Can Every Face of a Polyhedron Have Many Sides ?

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Dedicated to Joe Malkevitch, an old friend and colleague,
who was always partial to polyhedra

Abstract. The simple question of the title has many different answers, depending on the kinds of faces we are willing to consider, on the types of polyhedra we admit, and on the symmetries we require. Known results and open problems about this topic are presented.

The main classes of objects considered here are the following, listed in increasing generality:

Faces: convex n-gons, starshaped n-gons, simple n-gons — for $n \geq 3$.

Polyhedra (in Euclidean 3-dimensional space): convex polyhedra, starshaped polyhedra, acoptic polyhedra, polyhedra with selfintersections.

Symmetry properties of polyhedra P: *Isohedron* — all faces of P in one orbit under the group of symmetries of P; *monohedron* — all faces of P are mutually congruent; *ekahedron* — all faces have of P the same number of sides (*eka* — Sanskrit for "one"). If the number of sides is k, we shall use (k)-isohedron, (k)-monohedron, and (k)-ekahedron, as appropriate.

We shall first describe the results that either can be found in the literature, or obtained by slight modifications of these. Then we shall show how two systematic approaches can be used to obtain results that are better — although in some cases less visually attractive than the old ones.

There are many possible combinations of these classes of faces, polyhedra and symmetries, but considerable reductions in their number are possible; we start with one of these, which is well known even if it is hard to give specific references for precisely the assertion of Theorem 1.

Theorem 1. If P is an acoptic (k)-ekahedron of spherical type with simple polygons as faces, then $k \leq 5$.

Before recalling a proof of this assertion, we need to define the terms that appear in it and may be either not well known or ambiguous.

A *polyhedron* is a family of simple planar polygons which *in the context of this paper* are supposed to be edge-sharing (each edge belongs to precisely two faces), and form a *single circuit* of at least three faces around each vertex. We shall assume throughout that there are *no pairs of coplanar faces* and that the family is *strongly connected*; this last condition means that any two faces are connected by a chain of faces in which adjacent terms share an edge.

A polyhedron is said to be *acoptic* (that is, not selfintersecting) if it is homeomorphic to a compact manifold. If that manifold is a sphere, we say that the polyhedron is of *spherical type*. A polyhedron or polygon P is *starshaped* if there is a point x of P such that for each point y on the boundary of P the open segment xy does not meet the boundary of P. For spherical polygons the meaning of *starshaped* is the same but with arcs of great circles instead of segments.

With these definitions and understandings the proof is an immediate consequence of Euler's theorem, which implies that

$$3\,p_3 + 2\,p_4 + p_5 \geq 12 + \sum_{k \geq 6} (k-6)\,p_k\,,$$

where p_j is the number of faces with j sides. Hence each polyhedron in Theorem 1 has some faces with five or fewer sides, and thus $k \leq 5$.

On the other hand, the example of the regular (Platonic) pentagonal dodecahedron shows that the bound in Theorem 1 can be attained even under the much stronger assumptions of convexity of faces and polyhedra, and isohedrality (single orbit of faces under symmetries of the polyhedron). Moreover, there are many other polyhedra that are acoptic and (5)-isohedral with either convex pentagonal faces (an example is shown in Figure 1) — or with non-convex but starshaped faces; see Figure 2, taken from [11], where many other illustrations are shown as well.

Even staying with convex pentagons, by gluing together two or more isohedra of the same kind we can get many other numbers F of faces for (5)-monohedra. On the other hand, for each F ≥ 60 there are (5)-ekahedra with convex faces that are topological spheres. As shown recently [12], for each F ≥ 12 (except possibly F = 14) there are acoptic convex-faced (5)-ekahedra with F faces that are topological tori.

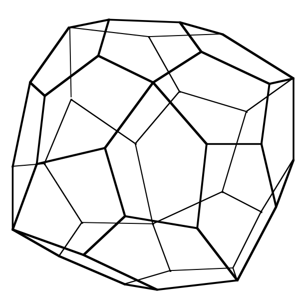


Figure 1. An acoptic isohedron with 24 pentagonal faces. It is a pentagonal icositetrahedron, a dual to the snub cube.

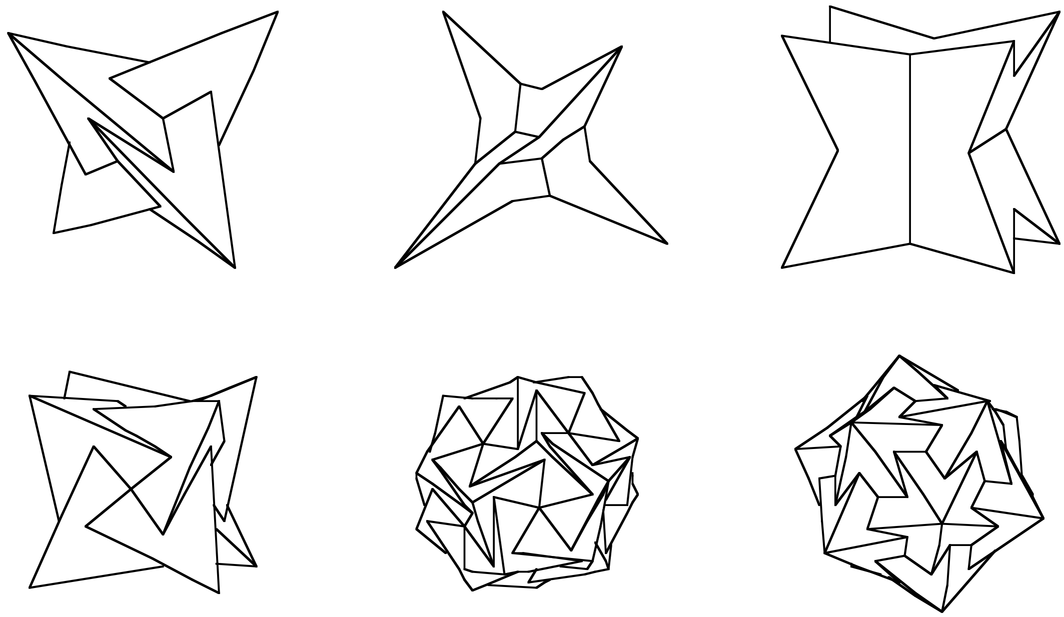


Figure 2. Acoptic, starshaped isohedra with 12, 24 or 60 starshaped pentagonal faces. From [11].

In contrast to Theorem 1, admitting nonconvex (but starshaped) faces makes it possible to obtain acoptic (6)-ekahedra that are topological tori, for many values of the number F of faces; for more details see [12].

If we admit (k)-isohedra with selfintersections, the bound in Theorem 1 does not apply even if we insist on convex faces. It is well known that $k = 6$ is possible; see Figure 3 which shows the *small triambic icosahedron* [19, p. 46], also known as the *triakis icosahedron* [5, p. 271]. However, even among selfintersecting polyhedra I do not know of any other isohedron with convex hexagonal faces, and I venture the following guesses:

Conjecture 1. There are no isohedra with convex hexagonal faces other than the small triambic icosahedron.

Conjecture 2. There are no (k)-isohedra with convex faces for any $k \geq 7$.

By gluing together copies of the triakis icosahedron one can obtain (6)-monohedra (with selfintersections) and arbitrarily large numbers of convex hexagonal faces. Wills [21] presents a very symmetric 3-valent acoptic (9)-ekahedron with 24 non-convex faces. As established by McMullen *et al.* [15], [16], there are acoptic (k)-ekahedra with convex faces, for arbitrarily large k.

Are there convex-faced acoptic (but not necessarily convex) *monohedra* with (*congruent*) hexagonal faces? Monohedra with simple but not necessarily convex hexagonal faces? I conjecture that there are none.

Conjecture 3. There exist no acoptic (k)-monohedra with simple faces for any $k \geq 6$.

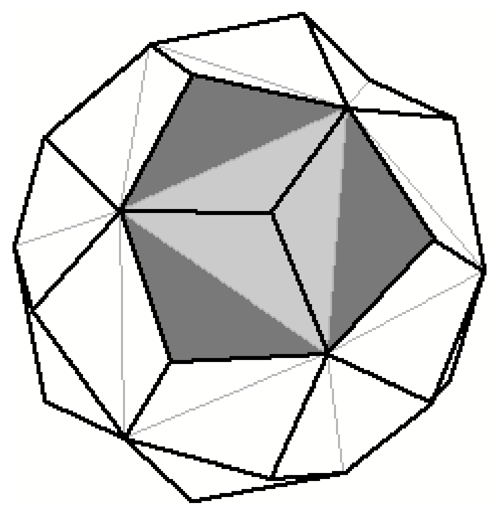


Figure 3. The *small triambic icosahedron* (also known as *triakis icosahedron*) is a selfintersecting isohedron with 20 convex hexagonal faces. One such face is shaded; the lighter shading indicates the part of that face which is "hidden" by the union of three other faces. This polyhedron can be interpreted as arising from a regular icosahedron by erecting on each face a 3-sided pyramid of such height that appropriate triangles become coplanar and determine a hexagon. It can also be described as the first stellation of the icosahedron. (See, for example, [5], [3], [19].) This orientable polyhedron has 60 edges and 32 vertices, hence it is of genus $g = 5$. (Note that the genus g of an orientable polyhedron is determined by its Euler characteristic; in the present case $2(1–g) = 20 – 60 + 32 = –8$, hence $g = 5$. A basic theorem of topology of surfaces implies that a polyhedron of genus g is a continuous image of a sphere to which g “handles” have been added.)

What changes occur if we do not insist on polyhedra that are acoptic — that is, if we admit selfintersections — but still require the faces to be simple and the polyhedron to be isohedral? This leads both to many polyhedra besides the triakis icosahedron, and to isohedra that have faces with more than six sides. The presentation of the known results is the main aim of the present note.

There are two distinct constructions, both fairly general, of such isohedra; we shall describe them next.

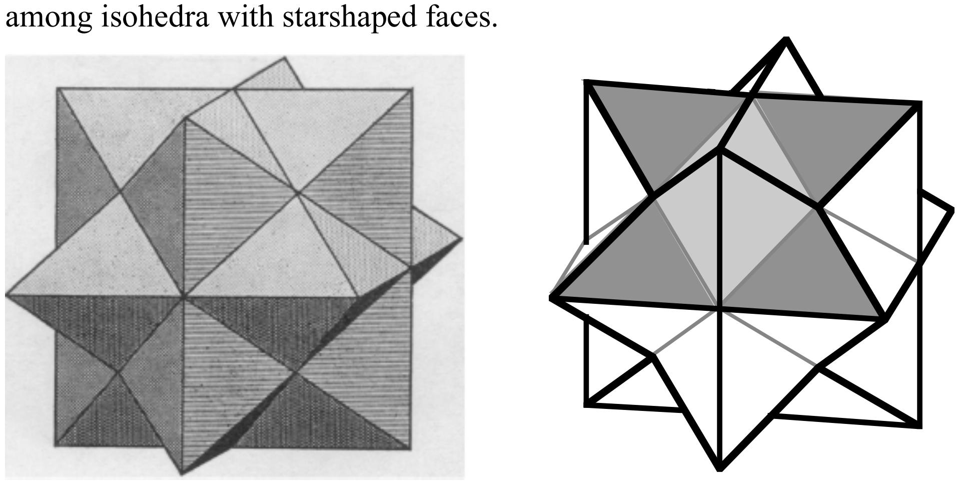
The first is based on the well-known stellation of given isohedra as an intermediate step. In many cases, from a stellation (which has a variety of "visible" pieces) we unite the coplanar parts with the portion of the plane they enclose, to reach a simple polygon. (In most situations that interest us, this polygon is starshaped.) We say that the isohedron that results by applying the symmetries of the starting isohedron is *isomeghetic* (that is, of equal extent) with the particular stellation used. The triakis icosahedron is a simple example of such constructions.

The isohedron of Figure 4a appears as the "first stellation" of the rhombic dodecahedron (see [6, page 127, or [13, Fig. 2] where an editorial comment states that it is "of course well known"). Our construction yields an isomeghetic (6)-isohedron with non-convex hexagons as faces shown in Figure 4b. In another guise, a different isomeghetic polyhedron appeared as one of the parallelogram-faced isohedra in [7, Fig.1]. In [2, Plate X, Fig. 13] another isomeghetic polyhedron with 24 quadrangles as faces is presented.

The "second stellation" of the rhombic dodecahedron (Figure 5, from [13, Fig. 3] can be interpreted as an (8)-isohedron with twelve nonconvex (simple) octagons as faces. (The editorial statement accompanying [13], to the effect that this polyhedron can be found in [2] appears based on a misunderstanding.) This polyhedron has 48 edges and 30 vertices, hence genus 4.

The two star-polyhedra (Figure 6) found by Kepler are stellations of the Platonic dodecahedron; they can be interpreted as selfintersecting (10)-isohedra having starshaped but nonconvex faces.

The second stellation of the icosahedron ([5, p. 271], [3, Plate 1.C], [19, p. 43], [14, No. 02]) which is also the compound of five octahedra, is isomeghetic to an isohedron with twenty faces and icosahedral symmetry, each face of which is a star-shaped 12-gon (Figure 7a). The final stellation of the icosahedron ([5, Plate 10], [3, Plate 3], [19, p. 65], [14, No. 59]) can be interpreted as a starshaped (18)-isohedron with 20 starshaped faces, (Figure 7b); see also Remark 3. Several other stellations of the icosahedron can be interpreted as (18)-isohedra. However, neither of these has faces with maximal number of sides, among isohedra with starshaped faces.



(a) (b)

Figure 4. (a) A polyhedron, identified as the first stellation of the rhombic dodecahedron; from [13]. (b) An isomeghetic (6)-isohedral polyhedron with twelve nonconvex hexagons as faces; see Remark 3 for an explanation of the term "isomeghetic". One of them is shown shaded; the part that is shaded in lighter color is obscured by parts of other faces. It has 36 edges and 20 vertices, and is therefore of genus 3.

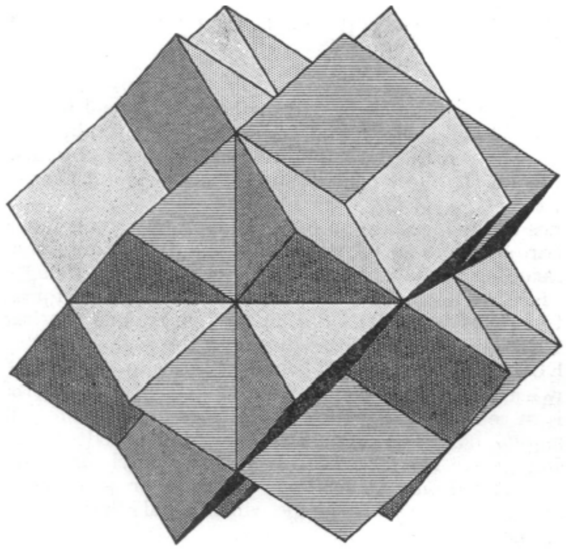
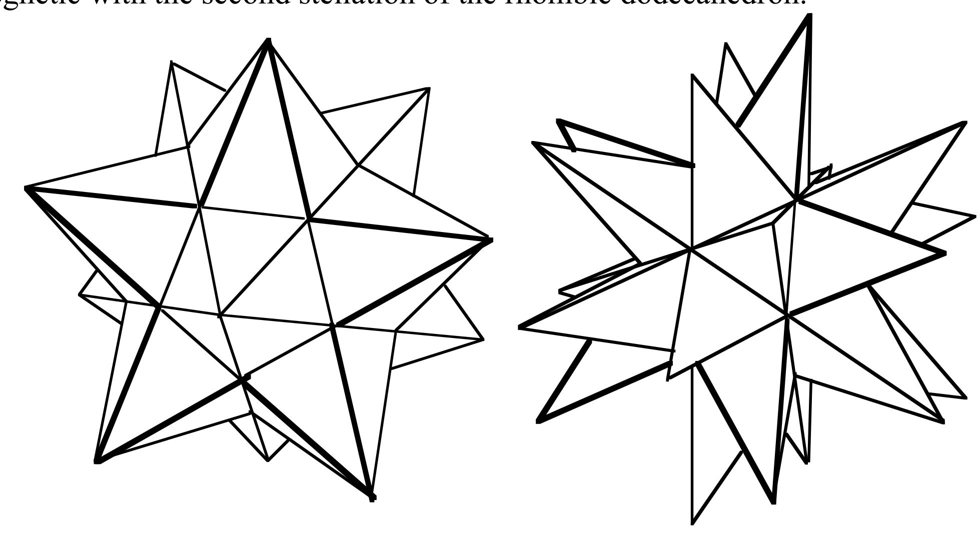


Figure 5. An (8)-isohedron with 12 star-shaped faces, each interpretable as the union of two squares that overlap in a sub-square with side equal to half those of the squares; it is isomeghetic with the second stellation of the rhombic dodecahedron.



(a) (b)

Figure 6. (a) A small stellated dodecahedron. (b) A great stellated dodecahedron. Each of these two regular star-polyhedra of Kepler can be interpreted as consisting of 12 star-shaped decagons as faces. One face is heavily drawn in each polyhedron.

The largest known number of sides of faces of an isohedron occurs in another polyhedron, one that is isomeghetic to a different stellation of the icosahedron. It is the stellation denoted E by [3, Plate II] and [5, page 273], and called No. 09 by Maeder [14]. The starshaped face is shown in Figure 8a; it has 30 edges, and this is the maximal number among all known isohedra. We show no image of it since it is quite confusing and unhelpful.

Conjecture 4. (k)-Isohedra with simple polygons as faces have k ≤ 30..

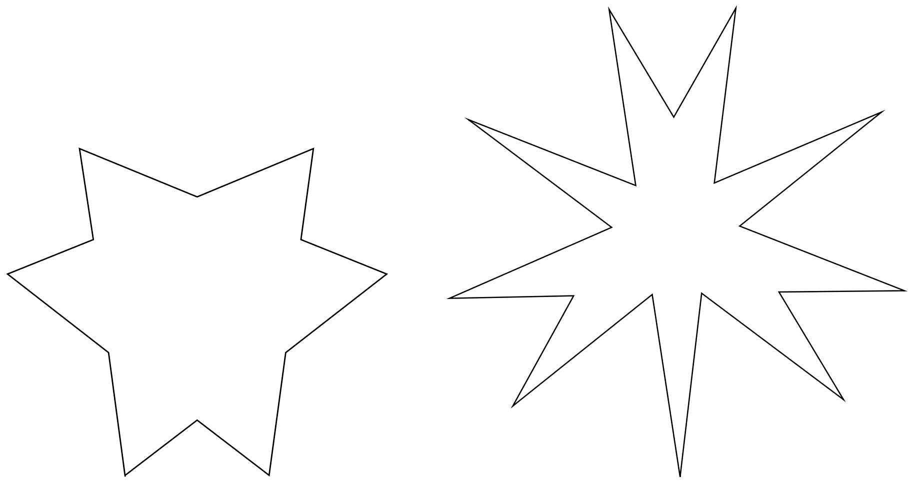
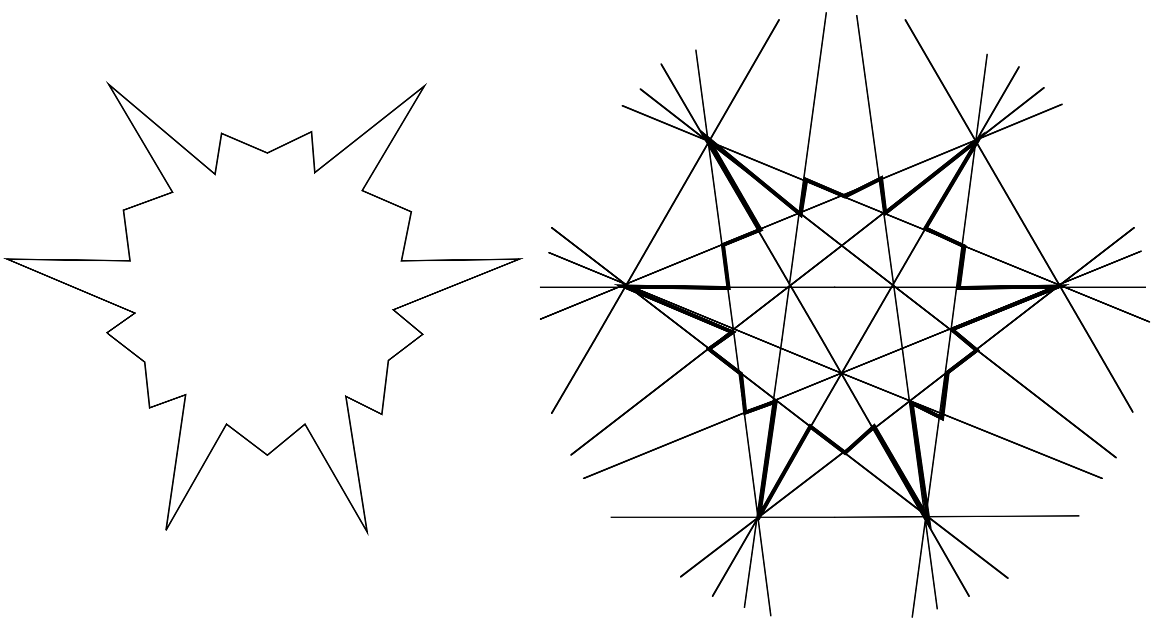


Figure 7. Two star-polygons with 12 resp. 18 edges, that can be used to construct isohedra with 20 faces and icosahedral symmetry, that are isomeghetic with the second resp. final stellation of the icosahedron.



(a) (b)

Figure 8. (a) The face of an isohedron obtained from the stellation denoted E in [5] and [3]. It has 30 sides, which is the largest known number of sides of a face of an isohedron. (b) Part of the stellation pattern of the icosahedron, with the face in (a) superimposed on the lines of the pattern.

The second general method of construction of isohedra with many-sided faces uses *Möbius nets*, obtained as follows.

The octahedral group of symmetries is generated by reflections in the nine planes of mirror symmetry of the cube (or the regular octahedron); similarly, the icosahedral group is generated by the 15 planes of mirror symmetry of the regular dodecahedron or icosahedron. The tetrahedral symmetry is generated by the six mirrors, each determined by an edge and the midpoint of the opposite edge. In each case, if these planes are intersected by a sphere centered at their common point, the resulting great circles generate the octahedral (resp. icosahedral, resp. tetrahedral) *Möbius net* on the sphere. Each such net is a tiling of the sphere by congruent triangles, and the symmetry group involved acts transitively on these triangles. These three nets are shown in Figure 9; Figure 10 shows stereographic projections of the Möbius nets, which are sometimes more convenient to use. In all this we are disregarding the similarly defined Möbius nets of the dihedral symmetry groups, since these are not interesting in the present context.

This technique was apparently first used in [7], and has since found application in several other works [4], [12], [17]... . (The use of Möbius nets to construct isogonal polyhedra, and in particular, uniform ones, is well known.) The idea of the construction is very simple.

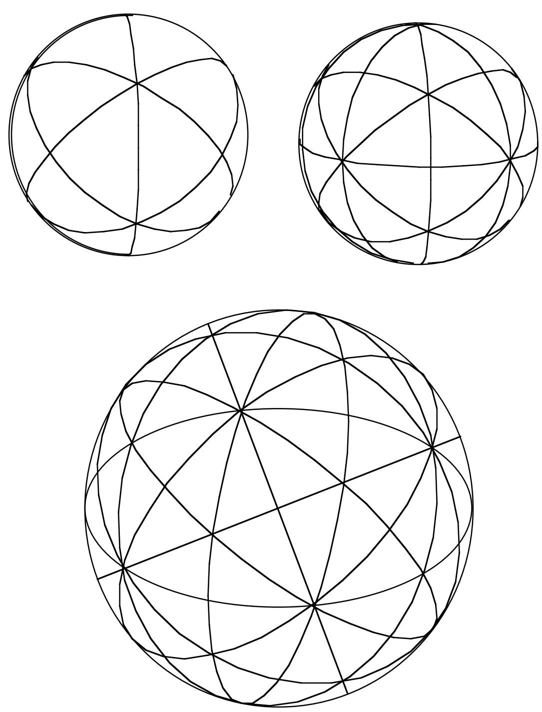


Figure 9. The tetrahedral, octahedral, and icosahedral Möbius nets.

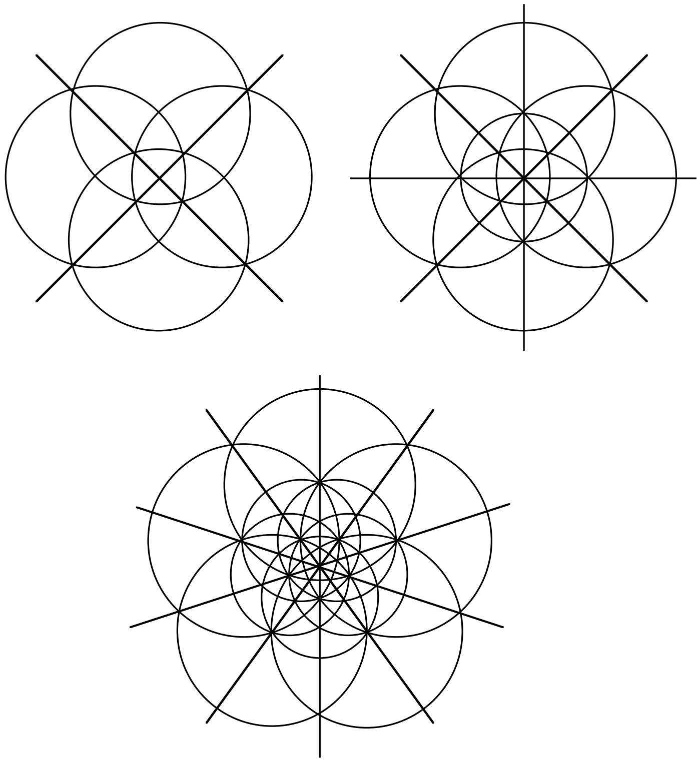
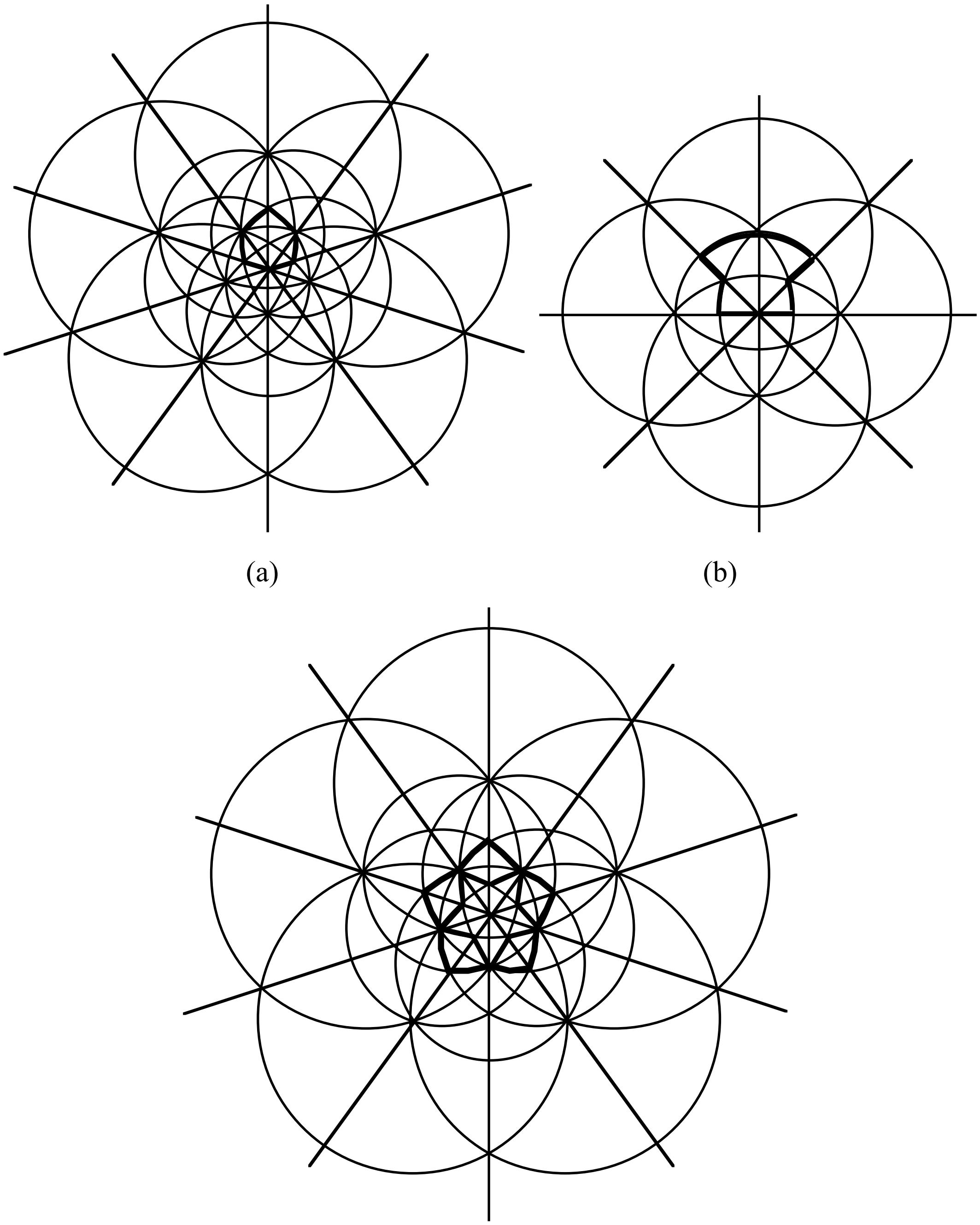


Figure 10. Stereographic projections of the three Möbius nets.

In one of the three Möbius nets, let S be a union of triangles contained in an open hemisphere, with the boundary of S a simple spherical polygon. Then images of S under the group generated by reflections of the net yield an isohedral tiling of the sphere. If S is projected onto a plane such that no two vertices are at the same distance from the center of the sphere, we obtain a simple polygon P. The reflections in the planes of symmetry then generate either a strongly connected isohedral polyhedron, possibly with selfintersections, or else several such polyhedra. In our search for polyhedra with large faces this construction is of the greatest interest. Clearly, if S is starshaped on the sphere, P is starshaped in its plane.

Several of the examples mentioned above can be obtained by this method — although they were originally found through other constructions. This is shown in Figure 11 by the sets S that correspond to the polyhedra in Figures 3, 4 and 6. In the following figures we show sets S in the three Möbius nets that lead to isohedra with faces that have many edges — in fact, in the tetrahedral and octahedral symmetry, more than any of the previously known examples.

We start with a set S in the tetrahedral net (Figure 12), leading to an isohedron with 24 hexagonal faces that are starshaped (Figure 13). Since an open hemisphere contains at most seven vertices of the tetrahedral Möbius net, the possibility of heptagonal faces is not excluded *a priori*. However, a detailed (but tedious) analysis shows that six is the maximal possible number of edges in the boundary of a set S. This establishes:



(a) (b)

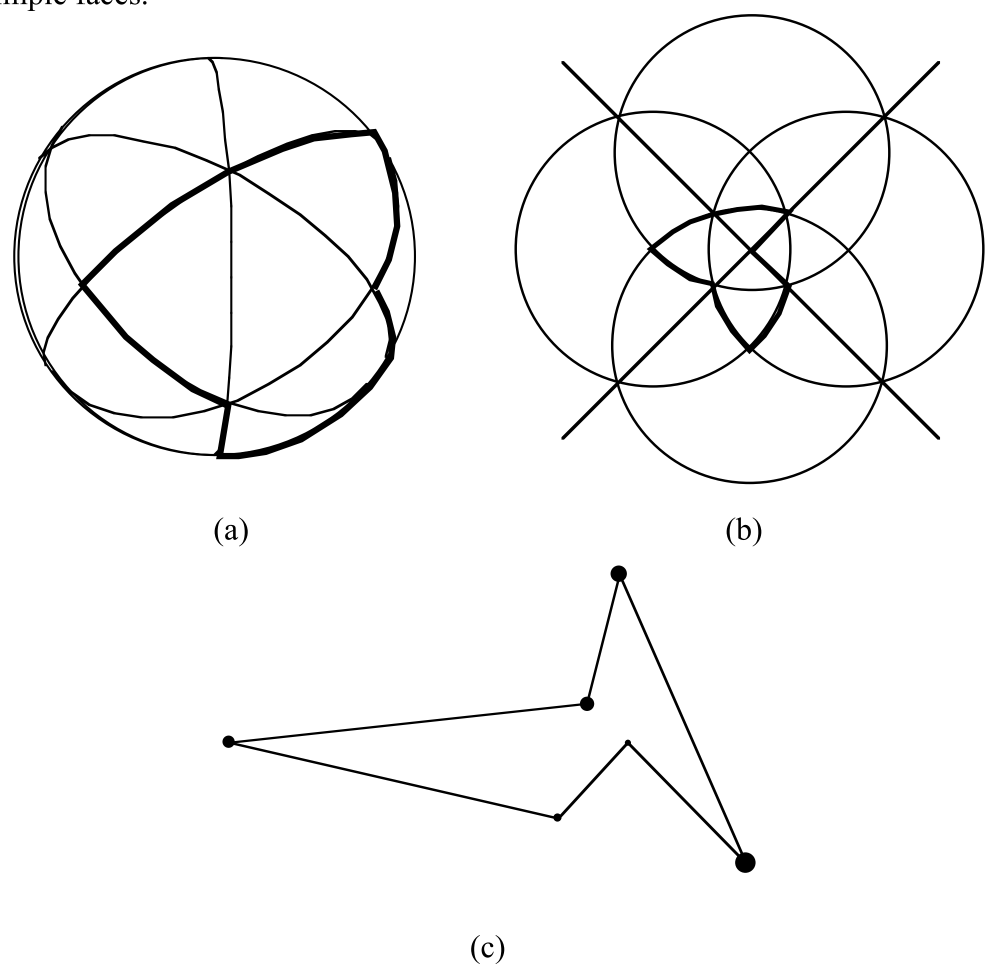
(c)

Figure 11. Examples of the Möbius nets construction that yield (a) the triakis icosahedron in Figure 3, (b) the polyhedron in Figure 4, and (c) the two regular star polyhedra of Kepler shown in Figure 6.

Theorem 2. The 24-faced (6)-isohedron described in Figures 12 and 13 has starshaped hexagonal faces; six is the largest number of sides of faces of an isohedron with tetrahedral symmetry.

Turning now to isohedra with octahedral symmetry, we note that the highest number of sides in previously known polyhedra is eight, achieved by the polyhedron in Figure 5. However, the construction using Möbius nets yields isohedra with more sides to each face. In Figure 14a we show a starshaped set S with eleven edges, and in Figure 14b a simple but non-starshaped set with 12 edges. In each case, our construction yields an isohedron with 48 faces. The depiction of these polyhedra is beyond my abilities.

Conjecture 5. The isohedra obtainable from the sets S in Figure 14 attain the maximal number of sides possible for isohedra with octahedral symmetry and starshaped resp. simple faces.



(a) (b)

(c)

Figure 12. (a) The unique polygon S in the tetrahedral Möbius net that has edges in the mirror lines, is contained in an open hemisphere, and has the maximal possible number of edges (six) among all such polygons. (b) The same polygon S in a stereographic projection of the Möbius net. (c) A starshaped planar polygon P obtained by projecting S onto a plane parallel to the equator of a hemisphere that contains S.

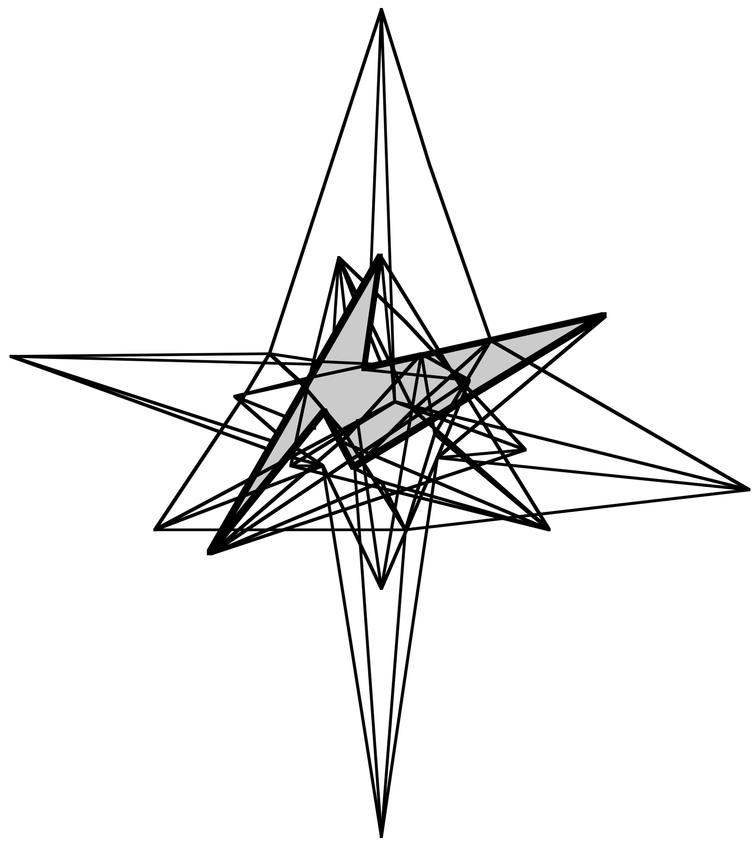
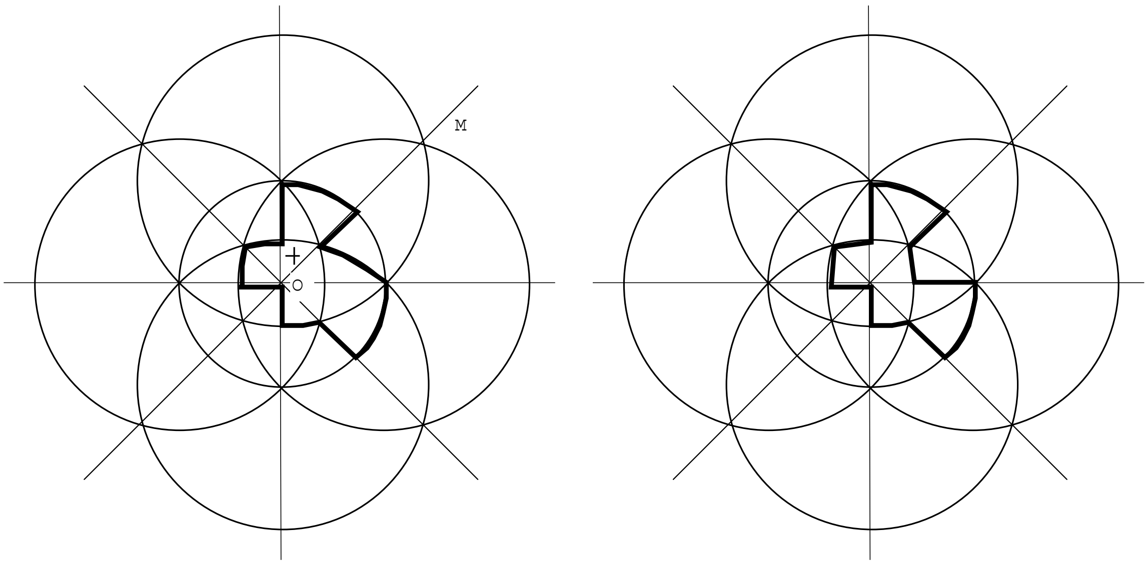


Figure 13. The (6)-isohedron with 24 faces generated from the polygon P in Figure 12(c) by the reflections in the mirrors of the tetrahedral Möbius net. All edges of the polyhedron are shown, but only one face is shown filled-in, since presenting all faces leads to an unintelligible figure.

While the method of Möbius nets yields the optimal available results in cases of tetrahedral and octahedral symmetry, the situation changes in the icosahedral case. The stellation E of the icosahedron, described above and illustrated in Figure 8 has starshaped faces with the maximal known number of sides (thirty), even among all isohedra with simple faces. The best I could find for the icosahedral Möbius net is the starshaped set S in Figure 15, with 19 sides, and the simple set S in Figure 16 with 27 sides.

Conjecture 6. The isohedra obtainable from the sets S in Figures 15 and 16 attain the maximal number of sides possible for isohedra arising from Möbius nets with icosahedral symmetry, and with starshaped resp. simple faces.



(a) (b)

Figure 14. (a) A starshaped polygon S in the octahedral Möbius net, with 11 sides. An isohedron with 48 faces P results by the method outlined in the text. (b) A simple, not starshaped, polygon S in the octahedral Möbius net, with 12 sides; it also leads to an isohedron with 48 faces P. These are the polyhedra with faces having the maximal known numbers of sides among those with octahedral symmetry and starshaped resp. simple faces.

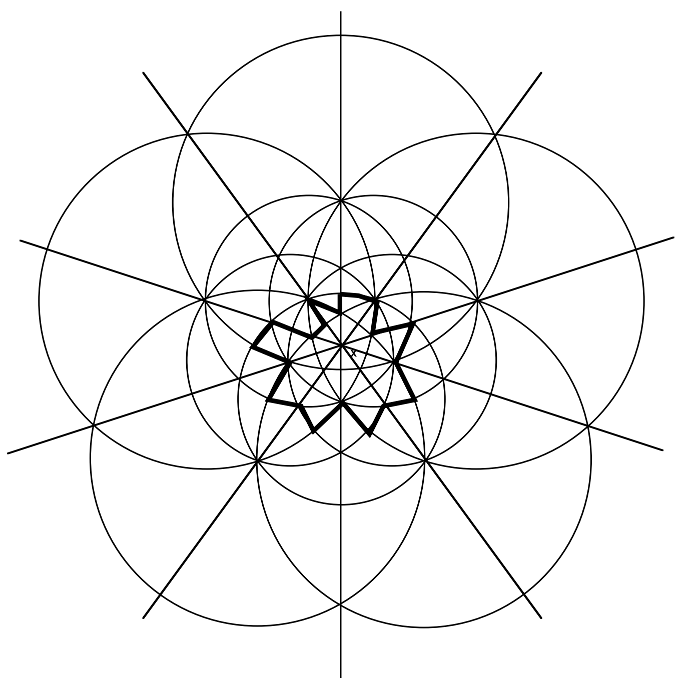


Figure 15. A starshaped set S in the icosahedral Möbius net, with 19 sides.

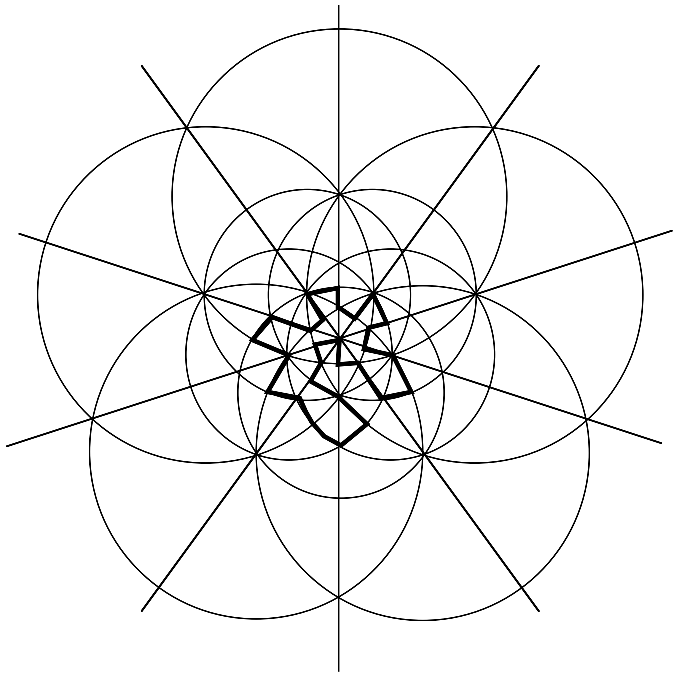


Figure 16. A simple set S in the icosahedral Möbius net, with 27 sides.

Remarks.

1. One of the results of McMullen *et al.* [15], [16], is the existence of 4-valent ekahedra with rather large genus, and with convex faces that have arbitrarily many sides. This is in contrast with the well-known result (see, for example, [1]) that 3-valent polyhedra with convex faces have genus 0 even without any symmetry assumption or restriction on the numbers of sides of the faces.

2. A related result explains why the above considerations did not include polyhedra with selfintersecting faces. In [8] isogonal prismatoids (polyhedra such that dihedral symmetry groups act transitively on their vertices) are constructed that are ekahedra with quadrangular faces and arbitrarily large valences. The polars of these polyhedra are isohedral, with dihedral symmetry groups and 4-valent vertices; they have faces of arbitrarily many sides – however, these faces are selfintersecting. It follows that they are not of any particular interest concerning questions we discussed here.

3. A recent discussion on the Internet prompts the following elaboration on the statements above concerning Kepler's star-polyhedra and the polyhedron discussed in connection with the 18-gon of Figure 7.

In each of these cases, as well as in many other instances, there are at least three different ways of looking at the polyhedral object. In the case of the small stellated dodecahedron (Figure 6a, see Figure 17) we can consider as faces the 12 pentagrams (one is emphasized below at left), or the 12 starshaped decagons (middle) that we selected in the text, or the 60 triangles (below right) that constitute the surface of the solid object. We ignored the first interpretation (which is the one under which the resulting polyhedron is regular) since we did not wish to consider as faces polygons with selfintersections. We also ignored the last interpretation, since we are interested in faces with a large number of sides. The term 'isomeghetic' (from Greek "meghethos" – extent) is used to indicate the relationship between these three interpretations of an object, and in other similar cases. The most important aspect of this situation is the realization that each of the interpretations is valid in its own context. This makes inappropriate the frequent controversies about the "right" and "wrong" interpretations — a controversy that goes back at least to Brückner's comments about Dostor, see [2, p. 16].

Similar triplet of interpretations can be given for the (18)-isohedron with faces one of which is shown in Figure 7b. As shown below in Figure 18, each face can be understood as a selfintersecting 9-gon, or as a starshaped 18-gon, and the resulting polyhedron can also be interpreted as a solid bounded by 180 skinny triangles of two shapes. The last interpretation is the one connected with the term echidnahedron (apparently introduced recently) for the last stellation of the icosahedron, see, for example, [20] or [18].

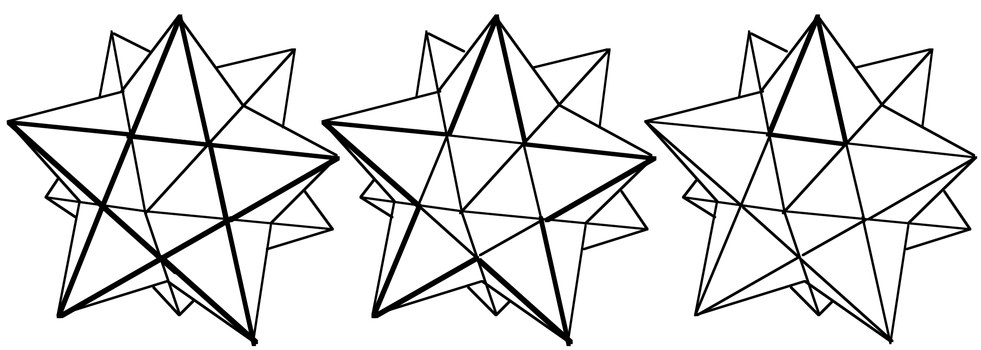


Figure 17. Three interpretations of the small stellated dodecahedron.

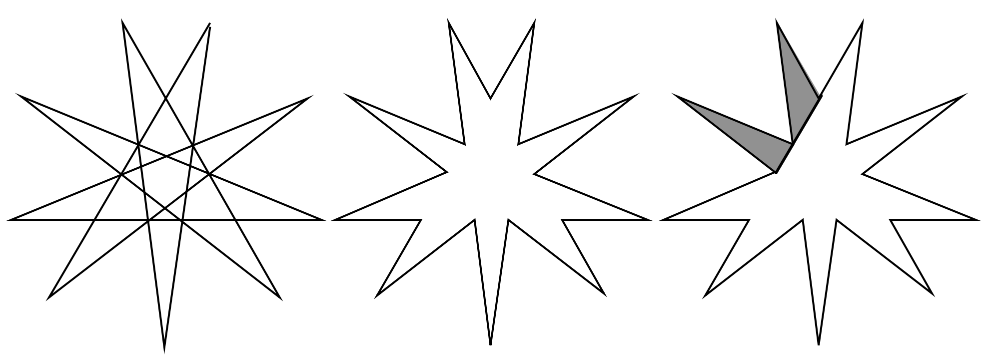


Figure 18. The three interpretations of the faces of the echidnahedron (the last stellation of the icosahedron).

4. When the number in Theorem 2 and the results on isohedra obtained from the octahedral Möbius net are considered together with the numbers of vertices in the respective nets, the following conjecture is almost inevitable:

Conjecture 7. The maximal number of sides of faces of isohedra with simple faces equals $r/2 - 1$, where r is the number of vertices in the corresponding Möbius net.

These numbers are 14, 26 and 62 respectively, and the explanation for the factor ½ is that any open hemisphere contains at most one half of the vertices. The subtraction of 1 is somewhat mysterious, but probably can be justified. As mentioned after Theorem 2, the bound is attained for the tetrahedral net. Conjecture 6 and Figure 14b show that the conjectured bound is attained for the octahedral net as well. The situation in the case of the icosahedral symmetry is surprising: the bound (in the present conjecture, and in Conjecture 4) is attained — but as far as known, only by an isohedron that is not arising by the Möbius net construction.

5. Interesting problems arise in the Möbius net construction we used if the set S of spherical triangles has any symmetries. In such a situation, if the plane used to construct the polygon P respects these symmetries, several outcomes are possible. In one view, the construction described above yields isohedra with sets of coinciding but distinguishable faces, that are adjacent by the same rules as the isohedra resulting from polygons in planes that are in general position (the kind of planes used above). This leads to isohedra that are polyhedra in the sense described in [9] and [10], more general than the definition used here. This interpretation can be justified by requiring continuity of the isohedra for continuous changes of the plane onto which the set S is projected. In the other view, the coinciding sets of faces are identified, and isohedra with smaller numbers of faces are produced. This comes at the price of loss of continuity. For example, "excavating" pyramids on the faces of a regular tetrahedron leads to polyhedra that are combinatorially equivalent to the triakis tetrahedron. However, in the approach of Shephard [17], this is not true for the case the excavating pyramids have equilateral triangles as faces. See [10] for additional comments and illustrations.

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