p \geq 3$ such a polyhedron has $2p$ quadrangular

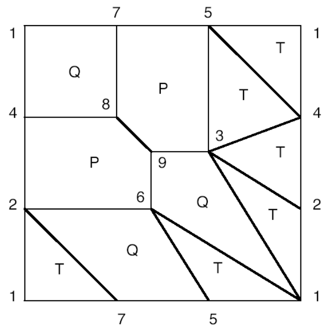


Figure 7: A toroidal map (denoted E in [6]) that is not geometrically realizable with convex faces.

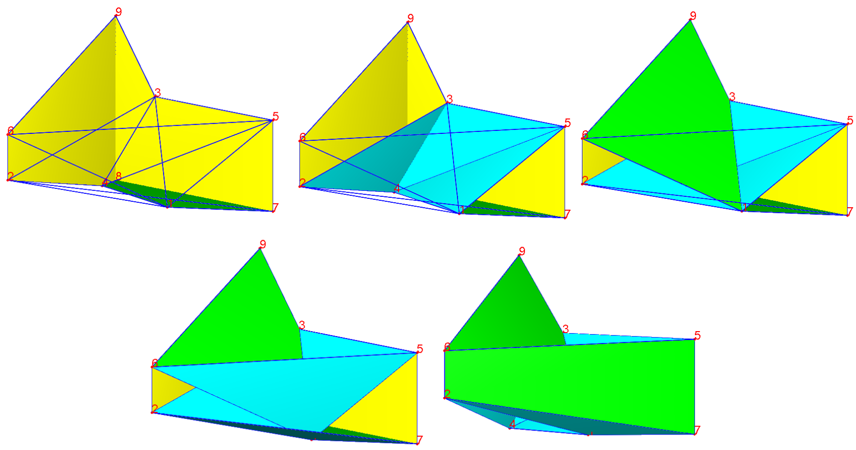


Figure 8: (a) Steps in the construction of the geometric realization of the map in Figure 7. The coordinates of the vertices, and the lists of faces, are given in Table 2 in the Appendix

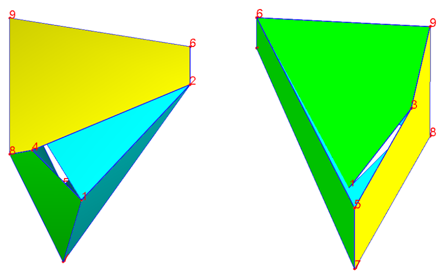


Figure 8: (b) Two additional views of the polyhedron constructed in (a).

faces, p meeting at each of two apices; the other vertices are 3-valent. By using two such polyhedra, with coinciding axes and planes of symmetry, and of appropriate sizes and positions, so that they mutually truncate all faces (which become convex pentagons), we obtain a quintangulation with $F = 4p$. An example is shown in Figure 9. By using appropriate chiral trapezohedra, a similar construction can be performed on chains of $q \geq 3$ trapezohedra, yielding quintangulations with $F = 2pq$ faces. In particular, by these constructions we obtain acoptic non-overarching convex-faced polyhedra with $F = 12, 16, 18, 20, 24, 28, \ldots$, with p-fold rotational symmetry. By attaching a copy of the pentagonal dodecahedron the number of faces increases by 10, hence yielding the missing values $F = 22, 26, 30, 34, 38, \ldots$. This establishes the proposition. □

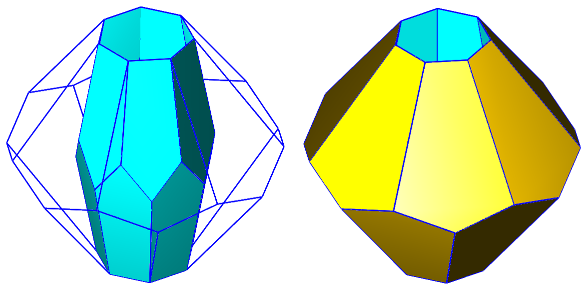


Figure 9: An example of a quintangulation with $F = 28$; here $p = 7, q = 2$.

Conjecture 4.2. *There is no quintangulation of the kind described in Proposition 4.1, with 14 faces, or with fewer than 12 faces.*

5. Hexangulations

For realizations without overarching faces we clearly have $F \geq 7$. In contrast to the cases considered above, here an infinite number of values of F are undecided. This is clearly the most difficult, and hence most interesting, of the kinds of polyhedra considered in this paper.

Proposition 5.1. *Acoptic hexangulations with F faces, with no overarching faces, exist for $F = 7$, and for all $F = pq$ with $p \geq 3$, $q \geq 3$.*

Proof. The hexangulation with $F = 7$ was found by Szilassi [20], and is reproduced in several other venues (see, for example, [1, 19, 21, 22]). One of the possible polyhedra is shown in Figure 10. □

The family of hexangulations with $F = pq$ for $p \geq 3, q \geq 3$ can be constructed by starting with the same trapezohedra (Catalan polyhedra that are polar to the antiprisms) as in the proof of Proposition 4.1. For

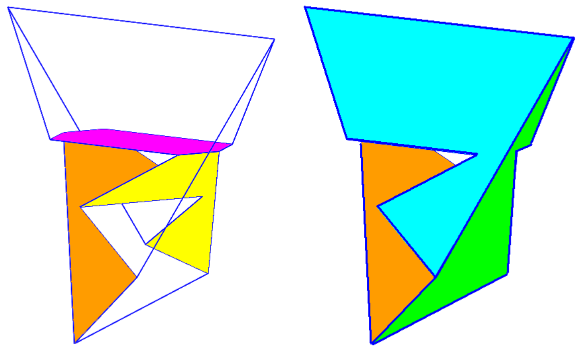


Figure 10: An example of a Szilassi torus, that is, a hexangulation with $F = 7$. The coordinates of the vertices, and the lists of faces, are given in Table 3 in the Appendix.

each $p \geq 3$ such a polyhedron P has $2p$ quadrangular faces, p of which meet at each of two apices of P. By intersecting such a polyhedron P with a p-sided prism, having its axis coinciding with the axis of P and rotated appropriately, the resulting tunnel has p hexagonal sides, and all $2p$ sides of P become hexagons as well. This construction (for $q = 3$) has been described for $p = 3$ by many writers, starting with Becker [3, 2]; see [7, 16, 21, 22] and others. The resulting polyhedron is shown in Figure 11. An example of this construction with $p = 5$ is shown in Figure 12. Since the construction works for all $p \geq 3$, this establishes the proposition for $q = 3$. For $q \geq 4$ it is only necessary to replace P by the convex polyhedron Q obtained by inserting, along the equatorial zigzag of P, $q - 3$ bands of hexagons, and continuing with the constructions as for $q = 3$. The case of $p = 3, q = 5$ is illustrated in Figure 13. The possibility of a geometric construction of Q as a convex polyhedron with a p-fold axis of rotational symmetry is guaranteed by Mani's [13] generalization of Steinitz's theorem on convex polyhedra (see also [11, p. 296a] and [14]).

Other methods of constructing hexangulations with $F = pq$ for $p \geq 3, q \geq 3$ faces are illustrated in Figures 14, 15 and 16.

The hexangulations with $F \geq 9$ faces exhibited so far are systematic (work for various numbers of faces) and exhibit considerable symmetry. However, for some of the missing values of F there are more complicated, asymmetric possibilities, somewhat analogous to the case of $F = 7$. One such example for $F = 10$ is shown in Figure 17. To understand its structure we show its gradual build-up; as documentation we present in Table 4 in the Appendix the coordinates of its vertices, and in Figure 18 its net.

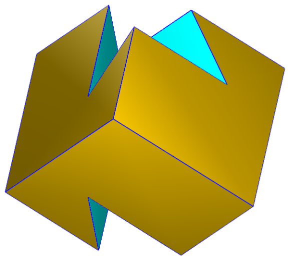


Figure 11: A hexangulation with $F = 9$ faces.

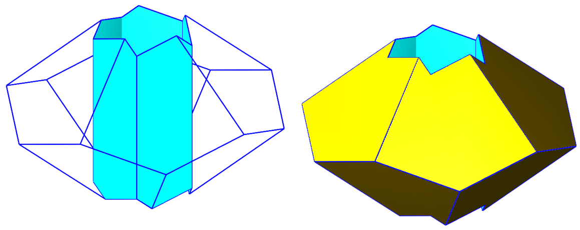


Figure 12: The construction of the text illustrated for $p = 5, q = 3$.

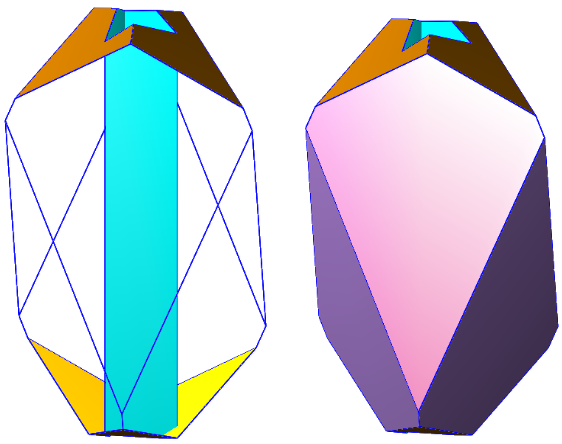


Figure 13: The construction of the text illustrated for $p = 3, q = 5$.

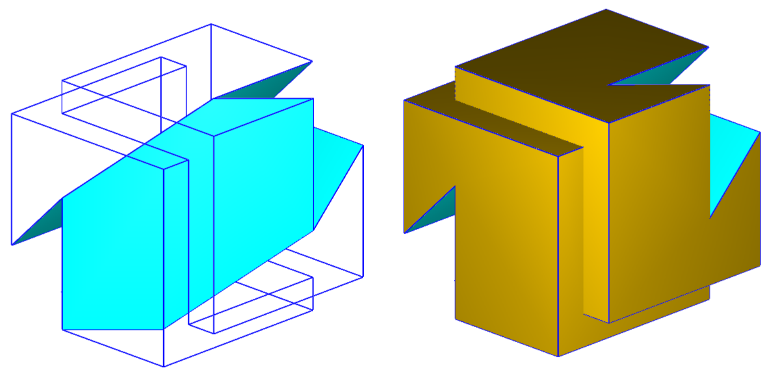


Figure 14: An alternative construction of a hexangulation with $p = 3, q = 5$. It generalizes the construction in Figure 11.

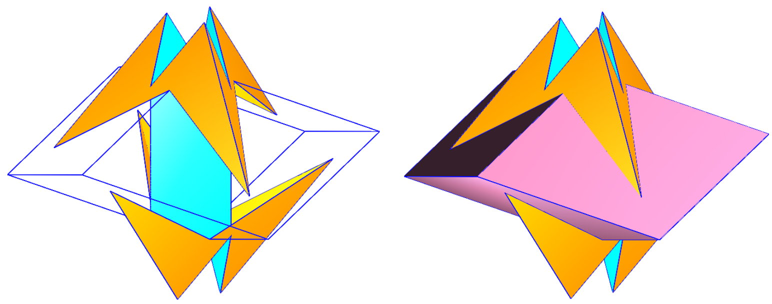


Figure 15: Another generalizable construction of a hexangulation with $F = 15$ faces.

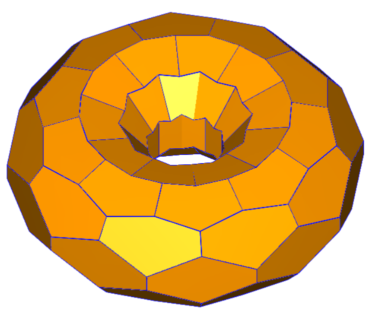


Figure 16: Another construction of a hexangulation with $F = 100$ faces. This can be generalized to other numbers of faces as well.

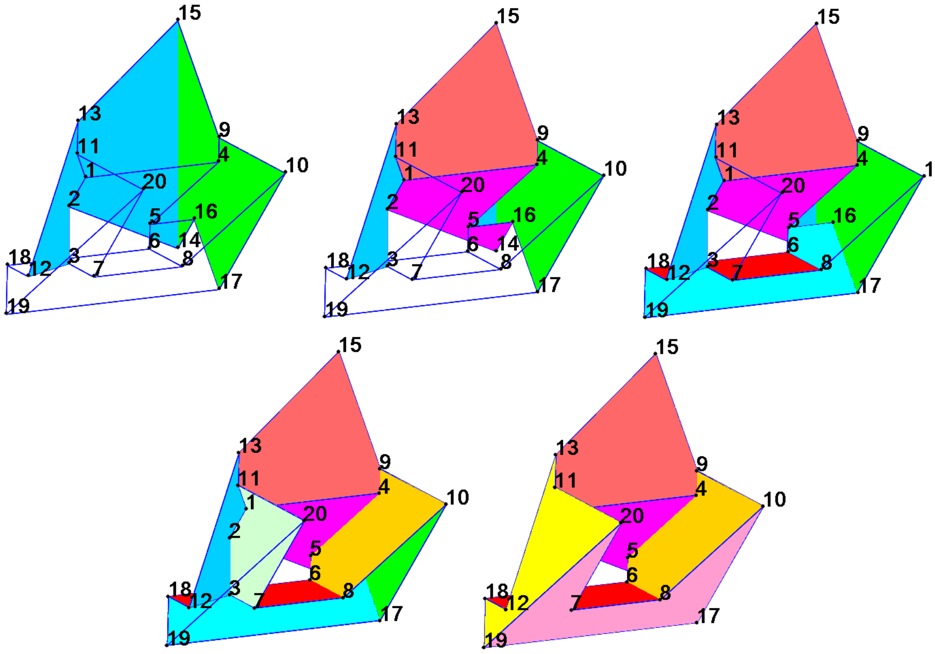


Figure 17: Several steps in the construction of a hexangulation with $F = 10$ faces. A net of the polyhedron is shown in Figure 18, and coordinates of the vertices are in Table 4 in the Appendix.

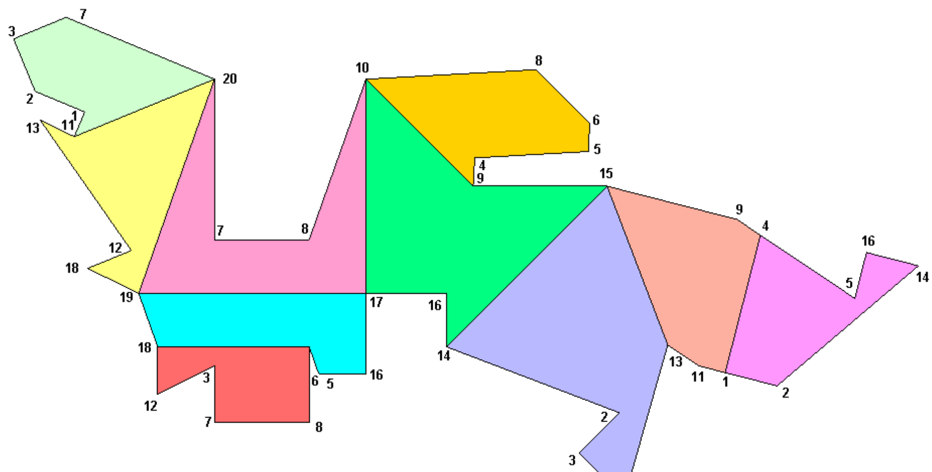


Figure 18: A net of the polyhedron shown in Figure 17.

Remark 5.2: Other hexangulations with $F = 4p$, $p \geq 3$, were found independently by Schwörbel [16] and by Szilassi [22]. A short description can be given by starting with a regular zigzag in 3-space consisting of $2p$ segments, and "thickening" it to a solid frame by L-shaped hexagons. This is illustrated in Figure 19 for the case $p = 3$.

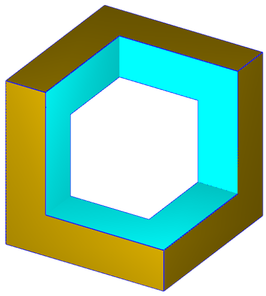


Figure 19: An illustration of the construction mentioned in Remark 5.2, using L-shaped faces. Here $p = 3$, and the hexangulation has $F = 4p = 12$ faces.

Conjecture 5.3. *There exist no hexangulations of the torus with F faces, with no overarching or coplanar faces, when $F = 8$ or a prime number. Such hexangulations exist for all $F = 2p$, where p is a prime number.*

Remark 5.4: If overarching faces are admitted, additional hexangulations are possible. Such a hexangulation with $F = 8$ was found by Schwörbel [16]. An improved drawing of Schwörbel's polyhedron is shown in Figure 20. It would be interesting to investigate whether it is possible to extend the construction mentioned in Remark 5.2 in order to find hexangulations with F equal to twice a prime number and with faces having L-shape analogous to the faces in Figures 19 and 20.

Remark 5.5: Schwörbel proved (see [16, 23]) that all triangulations, quadrangulations and hexangulations of the torus, in which the flags form one

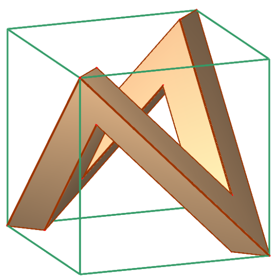


Figure 20: A hexangulation with $F = 8$ overarching faces.

transitivity class under combinatorial automorphisms, are geometrically realizable as acoptic polyhedra with possibly non-convex faces.

Remark 5.6: In contrast to the situation concerning quintangulations, it is well known that no hexangulation of the torus can have only convex faces (see [4] and [11, p. 253]). The polyhedra of the type represented by Figure 13 have only six non-convex faces.

Conjecture 5.7. *Every hexangulation of the torus has at least six non-convex faces.*

6. More general polyhedra

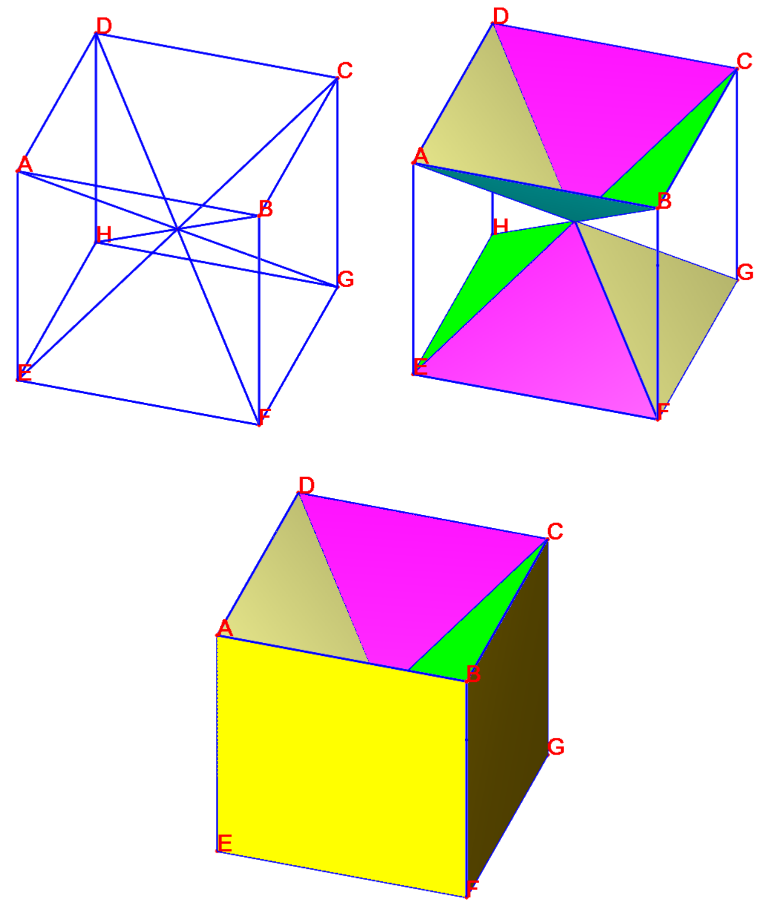
Many different kinds of geometric objects are called "polyhedra." The acoptic ones which we considered above are an immediate generalization of convex polyhedra. While much more general polyhedra have been studied (see, for example, [10]), in some sense it is reasonable to restrict the generality to polyhedra of the *Kepler-Poinsot type.* This terminology was used by Schulte and Wills [15] to indicate polyhedra that may have selfintersecting faces, as well as intersections of faces in ways other than what is stipulated for acoptic polyhedra. The name comes from the well-known regular star-polyhedra, discovered in part by Kepler in the 17th century, and by Poinsot in the 19th. However, other properties of acoptic polyhedra are assumed to be satisfied. This includes the single circuits of faces incident with a vertex, non-collinearity of adjacent edges of a face, and non-coplanarity of adjacent faces. As with the Kepler-Poinsot regular polyhedra, overarching faces are acceptable.

It is only natural that with a class of polyhedra wider than the acoptic ones, several additional kinds of tori are possible. This eliminates some of the restrictions in the propositions established above. Here are some of the additional polyhedra.

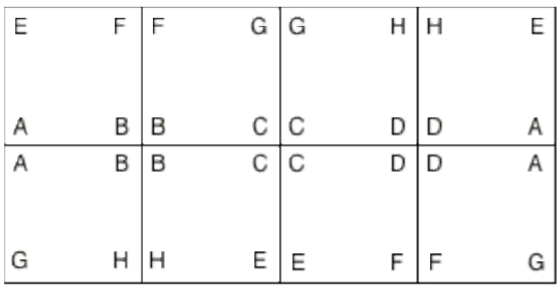
(i) In Figure 21(a) we show the construction of a Kepler-Poinsot type quadrangulation of the torus with $F = 8$ faces. As it has 16 edges and 8 vertices, its Euler characteristic is 0. To verify that this polyhedron is orientable, hence isomorphic to a quadrangulation of the torus, it is simplest to consider its map, shown in Figure 21(b). It may be conjectured that no quadrangulation of the Kepler-Poinsot type has fewer than 8 faces.

(ii) Another example is shown in Figure 22. It is hexangulation with $F = 11$ faces. It has two pairs of intersecting faces, and one selfintersecting face.

Conjecture 6.1. *The polyhedra of Kepler-Poinsot type admit hexangulations with F faces for all $F \geq 7$.*



(a)



(b)

Figure 21: A quadrangulation of the Kepler-Poinsot type with $F = 8$ faces.

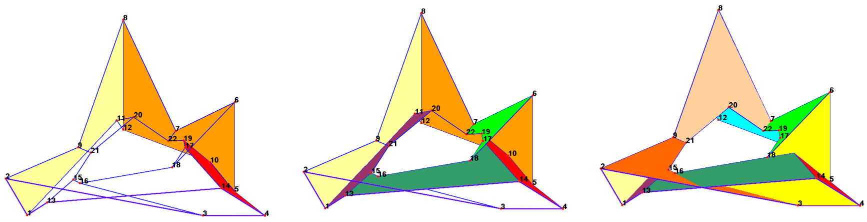


Figure 22: A hexangulation with $F = 11$ faces. The coordinates of the vertices, and the lists of faces, are given in Table 5 in the Appendix.

7. Acknowledgements

Branko Grünbaum acknowledges the hospitality of the Whiteley Center at the Friday Harbor Labs of the University of Washington, during the preliminary stages of this work, in 2006. Both authors appreciate the possibility to render the diagrams of the polyhedra in the Euler3D software, with which Lajos Szilassi drew all the figures.

"Euler3d" is free software designed to display polyhedra in 3-dimensional space. It is currently available for the Windows operating system and may be freely downloaded from `http://www.euler3d.hu/index.php?lang=EN`.

The polyhedron to be displayed has to be described by the numerical coordinates of its vertices. The faces are given by the ordered vertices belonging to the face. When a face is registered, the program runs through the following steps:

(1) Checks that the first three points determine a plane.
(2) If so, then all subsequent vertices must belong to this plane.
(3) Checks that the ordered sequence of vertices forms a simple polygon.
(4) If so, the face is drawn according the given order of vertices.

The software is further able to determine the given polyhedron's dual with respect to any base sphere except if the center of the sphere lies on a plane determined by one of the faces. It is also possible to import objects described in the VRML1 format. In particular, the software is capable of displaying surfaces produced by MAPLE. Soon an upgrade will be available with the additional capability of the user renaming the interface's command options in any language.

Since the above was written, L. Szilassi has found constructions for hexagonalizations with 8 and 11 faces that consist of selfintersection-free hexagons, have no intersecting faces, and no overarching faces. Hence Conjecture 5.3 is resolved in the negative for these values. The construction is announced and illustrated in an abstract "Some Regular Toroid," submitted to the 11th Annual Bridges Conference (Leeuwarden, The Netherlands, July 2008).

A. Appendix

Table 1: Coordinates of the vertices of the polyhedron in Figure 5, and the list of faces in terms of vertices.

	Vertices	Faces
1	(-32, 0, 4)	(1, 2, 3, 4)
2	(16, -30, 4)	(4, 2, 5, 6)
3	(-14, 0, 4)	(4, 6, 7, 1)
4	(16, 30, 4)	(2, 1, 7, 5)
5	(16, -15, -11)	(2, 4, 8, 9)
6	(16, 15, -11)	(2, 9, 10, 3)
7	(-8, 0, -11)	(3, 10, 8, 4)
8	(8, 10, 0)	(7, 10, 9, 5)
9	(8, -10, 0)	(7, 10, 8, 6)
10	(-8, 0, 2)	(6, 5, 9, 8)

Table 2: Coordinates of the vertices of the polyhedron in Figure 8, and the list of faces in terms of vertices.

	Vertices	Faces	
1	(0, -2, -4)	(3, 5, 7, 8, 9)	non-convex
2	(4, 4, -2)	(2, 6, 9, 8, 4)	
3	(2, -4, 0)	(1, 4, 8, 7)	
4	(4, -3, -4)	(2, 4, 3)	
5	(-4, -4, 0)	(3, 4, 5)	
6	(4, 4, 0)	(1, 5, 4)	
7	(-4, -4, -4)	(1, 2, 3)	
8	(4, -4, -4)	(1, 3, 9, 6)	
9	(4, -4, 3.2)	(1, 6, 5)	
		(1, 7, 2)	
		(2, 7, 5, 6)	

Table 3: Coordinates of the vertices of the polyhedron in Figure 10, and the list of faces in terms of vertices.

Vertices		Faces
1	(-24, 0, 24)	(1, 2, 14, 10, 8, 6)
2	(24, 0, 24)	(1, 6, 4, 3, 11, 13)
3	(0, -25.2, -24)	(3, 5, 7, 8, 10, 11)
4	(0, 25.2, -24)	(4, 6, 8, 7, 9, 12)
5	(4, -10, -16)	(2, 5, 3, 4, 12, 14)
6	(-4, 10, -16)	(13, 9, 7, 5, 2, 1)
7	(-7.5, -7.5, -6)	(9, 13, 11, 10, 14, 12)
8	(7.5, 7.5, -6)	
9	(9, -5, 4)	
10	(-9, 5, 4)	
11	(-14, 0, 4)	
12	(14, 0, 4)	
13	(-14, -5, 4)	
14	(14, 5, 4)	

Table 4: Coordinates of the vertices of the polyhedron in Figure 17, and the list of faces in terms of vertices.

Vertices		Vertices		Faces
1	(3, -12, 9)	11	(0, -12, 12)	(2, 3, 12, 13, 15, 14)
2	(-3, -12, 3)	12	(1.5, -21, -6)	(1, 4, 9, 15, 13, 11)
3	(-3, -12, -6)	13	(-4, -10, 16)	(1, 2, 14, 16, 5, 4)
4	(3, 10.5, 9)	14	(-15, 12, -9)	(9, 10, 17, 16, 14, 15)
5	(-9, 4.5, 3)	15	(-15, 12, 27)	(4, 5, 6, 8, 10, 9)
6	(-6, 3, -6)	16	(-9, 12, -3)	(5, 6, 18, 19, 17, 16)
7	(6, -12, -6)	17	(0, 12, -12)	(7, 8, 10, 17, 19, 20)
8	(6, 3, -6)	18	(-6, -21, -6)	(3, 7, 8, 6, 18, 12)
9	(0, 12, 12)	19	(0, -24, -12)	(11, 13, 12, 18, 19, 20)
10	(24, 12, 12)	20	(24, -12, 12)	(1, 2, 3, 7, 20, 11)

Table 5: Coordinates of the vertices of the polyhedron in Figure 22, and the list of faces in terms of vertices.

Vertices				Faces
1	(-2.666665,	6.666666,	-1.333332)	(7, 8, 9, 21, 20, 22)
2	(-4.235293,	8.235295,	2.117647)	(1, 11, 20, 21, 15, 13)
3	(-2.666666,	-2.222221,	-3.111110)	(1, 2, 3, 4, 14, 13)
4	(-2.666666,	-5.365853,	-3.739836)	(1, 2, 9, 8, 12, 11)
5	(-2.666666,	-3.833332,	-2.037036)	(2, 3, 16, 15, 21, 9)
6	(-2.666666,	-3.833332,	3.166666)	(13, 14, 17, 18, 16, 15)
7	(1.035625,	-1.982186,	-0.535624)	(4, 5, 10, 19, 17, 14)
8	(4.333333,	-0.333332,	4.127334)	(5, 6, 7, 8, 12, 10)
9	(1.044932,	2.955068,	-0.522465)	(3, 4, 5, 6, 18, 16)
10	(-1.117646,	-3.058822,	-1.176469)	(6, 7, 22, 19, 17, 18)
11	(3.866667,	0.133333,	-1.333332)	(10, 12, 11, 20, 22, 19)
12	(4.333333,	-0.333332,	-2.266665)	
13	(-3.212598,	5.826772,	-0.409448)	
14	(-3.839387,	-2.791586,	-0.879540)	
15	(-2.482758,	4.275863,	0.137931)	
16	(-2.666666,	4.000000,	0.000000)	
17	(-1.974025,	-1.532467,	0.519481)	
18	(-2.666666,	-0.666666,	0.000000)	
19	(-0.974358,	-1.769230,	0.256410)	
20	(1.610954,	-0.027788,	0.277890)	
21	(0.840477,	2.376870,	-0.811564)	
22	(0.798084,	-1.527533,	-0.871507)	

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