Small unstretchable simplicial arrangements of pseudolines

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The aim of this note is to present five unstretchable simplicial arrangements of pseudolines, one with 15 pseudolines and four with 16. Three of the five are from [4] (see Figures 4 and 5), while two (Figure 6) have not been published previously. My conjecture is that these are the only unstretchable simplicial arrangements with 16 or fewer pseudolines.

Several words in the above statement require explanation, which will be provided soon. It is my hope that the attractive diagrams, together with the absence of any clear path to the proof or disproof of the several conjectures formulated here, will challenge the curiosity of the reader. Brief indications of the background and history will also be provided.

The setting of the paper is the *real projective* plane $\mathbb{P}^2$. It is very convenient to use as a model of this plane the *extended Euclidean plane* $\mathbb{E}^{2+}$, that is the usual Euclidean plane $\mathbb{E}^2$ augmented by *points-at-infinity* (also known as *ideal points*) that are in 1-to-1 correspondence with families of all mutually parallel lines, together with the *line-at-infinity* that consists of all the points-at-infinity.

An *arrangement* of lines in $\mathbb{E}^{2+}$ is a family of lines, together with the complex formed by the points (*vertices*) at which the lines intersect each other, the segments (*edges*) into which the vertices divide the lines, and the connected components (*cells*) of the complement of the union of the lines. Throughout, we denote the numbers of vertices, edges and cells by v, e, f, respectively. This is illustrated in Figure 1. Two arrangements are *isomorphic*, and considered as not distinct, if there is a dimension- and incidence-preserving 1-to-1 mapping between their vertices, edges and cells.

A *pseudoline* is a simple polygonal line that coincides with a straight line except possibly along a finite segment. Clearly, straight lines are particular cases of pseudolines. A *pseudoline arrangement* is a family of pseudolines that meet pairwise transversally in a single point. An example is shown in Figure 2. A pseudoline arrangement is *unstretchable* provided there is no arrangement of straight lines isomorphic with it. The example in Figure 2 is unstretchable because, by the well-known theorem of Pappus from projective geometry, if the pseudoline were replaced by a straight line, the central point would have to be incident with that line, hence also with the pseudoline shown.

An arrangement (of line or pseudolines) is *simplicial* provided each cell is a triangle, that is, is incident with precisely three vertices and three edges. Examples of simplicial arrangements of lines are shown in Figure 3. At this time three infinite families of such arrangements, as well as 90 individual "sporadic" ones, are known.

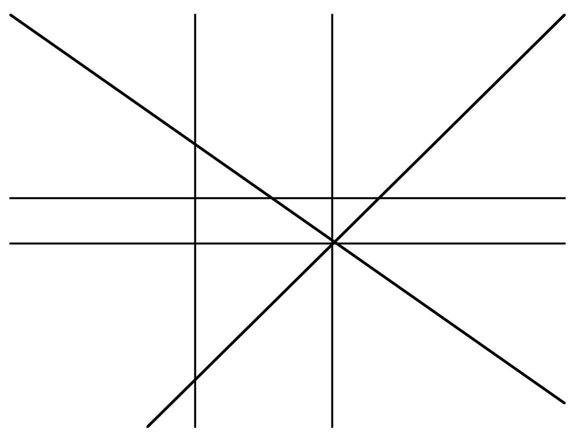


Figure 1. In the extended Euclidean plane, we show the six lines of an arrangement with (v, e, f) = (10, 22, 13). Two vertices are points-at-infinity, four cells cross the line-at-infinity, and v - e + f = 1, as appropriate for complexes in the projective plane. If the line at infinity is included, then the arrangement has seven lines and (v, e, f) = (12, 28, 17); there are four vertices and four edges at-infinity.

It is well-known (see, for example, [2]) that all arrangements with at most eight lines are stretchable. The example in Figure 2 shows that this is a best possible result. This example is not simplicial, and a natural question concerns the smallest size of a simplicial unstretchable arrangement.

The smallest known simplicial unstretchable arrangement is shown in Figure 4. The next smallest known arrangements have 16 lines each, and are shown in Figures 5 and 6.

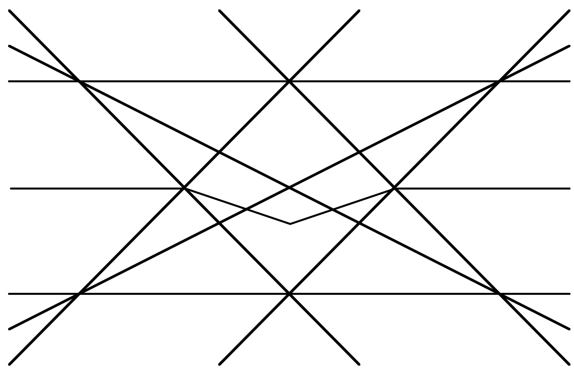


Figure 2. An arrangement of nine pseudolines; note that eight of them are actually straight lines. This arrangement is unstretchable.

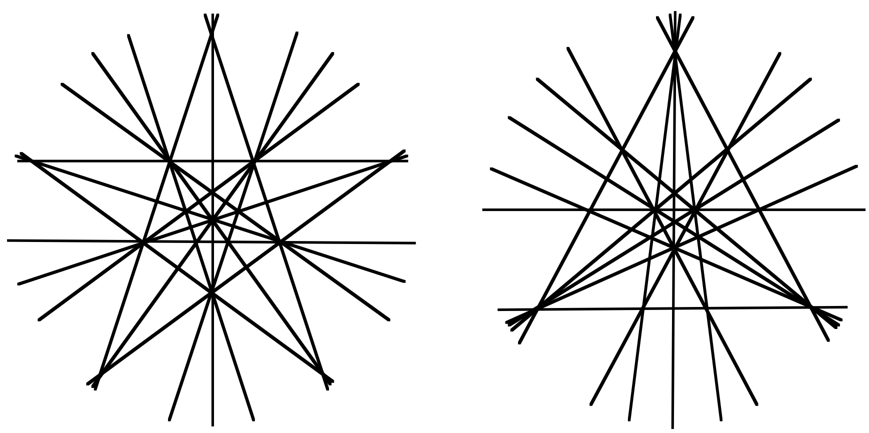
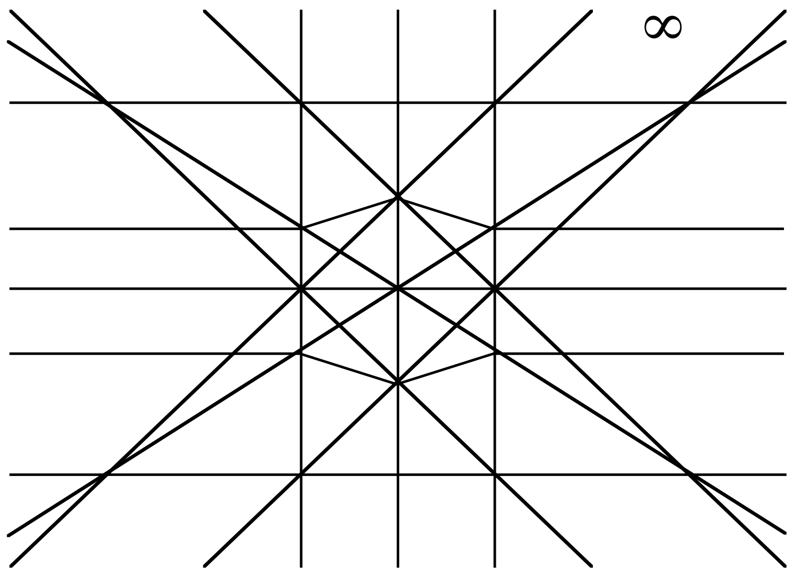


Figure 3. Two isomorphic simplicial arrangements of 15 lines.

The arrangements in Figures 4 and 5 appeared in [4]; the two arrangements in Figure 6 are new. They were found since the publication of [4], and have not been published before.

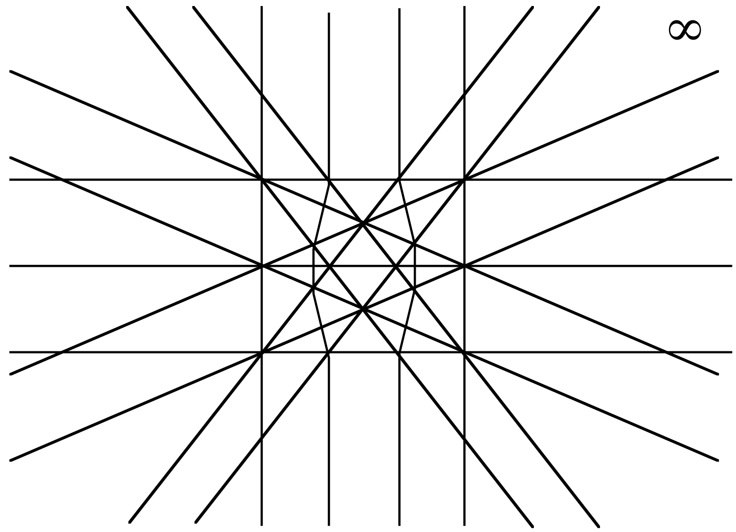
A bit of history. Simplicial arrangements of lines were first considered by E. Melchior [7] in 1941; although the article was published in the infamous journal *Deutsche Mathematik*, its contents is strictly mathematical. A much more extensive collection of simplicial arrangements of lines was published [3] in 1971. An updated and corrected version appears in [5].

Arrangements of pseudolines were introduced by F. Levi [6], and simplicial arrangements of pseudolines were considered in [4]. General constructions of such configurations appear in L. Berman [1].

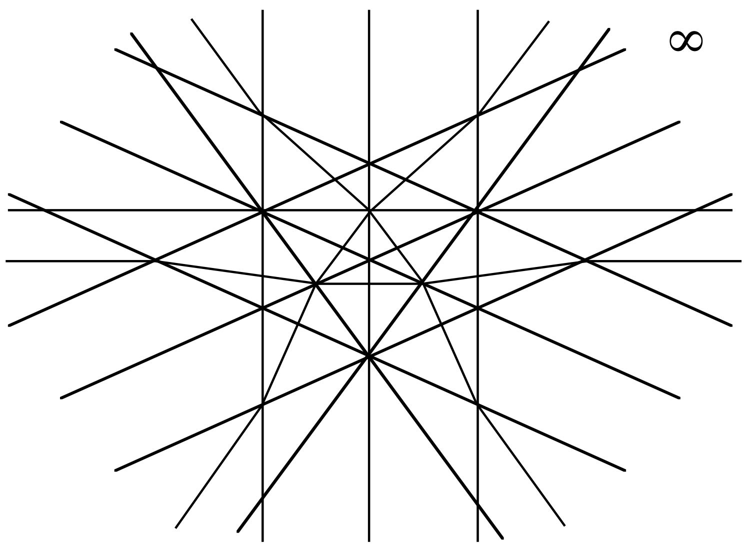


$B_1(15)$

Figure 4. The smallest known simplicial unstretchable arrangement. The symbol ∞ indicates (here, and in the following illustrations) that the line-at-infinity is included in the arrangement. The notation $B_1(15)$ is supplied for ease of reference.

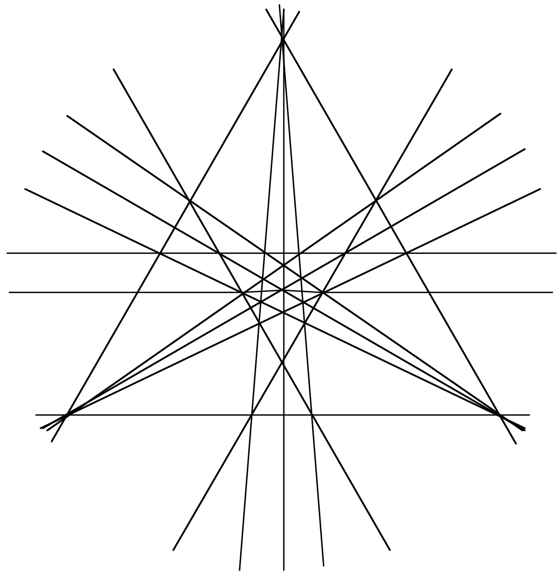


B$_1$(16)

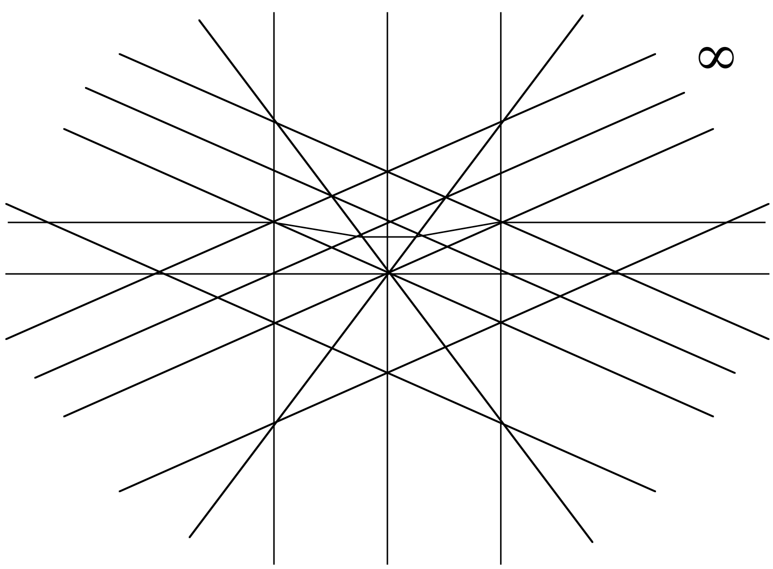


B$_2$(16)

Figure 5. The two simplicial unstretchable arrangements with 16 lines, from [4].



$B_3(16)$



$B_4(16)$

Figure 6. Two new simplicial unstretchable arrangements with 16 lines.

In connection with the material presented here there remain many open questions.

Conjecture 1. All simplicial arrangements with at most 14 pseudolines are stretchable.

Conjecture 2. Up to isomorphism, the only simplicial unstretchable arrangements of 15 or 16 pseudolines are the ones in Figures 4, 5, and 6.

Conjecture 3. There are no simplicial arrangements of straight lines besides the three infinite families and 90 sporadic arrangements listed in [5].

In view of the very different appearance of the isomorphic simplicial arrangements of lines in Figure 3, it is reasonable to ask whether the four arrangements of 16 pseudolines are really distinct. The lack of isomorphism can be seen easily by considering the numerical data in Table 1. Besides the values of (v, e, f) we also give the relevant values of t_k and r_k; the former is the number of vertices incident with precisely k of the pseudolines, and the latter is the number of pseudolines incident with precisely k of the vertices.

	(v, e, f)	(t_2,t_3,t_4,t_5,t_6)	(r_4,r_5,r_6,r_7,r_8)
$B_1(15)$	(33,96,64)	(12,14,6,0,1)	(1,0,8,4,2,0)
$B_1(16)$	(38,111,74)	(14,14,9,1,0)	(0,0,3,11,2)
$B_2(16)$	(37,108,72)	(15,13,6,3,0)	(0,0,7,3,6)
$B_3(16)$	(38,111,74)	(12,20,3,3,0)	(0,0,5,7,4)
$B_4(16)$	(37,108,72)	(13,18,3,2,1)	(0,2,5,4,5)

Table 1. The values of some integer invariants of the five arrangements. These show the essential difference of the four arrangements of 16 pseudolines..

Two aspects of the claims in connection with the unstretchable simplicial arrangements presented above were left without a proof. The first is the question whether they actually exist – that is, whether the imputed incidences actually happen. The other concerns the claim of unstretchability. Both can be resolved, in each case, by selecting suitable points or lines and following up on implied incidences. This shows that the vertices and straight lines meet as indicated, and that the vertices supposed to lie on pseudolines cannot be collinear.

References.

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