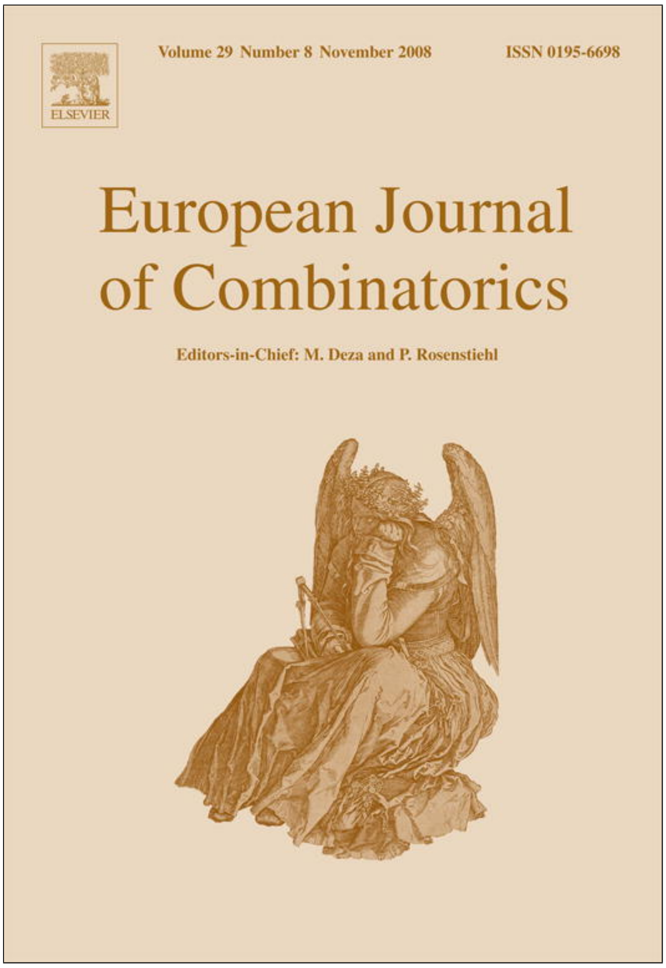
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Musings on an example of Danzer's

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Dedicated with best wishes to Ludwig Danzer, a friend for almost half a century, on his eightieth birthday

Abstract

Some twenty years ago Ludwig Danzer provided the example of a (35_4) geometric configuration for which no visually attractive presentation was found despite its great combinatorial symmetry. For a long time no other (35_4) configurations was known. This changed recently, and the newly found symmetric configurations, and missteps in finding them, are presented in some detail.

1. Introduction

Some twenty years ago, John Rigby and I were preparing our paper [9] for publication. This paper contained the first graphic depiction of several configurations (n_4), among them the configurations (21_4) and (24_4). The former (reproduced here as Fig. 1) had been studied extensively, starting with F. Klein in 1875; see Coxeter [4] for details of the history. It had been the subject of many representations, but none by points and lines in the real Euclidean plane. Our representation of these and some other (n_4) configurations led us to consider whether every configuration of this kind that is realizable in the Euclidean plane has n divisible by 3. A counterexample to this possibility was communicated to us by L. Danzer. It is a configuration (35_4), obtained by choosing seven 3-spaces in general position in the 4-dimensional space; they meet by fours and by threes in 35 points and 35 lines, which form the configuration (35_4) in 4-space. A projection yields the desired configuration in the plane.

Combinatorially, Danzer's configuration can be interpreted as defined by all 3-sets and all 4-sets that can be formed by the elements of a 7-element set; each "point" is represented by one of the 3-sets, and it is incident with those lines (represented by 4-sets) that contain the 3-set.

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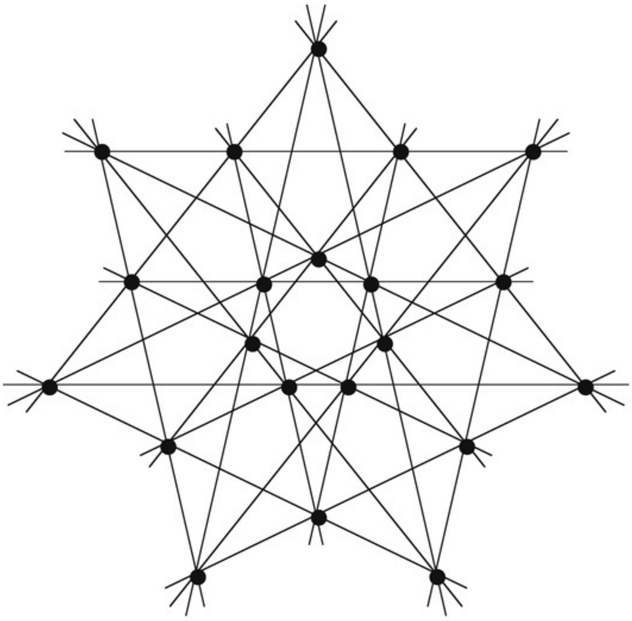

Fig. 1.

Following the publication of [9], the study of geometric configurations (n_4) gradually took off, with reasonably nice geometric representations found for many values of n. Details can be found in [7] and [8], where references to other publications may be found as well. However, the case $n = 35$ resisted any such representation till very recently. The following pages will present such representations, after detailing some unsuccessful – but educationally valuable – attempts. It seems that any representation of Danzer's configuration is of necessity so cluttered and unhelpful for visualization that no attempt to present it has ever been made.

2. The first approach

Generalizing the constructions in [9], Boben and Pisanski [2] developed the theory of *polycyclic configurations*. As a thumbnail sketch of part of their idea we may say that such a configuration (n_4) with $n = k \cdot m$, consists of k concentric sets of m points each, where each set is formed by the vertices of a regular m-gon, the sizes and positions of the various sets being such that quadruplets lie on lines with which they form a configuration (n_4). (See below, in Remark (iii) additional explanations of the concepts involved here.) A notation for these polycyclic configurations (n_4), introduced in [2] and slightly modified in [7], is of the form $m(s_1, t_1, s_2, t_2, \ldots, s_k, t_k)$, where the integers s_i and t_i describe the configuration in a specific way, and are subject to various constraints that do not need to occupy us here. Details are available in [2] and [7]. As an example, the (21_4) configuration in Fig. 1 has description 7(3,2,1,3,2,1).

It is easy to guess that since $35 = 5 \cdot 7$, a polycyclic configuration with five sets of vertices of regular heptagons would yield an easily representable configuration (35_4). In fact, Fig. 2 shows the configuration 7(2,1,2,1,3,2,1,2,1,3), as drawn with a program in Mathematica®. At first glance it seems to fulfill the wish to have a simple and graphically understandable interpretation of a (35_4) configuration. However, the attentive reader may notice a difference in the way configurations are presented in Figs. 1 and 2. The latter is drawn according to the tradition started with the earliest drawings of any configurations, and prevailing with many writers to this day. In this tradition the lines of the configurations are presented as the shortest segments that connect

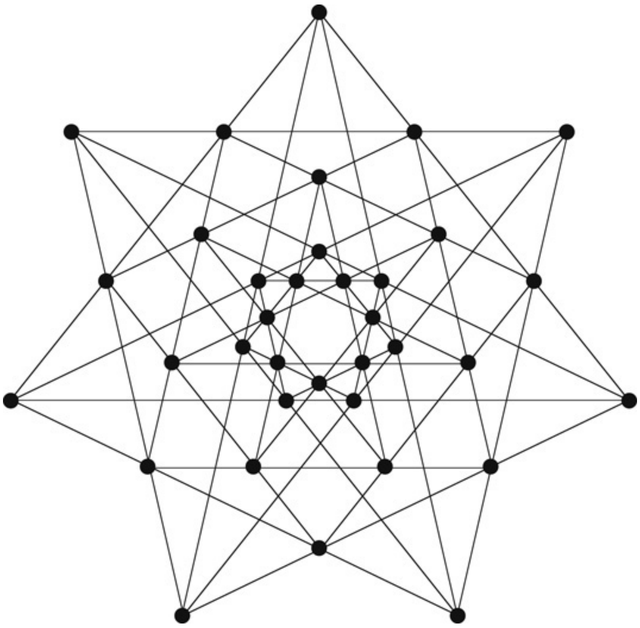

Fig. 2.

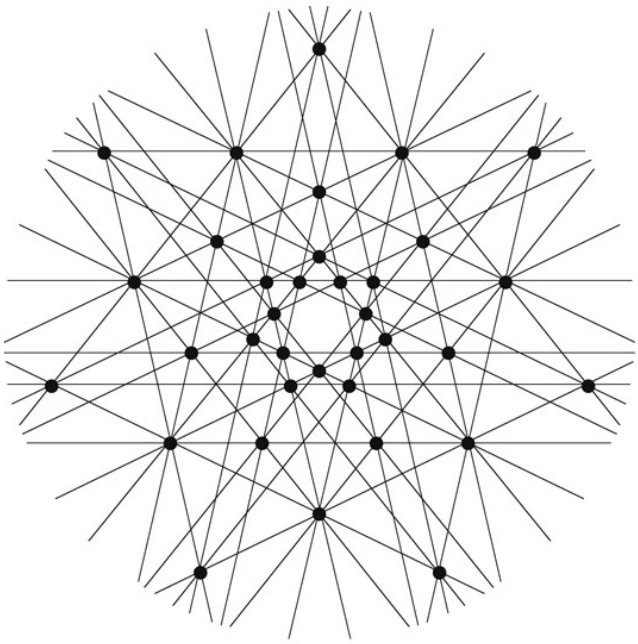

Fig. 3.

all the points that are supposed to be incident with each line. But this representation is sometimes misleading. Fig. 3 shows the same points and lines as are present in Fig. 2, but with lines extended beyond all relevant points of the configuration. As is visible from Fig. 3, these points and lines do not form a configuration since some lines pass through six of the points, and some points are on six of the lines. This situation is a consequence of Lemma 31 of [2]. It is possible that a lessening of symmetry in Fig. 3 may lead to a proper configuration, but this has not been achieved so far. It may be noted that there are two other polycyclic symbols for (35_4) that may be considered, but both lead to the same type of non-configuration as shown in Fig. 3.

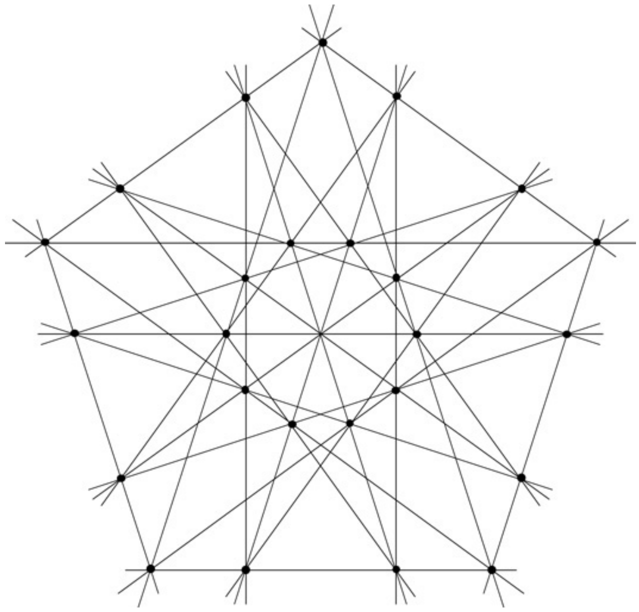

Fig. 4.

3. New construction methods

Following the failure of finding reasonable presentations of any polycyclic configuration (35_4) (in the sense of Section 2) it seemed doubtful that any such presentation could be found. This changed early in 2006, when Jürgen Bokowski found a new method for constructing a configuration (25_4), with 5-fold symmetry but not falling within the scope of polycyclic configurations as defined above; this configuration is shown in Fig. 4. It was then very easy to modify his construction to obtain a (35_4) configuration, shown in Fig. 5.

This construction method turned out to be modifiable in a variety of ways. Some of the other configurations (35_4) obtained by such modifications are shown in Figs. 6–8. All these have 7-fold symmetry, and given the method of construction, their existence can be inferred by continuity. In Fig. 9 we show another variant of the construction, yielding a configuration (35_4) with 5-fold symmetry.

In contrast to the case of polycyclic configurations, for the configurations considered in this section no comprehensive theory or notation has been devised so far.

Thus the quest for a graphic representation of a (35_4) configuration has finally been achieved, severalfold. But it is of some interest to note additional aspects.

4. Remarks

(i) None of the geometric configurations (35_4) described above is isomorphic (combinatorially equivalent) to the (35_4) configuration described by Danzer. That configuration has a group of automorphisms that is transitive on the flags (incident point–line pairs) of the configuration. It is easy to show, in one of several ways, that this is not the case for our configurations. We found it convenient to compare the *deleted neighborhoods* of points in the configurations considered. By "deleted neighborhood" of a point V in a configuration C we mean the set of points on lines through V, together with the set of configuration lines not through V that are determined by two or more of these points. It is easy to verify that for the Danzer configuration the deleted neighborhood of every point consists of four disjoint "triangles" (or, in a more appropriate terminology, trilaterals). This implies, in particular, that every point of the deleted neighborhood

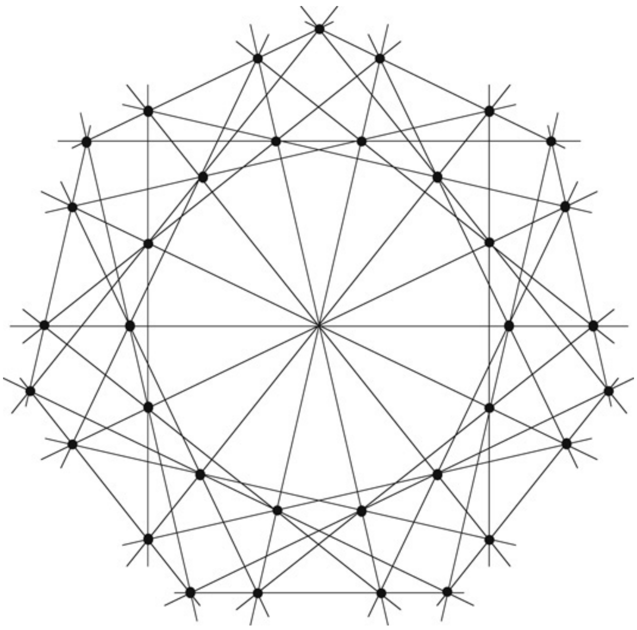

Fig. 5.

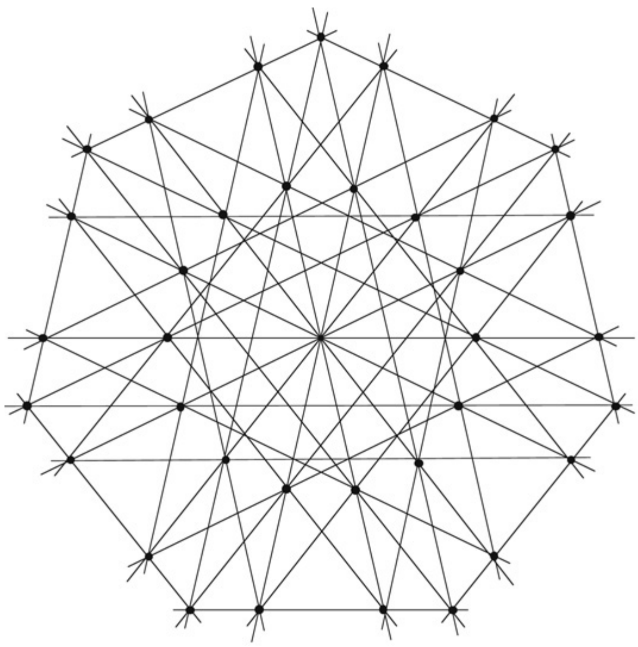

Fig. 6.

is incident with two lines of that neighborhood. In contrast, for each of the configurations in Figs. 5 to 9, the deleted neighborhood of the topmost point contains points that are incident with a single line of the neighborhood. Like many other aspects of the configurations obtainable by the construction described in Section 3, their automorphism groups have not been investigated so far.

(ii) Concerning the "failed configuration" in Fig. 3, it may be noted that a slight change may lead to a visually acceptable presentation of the configuration. The change consists in substituting "pseudolines" (curved lines) for (straight) lines, as shown in Fig. 10. We shall not dwell here on the topic of pseudolines; information may be found in [6], [5] or [3]. Even so, it may be noted that the study of pseudoline configurations contributed in at least two definite

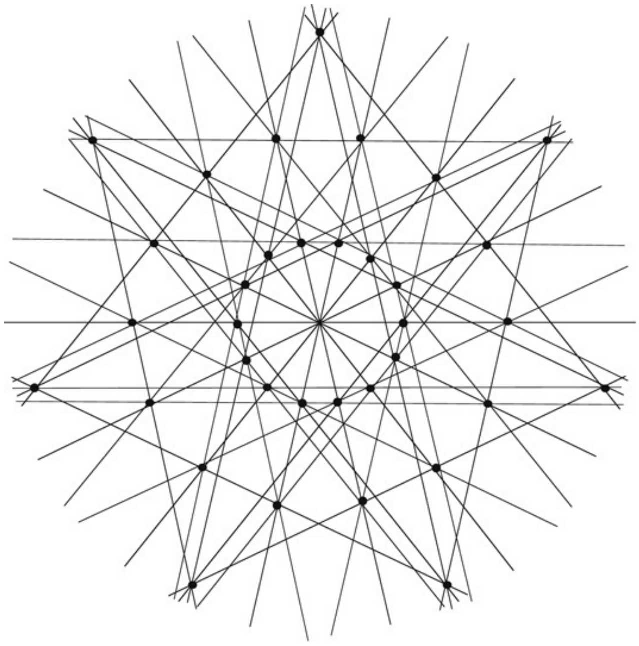

Fig. 7.

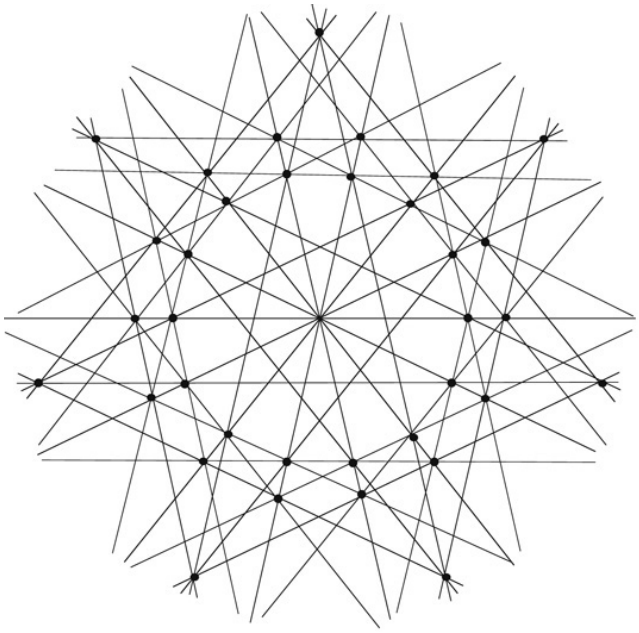

Fig. 8.

ways to the investigation of configurations of points and lines. As Bokowski informed me in a personal message, he found his configuration (25_4) (Fig. 4 above) while working on pseudoline configurations. The second instance came when I tried to construct a (20_4) configuration. What I first managed to do was to construct a (20_4) configuration of pseudolines, by merging two (10_3) configurations. Looking at the result, I wondered whether one could straighten the pseudolines. As it turned out, this is possible, resulting in the configuration shown in Fig. 11; the two (10_3) configurations are clearly visible in the diagram. This configuration contradicts the long-standing conjecture [9, p. 343] that the (21_4) configuration of Fig. 1 is the (n_4) geometric

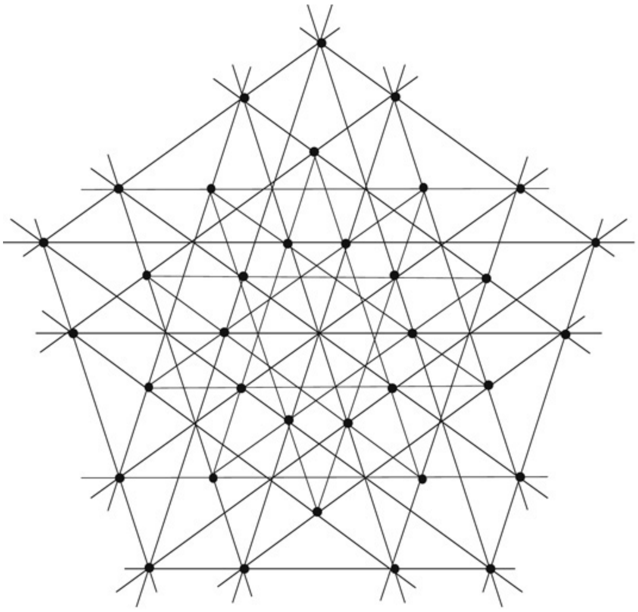

Fig. 9.

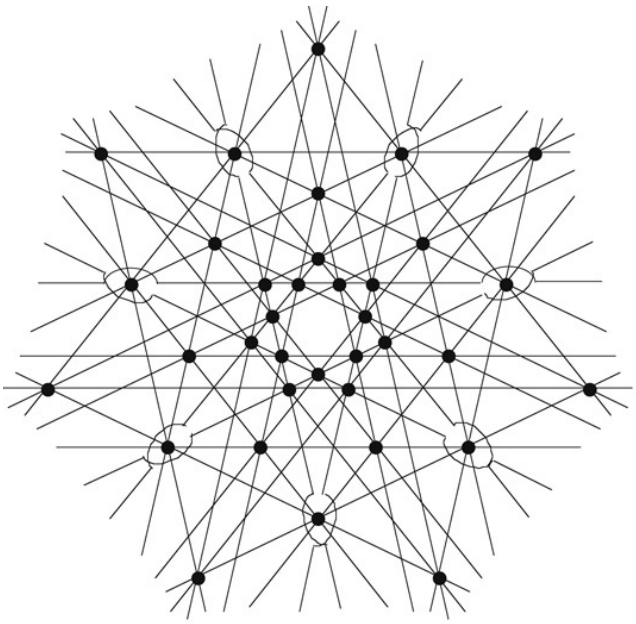

Fig. 10.

configuration with the smallest n. The construction of the (20_4) configurations can be extended to configurations (n_4) for all $n = 4$ k.

(iii) There is some confusion about the term "polycyclic configuration", caused by the present author's selective and defective reading and understanding of the definition in [2]. It has become apparent – with my attention drawn to the facts by Berman and by Pisanski – that the definition given in [2] is more general than the interpretation I gave it in [7] and other papers, as well as in Section 2. Their definition does not involve the *full* symmetry group of the geometric configuration, but depends only on the existence of *some* non-trivial symmetry under which all orbits (of points and lines) have the same size. My definition of polycyclic configurations (given above, as well as in [7] and other publications) coincides with what Berman [1] has called

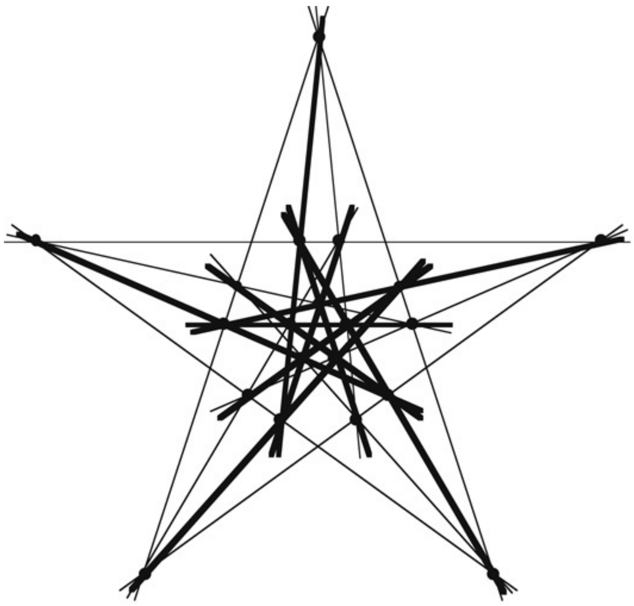

Fig. 11.

“celestial” configurations; it is the only kind presented in the examples of [2]. The configurations presented in Figs. 4–9 and 11 are polycyclic (in the sense of [2], since under a rotational symmetry they have orbits of the same sizes), but not celestial (since under the full symmetry group the orbits have different sizes).

(iv) The method of representing configurations by the shortest segments that connect all points incident with the carrier line (which we utilized in Fig. 2) may possibly be the source of the error committed by Steinitz in [11]. The main result of that thesis asserts that if one disregards one arbitrarily chosen flag in any connected configuration (n_3), there is a collection of points and straight lines that realizes all the other incidences and only the required incidences. As noticed (to my knowledge) first by Pisanski [10], this is not correct since there may be uncalled-for incidences in *every* realization (with one flag omitted) of certain configurations. As observed by Pisanski, if one uses representations by segments (instead of whole lines), superfluous incidences can be avoided in all cases studied so far.

(v) Despite all the advantages that the representation of configurations by sufficiently extended lines has over representations using shortest segments, the latter kind is at times to be preferred. The reason is that the extended lines tend to clutter up the presentation even in moderately sized diagrams. For example, the structure of the configurations in Figs. 7 and 8 would be more readily discerned if only the necessary segments were present. Thus, it should depend on the particular situation, the presentation of which is chosen, with the writer aware of the advantages and shortcomings of each method.

(vi) Danzer did not claim that the configuration he communicated to us is his invention. Analogous constructions go far back in history to the various works of Cayley, Schläfli, Cremona and others. However, I am not aware of any previous explicit occurrence of the (35_4) configuration, and in my mind I still call it “Danzer's configuration”.

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