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EQUIPARTITE GRAPHS

BY

BRANKO GRÜNBAUM
Department of Mathematics, University of Washington
Seattle, WA 98195, USA
e-mail: grunbaum@math.washington.edu

AND

TOMÁŠ KAISER *
Department of Mathematics, University of West Bohemia
Univerzitní 8, 306 14 Plzeň, Czech Republic
and
Institute for Theoretical Computer Science (ITI), Charles University,
Prague, Czech Republic.
e-mail: kaisert@kma.zcu.cz

AND

DANIEL KRÁL'
Department of Applied Mathematics
and Institute for Theoretical Computer Science (ITI)
Charles University,
Malostranské nám. 25, 118 00 Prague 1, Czech Republic
e-mail: kral@kam.mff.cuni.cz

AND

MOSHE ROSENFELD**
Computing and Software Systems Program, University of Washington,
Tacoma, WA 98402, United States
e-mail: moishe@u.washington.edu

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ABSTRACT

A graph G of even order is **weakly equipartite** if for any partition of its vertex set into subsets V_1 and V_2 of equal size the induced subgraphs $G[V_1]$ and $G[V_2]$ are isomorphic. A complete characterization of (weakly) equipartite graphs is derived. In particular, we show that each such graph is vertex-transitive. In a subsequent paper, we use these results to characterize equipartite polytopes, a geometric analogue of equipartite graphs.

1. Introduction

Classifications of combinatorial objects possessing a variety of symmetries have been extensively studied. In this paper, we study a new kind of symmetric graphs: **equipartite graphs**.

Definition: A graph G of order $2n$ is **weakly equipartite** if for any partition of $V(G)$ into two sets A and B of n vertices each, the subgraphs of G induced by A and B are isomorphic. If there is an automorphism of G mapping A onto B then G is **equipartite**.

Clearly, each equipartite graph is weakly equipartite. These notions were introduced in [5] and are motivated by the study of a related notion of equipartite polytopes.

Figures 1 and 2 show the list of all equipartite graphs of orders six and eight.

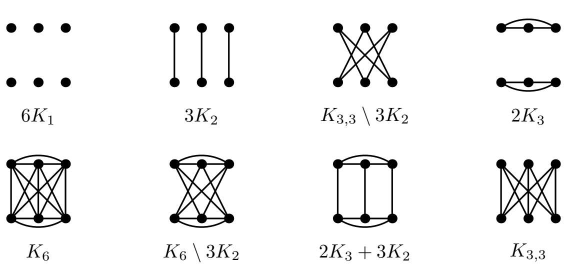

Figure 1. Equipartite graphs of order six.

In this paper, we obtain a complete characterization of weakly equipartite graphs. Our characterization yields that every weakly equipartite graph is actually equipartite and also vertex-transitive. These results enable us to fully

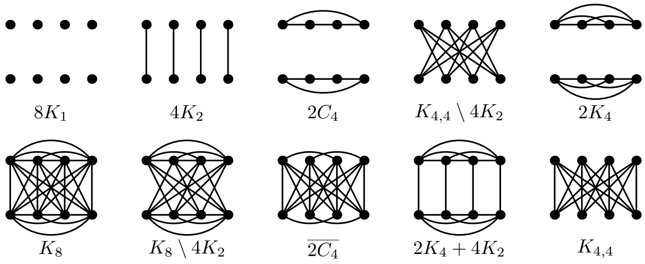

Figure 2. Equipartite graphs of order eight.

characterize equipartite polytopes and prove, in particular, that each equipartite d-polytope has at most $2d + 2$ vertices. These results are contained in a subsequent paper [3] by the present authors.

We prove that for $n = 3$ and $n \geq 5$ there are exactly 8 equipartite graphs of order $2n$; all generated by taking graph sums of subsets of three graphs: $\{nK_2, K_{n,n} \setminus nK_2, 2K_n\}$. Curiously, there are 10 equipartite graphs of order 8.

2. Preliminaries

We use standard graph theory terminology which can be found, e.g., in [2, 6]. A **closed neighborhood** $N(v)$ of a vertex v in G is the set consisting of the vertex v and all its neighbors. If $A \subseteq V(G)$, then $G[A]$ stands for the subgraph induced by the vertices of A; $A^c = V(G) \setminus A$. A set $A \subseteq V(G)$ is **dominating** if each vertex of G is contained in A or adjacent to a vertex of A. Throughout the paper, we often consider partitions of $V(G)$ into two equal-size sets A and B. If $a \in A$ and $b \in B$, then the partition $V(G)$ into the sets A' and B' where $A' = (A \setminus \{a\}) \cup \{b\}$ and $B' = (B \setminus \{b\}) \cup \{a\}$ is said to be obtained by **switching** the vertices a and b, or, for short, that we **switch** the vertices a and b.

A union of k vertex-disjoint copies of a graph G is denoted by kG. We write $G + H$ for an edge-disjoint union of two graphs G and H on the same vertex set; the pairs of corresponding vertices will always be clear from the context. Similarly, $G \setminus H$ stands for a graph G without a subgraph isomorphic to H (again, the graph $G \setminus H$ will be uniquely determined by the context). This notation is used in Figures 1 and 2.

A permutation group Γ acting on a set A_0 of size $2n$ has the **interchange property** [1] if for every n-element subset $A \subseteq A_0$, there is a group element

$g \in \Gamma$ which interchanges A with its complement. Note that a graph G is equipartite if and only if its symmetry group, acting as a permutation group on the vertices of G, has the interchange property. Theorem 1 from [1] readily translates to our setting as Lemma 1:

Lemma 1: *If a graph G with $2n$ vertices is equipartite, then G is vertex-transitive.*

In the sequel, we show that even weakly equipartite graphs are vertex transitive and also equipartite.

Let us now state two lemmas on (weakly) equipartite graphs. The proof of the first lemma follows directly from the definition.

Lemma 2: *The complement of a weakly equipartite graph is weakly equipartite.*

Lemma 3: *Every weakly equipartite graph G of order $2n$ is regular.*

Proof. Consider a graph G of order $2n$ that is not regular. Let $v_1, \cdots, v_{2n}$ be the vertices of G and let d_i be the degree of the vertex v_i. We can assume that $d_1 \geq d_2 \geq \cdots \geq d_{2n}$. Since G is not regular, $d_1 > d_{2n}$. Split the vertex set of G into two parts $A = \{v_1, \ldots, v_n\}$ and $B = \{v_{n+1}, \ldots, v_{2n}\}$. Let m_{AB} be the number of edges ab of G with $a \in A$ and $b \in B$. The numbers of edges of the subgraphs $G[A]$ and $G[B]$ are $m_A = (d_1 + \cdots + d_n - m_{AB})/2$ and $m_B = (d_{n+1} + \cdots + d_{2n} - m_{AB})/2$, respectively. Since $d_1 \geq d_2 \geq \cdots \geq d_{2n}$ and $d_1 > d_{2n}$, we have $m_A > m_B$. But then the graphs $G[A]$ and $G[B]$ are not isomorphic and G is not weakly equipartite. ∎

We further restrict the vertex degrees that can appear in weakly equipartite graphs. Note that Lemma 4 excludes the existence of a 2-regular weakly equipartite graph of order $2n \geq 12$ but it does not exclude the existence of such graphs of orders 4, 6, 8 and 10 (in fact, $2C_4$ is a 2-regular equipartite graph of order 8).

Lemma 4: *If G is a weakly equipartite graph of order $2n$, then G is d-regular where*

$$d \in \{0, 1, n-3, n-2, n-1, n, n+1, n+2, 2n-2, 2n-1\}.$$

Proof. To show that a d-regular graph G of order $2n$ is not weakly equipartite it is enough to show that there is a dominating set $A' \subset V(G)$ of size $\leq n$

that contains $N(v)$ for some vertex $v \in G$. Indeed, if such a set exists, then we can add to it $n - |A'|$ vertices to obtain a set A with n vertices such that $\Delta(G[A]) = d$ while $\Delta(G[A^c]) < d$ (since every vertex in A^c has a neighbor in A) and the two graphs are not isomorphic.

Fix a weakly equipartite d-regular graph G of order $2n$. By Lemma 2, we can assume that $d \leq n$. We note that if $n \leq 5$, then the statement of the lemma trivially holds since the set in the statement contains all integers between 0 and $2n - 1$. So assume that G is of order $2n \geq 12$, and regular of degree d, $3 \leq d \leq n-4$ (the simple case $d = 2$ will be treated at the end separately). We shall show that such a graph cannot be weakly-equipartite.

Let $n = k(d+1)+r$; $r < d+1$. Let $\{N(v_1), \dots, N(v_m)\}$ be a largest possible set of mutually disjoint closed neighborhoods in G.

Claim: $m(d + 1) > n$. *Otherwise, we can add vertices to* $\bigcup_1^m N(v_i)$ *to obtain a set* A *with* n *vertices and* G*, being weakly equipartite, implies that* $G[A] \cong G[A^c]$ *and* $G[A^c]$ *contains* m *mutually disjoint copies of closed neighborhoods contradicting the maximality of* m*. The same argument shows that* $m = 2k = 2\lfloor \frac{n}{d+1} \rfloor$.

For notational convenience, we assume first that $r = 0$. Since $d \leq n - 4$ we must have $k \geq 2$. Let $A_0 = \bigcup_1^k N(v_i)$, $|A_0| = n$. Note that $\{v_1, \dots, v_k\}$ is a dominating set in $G[A]$; hence, if G, is weakly equipartite, then $G[A^c]$ also contains a set $\{v_{k+1}, \dots, v_{2k}\}$ that dominates $G[A^c]$. It follows that $D = N(v_1) \cup \{v_2, \dots, v_{2k}\}$ is a dominating set in G of size $(d + 1) + (2k - 1)$. Since $n = k(d+1)$, $k \geq 2$ and $d+1 \geq 4$ we have $n = k(d+1) = (d+1)+(k-1)(d+1) \geq (d+1)+4(k-1) \geq (d+1)+(2k-1)$. Hence G contains a dominating set D of size $\leq n$ that includes a closed neighborhood so G cannot be weakly equipartite.

If $r > 0$ let $A_0 = \bigcup_1^k N(v_i)$. In this case, $|A_0| < n$ so we add to it $n-|A_0|$ vertices from $N(v_{k+1})$ containing v_{k+1} to obtain a set A with n vertices. As above, $G[A] \cong G[A^c]$ and thus $G[A^c]$ contains a dominating set $\{v_{k+2}, \dots, v_{2k+2}\}$. It follows that $D = N(v_1) \cup \{v_{k+2}, \dots, v_{2k+2}\}$ is a dominating set of vertices in G of size $d+1+2k+1$. If $k = 1$, then $|D| = d+4$, and since $n \geq d+4$, $|D| \leq n$. For $k > 1$, $n = k(d+1)+r \geq (d+1)+(k-1)(d+1)+1 \geq (d+1)+2k+1 = |D|$ and again, G is not weakly equipartite.

When G is 2-regular, it is easy to see that the cycles C_{2k} are not weakly equipartite for $k \geq 4$. Indeed, take for one set an arc of length $k - 2$ and add to it the vertex in the middle of its complementary arc. You get one subgraph consisting of a path with an isolated vertex and the other graph will consist of two disjoint paths of length ≥ 1 each. If G is a collection of cycles of total order ≥ 10 it is easy to see that it cannot be weakly equipartite. We leave the simple argument to the reader. ∎

3. Weakly equipartite graphs with small degrees

The proof of the theorem that characterizes weakly equipartite graphs is split into several steps. We have already observed some general properties of weakly equipartite graphs, in particular, that they are regular graphs with very restricted degrees. Next, we focus on d-regular graphs of order $2n$ with $d \leq n-1$. We distinguish two cases based on whether the graph is disconnected or connected. In Subsection 3.1, we show that the only disconnected weakly equipartite graphs are $2nK_1$, nK_2, $2C_4$ and $2K_n$. In Subsection 3.2, we establish that, in most cases, the only connected weakly equipartite bipartite graph of order $2n$ with degrees smaller than n is the graph $K_{n,n} \setminus nK_2$. Our results are then combined to provide a full characterization of equipartite and weakly equipartite graphs in the next section.

3.1. Disconnected weakly equipartite graphs. First, we show that the orders of all the components of a disconnected weakly equipartite graph are the same.

Lemma 5: *If G is a disconnected weakly equipartite graph, then all its components have the same order.*

Proof. Consider a weakly equipartite graph G of order $2n$ with k components and let $n_1 \geq \cdots \geq n_k$ be their orders. In addition, let Γ_i be the component of order n_i. Choose k_0 to be the smallest index such that $n_1 + \cdots + n_{k_0} \geq n$. If $n_1 + \cdots + n_{k_0} > n$, let W be a subset of vertices of Γ_{k_0} of size w, where $w = n - n_1 - \cdots - n_{k_0-1}$, such that the subgraph $\Gamma_{k_0}[W]$ is connected. Split the vertex set of G into two parts A and B as follows:

$$A = V(\Gamma_1) \cup \cdots \cup V(\Gamma_{k_0-1}) \cup W$$
$$B = V(G) \setminus A.$$

The number of components of $G[A]$ is k_0 by the choice of the set A. Since G is weakly equipartite, $G[B] \cong G[A]$. In particular, the number of its components is also k_0 and it contains a component Γ_j of order n_1. However, such a component of $G[B]$ is also a component of G and therefore its index $j > k_0$. It follows that the first $k_0 - 1$ components of $G[A]$ are all of size n_1, hence $G[B]$ also has $k_0 - 1$ components of order n_1 and so is the order of Γ_{k_0}. ∎

To prove Lemma 7 below, we need the following lemma [4, Lemma 1.15]:

Lemma 6: *Let G be a 2-connected graph that is not complete. If G is not a cycle, then G contains two nonadjacent vertices u and v such that the graph $G \setminus \{u, v\}$ is connected.*

Lemma 7: *If G is a disconnected weakly equipartite d-regular graph G with $d > 2$, then G is a disjoint union of two cliques of the same order.*

Proof. Let G be a disconnected weakly equipartite d-regular graph of order $2n$ and $k \geq 2$ components. By Lemma 5, the order of each component is equal to $2n/k$. By Lemma 4, $d \geq n - 3$. If all the components have order $d + 1$, they are complete graphs. Hence, the order of every component is at least $d + 2 \geq n - 1$. If there are more than two components, then $3(n-1) \leq 2n$ which implies $n \leq 3$ and $d \leq n - 1 \leq 2$, contrary to our assumption $d > 2$. We conclude that G consists of two components of order n each.

Let Γ_1 and Γ_2 be the two components of G. We show that Γ_1 is a complete graph and since G is weakly equipartite so is Γ_2. If Γ_1 contains a cut-vertex v_1 then it contains vertices of degree $\leq (n-1)/2$. Since $d \geq n - 3$ we get that $(n-1)/2 \geq n - 3$ or $n \leq 5$. It is easy to check that no regular graphs of order $2n = 10$ or 8 regular of degree $d \geq 3$ with two isomorphic connected components each have a cut vertex. We can conclude that Γ_1 contains no cut-vertex, i.e., Γ_1 is 2-connected.

Assume now that Γ_1 is not a complete graph. Note that since $d > 2$ it is not a cycle. By Lemma 6, Γ_1 contains two nonadjacent vertices u and v such that $\Gamma_1 \setminus \{u, v\}$ is connected. Let $u'v'$ be any edge of Γ_2. Switch u and u', and v and v' to get a partition of $V(G)$ into A and B:

$$A = \{u', v'\} \cup (V(\Gamma_1) \setminus \{u, v\})$$
$$B = \{u, v\} \cup (V(\Gamma_2) \setminus \{u', v'\}).$$

Observe that the number of components of $G[A]$ is two and the number of components of $G[B]$ is at least three hence G is not weakly equipartite, contradicting our assumptions. So Γ_1 and Γ_2 are complete graphs. ∎

We are now ready to characterize all disconnected weakly equipartite graphs. Note that each disconnected weakly equipartite graph is also equipartite.

Theorem 8: *Any disconnected weakly equipartite graph G is one of the following graphs:*

$$2nK_1,\ nK_2,\ 2C_4 \text{ and } 2K_n.$$

Proof. It is straightforward to verify that the graphs $2nK_1$, nK_2, $2C_4$ and $2K_n$ are weakly equipartite. Consider a weakly equipartite disconnected graph G. By Lemma 3, the graph G is d-regular for some d, and, by Lemma 5, all the components of G have the same order.

If $d = 0$, then G is $2nK_1$. If $d = 1$, then G is nK_2. On the other hand, if $d > 2$, then $G = 2K_n$. Hence, we can assume that $d = 2$ and G is a disjoint union of cycles of the same length $\ell \geq 3$.

If G contains more than two cycles, we can partition its vertices into a set A that contains a cycle plus vertices from each of the other cycles. The graph $G[A]$ will contain a cycle while $G[A^c]$ will not.

The graphs $2C_k$, $k > 5$ and G are not weakly equipartite. To see this take a partition that consists of a path of length $k - 3$ from one cycle and add to it a pair of vertices that are not connected by an edge from the second cycle. This graph will have 2 isolated vertices while the other will not. Hence, the only 2-regular disconnected weakly equipartite graphs are: $2K_3$ and $2C_4$. ∎

3.2. Connected weakly equipartite graphs. First, we prove a lemma describing a very special structure that each d-regular weakly equipartite graph of order $2n$ with $d \leq n - 1$ contains.

Lemma 9: *Let G be a weakly equipartite d-regular graph of order $2n$ with $d \leq n - 1$ and let v_0 be an arbitrary vertex of G. Then, there is a subset $A \subset V(G)$ with $|A| = n$ such that $N(v_0) \subseteq A$ and $G[A] \setminus N(v_0)$ is a set of isolated vertices in $G[A]$.*

Proof. Let A_0 be an n-vertex subset $V(G)$ containing $N(v_0)$ that minimizes $\sum d_{G[A_0]}(v) : v \in G[A_0] \setminus N(v_0)$. If A_0 is of the form described in the statement of the lemma, we are done. Otherwise, there exists a vertex $v \in G[A_0] \setminus N(v_0)$ joined by an edge to another vertex of A_0. As G is weakly equipartite, let v_0' be the counterpart of the vertex v_0 in $G[A_0^c]$. Note that all d neighbors of v_0' are contained in A_0^c.

The set A'_0 obtained by switching v and v'_0 contains $N(v_0)$. In addition, the vertex v'_0 is an isolated vertex in $G[A'_0]$. Therefore, $\sum d_{G[A'_0]}(v) : v \in G[A'_0] \setminus N(v_0) < \sum d_{G[A_0]}(v) : v \in G[A_0] \setminus N(v_0)$ contradicting the choice of the set A_0. ∎

Now we show that all weakly equipartite connected regular graphs of order $2n$ with maximum degree at most $n-1$ are bipartite.

Lemma 10: *If G is a weakly equipartite connected d-regular graph of order $2n$ with $n-3 \leq d \leq n-1$, then G is bipartite.*

Proof. Let $k = n - 1 - d$. Note that k is 0, 1 or 2. Fix a set $A \subseteq V(G)$ of size n as described in Lemma 9. Let γ_A be a vertex of degreed d contained in A, $\Gamma_A = N(\gamma_A)$ and X_A the independent set consisting of the k isolated vertices of $G[A]$. Let $B = A^c$. Since the graph G is weakly equipartite, the subgraph $G[B]$ is isomorphic to $G[A]$. Let Γ_B, γ_B and X_B be isomorphic images of Γ_A, γ_A and X_A in $G[B]$, respectively. In addition, let $\Gamma'_A = \Gamma_A \setminus \gamma_A$ and $\Gamma'_B = \Gamma_B \setminus \gamma_B$ (see Figure 3).

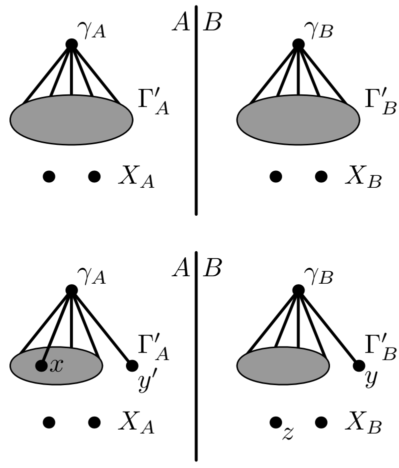

Figure 3. Notation used in the proof of Lemma 10.

Clearly, Γ'_A and Γ'_B are isomorphic. We show that both graphs Γ'_A and Γ'_B consist of isolated vertices by considering three distinct cases. This will imply that G is bipartite since its vertex set can be partitioned into two independent sets $\{\gamma_B\} \cup \Gamma'_A \cup X_A$ and $\{\gamma_A\} \cup \Gamma'_B \cup X_B$.

First, assume for the sake of contradiction that the graphs Γ'_A and Γ'_B are connected. Since G is connected, Γ'_A contains a vertex x adjacent to a vertex from the set B which cannot be γ_B. Consider the set A^* obtained from A and B by switching x and γ_B. The subgraph $G[A^*]$ consists of $k+2$ components precisely: one of them is formed by the vertex γ_A and its $d-1$ neighbors in A^* and the remaining components are isolated vertices, namely, the vertex γ_B and the k vertices of X_A. On the other hand, the subgraph $G[A^{*c}]$ consists of at most $k+1$ components. To see this, note that $G[B \setminus \{\gamma_B\}]$ consists of $k+1$ components: (Γ'_B and the isolated vertices of X_B) and since x is joined by an edge to a vertex of B and this vertex cannot be γ_B (because the neighbors of γ_B are in B) the vertex x is not isolated in $G[A^{*c}]$. Hence, $G[A^{*c}]$ consists of at most $k+1$ components contradicting our assumption that G is weakly equipartite.

Assume now that the graphs Γ'_A and Γ'_B are formed by at least two components each, and not all are isolated vertices.

Choose x to be any non-isolated vertex of Γ'_A. Since the graph G is d-regular, x is adjacent to a vertex of B. We consider the sets A' and B' obtained from A and B by switching the vertices x and γ_B. Note that $G[A']$ is formed by $k+2$ components: one of them is formed by the vertex γ_A and its $d-1$ neighbors in A' and the remaining components are isolated vertices, namely, the vertex γ_B and the k vertices of X_A. Since the graph $G[A']$ contains a vertex of degree $d-1$ (the vertex γ_A), $G[B']$ also contains a vertex x_0 of degree $d-1$. Since the degree of x in $G[B']$ is at most $d-2$ (x has at least two neighbors in Γ_A) and the degrees of the vertices of X_B are at most one (they can be only adjacent to x), x_0 must belong to Γ'_B. If x_0 were not adjacent to x, then its $d-1$ neighbors would have to be all the vertices of Γ'_B and Γ'_B would be formed by a single component contrary to our assumption. Hence, x_0 is adjacent to x and its remaining neighbors are the remaining $d-2$ vertices of Γ'_B. We conclude that Γ'_B consists of precisely two components: one formed by $d-1$ vertices and the other is an isolated vertex y.

Since $G[A']$ contains $k+1$ isolated vertices and $G[B \setminus \{\gamma_B\}]$ contains $k+1$ isolated vertices (y and X_B), the vertex x cannot be adjacent to y. Since we can choose as x any vertex of the component of Γ'_A of order $d-1$, we conclude that y can be adjacent only to its counterpart y' in A, the vertices of X_A and the vertex γ_B. Similarly, an isolated vertex z of B is adjacent to no neighbor of γ_A with a possible exception of y'. Consequently, if $k = 0$, the degree of y does not exceed two and if $k = 1$ the degree of z does not exceed two, which contradicts our assumption that $d \geq 3$. Hence, $k = 2$. Let z be one of the isolated vertices in $G[B]$. Since z can be adjacent only to y' and the k vertices from X_A, its degree is 3, so $d = 3$ and since $d \geq n - 3$ the only remaining possibility is a connected cubic graph of order 12. In Lemma 12 we prove that no connected cubic graphs of order 12 are weakly equipartite. ∎

We can now characterize weakly equipartite connected bipartite regular graphs:

Lemma 11: *Let G be a weakly equipartite, bipartite connected d-regular graph of order $2n$ with $3 \leq d \leq n - 1$. Then, $G = K_{n,n} \setminus nK_2$.*

Proof. Let V_1 and V_2 be the two independent sets that partition G. Since G is regular, we have $|V_1| = |V_2|$. By Lemma 4, $d \in \{n-3, n-2, n-1\}$. If $d = n-1$, then $G = K_{n,n} \setminus nK_2$. We now exclude the cases $d = n - 3$ and $d = n - 2$.

As the first step, we find vertices w and w' of V_1 and x and y of V_2 such that w is adjacent neither to x nor y and w' is adjacent to both x and y. If $d = n-2$, choose w to be a vertex of V_1 and let x and y be the two vertices of V_2 which are not adjacent to w. Note that $n \geq 5$ because $d \geq 3$. Since $2d = 2n - 4 > n$, x and y share a common neighbor $w' \in V_2$.

If $d = n - 3$, let w be again a vertex of V_1 and let x, y and z be the three vertices of V_2 which are not adjacent to w. Note that $n \geq 6$ because $d \geq 3$. Since $3d = 3n - 9 > n$, at least two of the vertices x, y and z have a common neighbor. Assume that x and y are such two vertices and w' is their common neighbor.

We now proceed jointly for both cases. Consider the sets A and B obtained from V_1 and V_2 by switching w and x, and w' and y. The degrees of x and y in $G[A]$ are exactly $d-1$ by the choice of w and w'. Each vertex of V_1 is adjacent to at least one vertex of $V_2 \setminus \{x, y\}$ (recall that d is at least three) and thus each vertex of V_1 has degree at most $d - 1$ in $G[A]$. Hence $\Delta(G[A]) \leq d - 1$. On the other hand, $G[B]$ contains a vertex of degree d (the vertex w). Hence, the subgraphs $G[A]$ and $G[B]$ are not isomorphic. Therefore, d is neither $n-3$ nor $n - 2$. ∎

4. Characterization of equipartite and weakly equipartite graphs

Before we prove Theorem 13, we note that 2-regular graphs were dealt with in Lemma 4 and there is a single size of 3-regular weakly equipartite graphs not covered by results in the previous two sections:

Lemma 12: *There is no weakly equipartite cubic graph of order* 12.

Proof. Let G be a cubic graph of order $2n = 12$. Assume for the sake of contradiction that G is weakly equipartite.

Assume first that the graph G is triangle-free. Let v_0 be any vertex of G and let A be a set of vertices of G as in Lemma 9, i.e., the subgraph $G[A]$ is isomorphic to $K_{1,3} + K_1 + K_1$. Let $B = V(G) \setminus A$. Since the graph G is weakly equipartite, the subgraph $G[B]$ is isomorphic to $G[A]$. Let v_0' be the counterpart of v_0 in $G[B]$. Since G is cubic, all the neighbors of v_0 are in A and all the neighbors of v_0' in B. In particular, the sets $A' = (A \setminus \{v_0\}) \cup \{v_0'\}$ and $B' = (B \setminus \{v_0\}) \cup \{v_0'\}$ are independent. Since $A' \cup B' = V(G)$, the graph G is bipartite. Since the girth of such graphs is 4, we can find vertices w, w' in one partition and x, y in the other partition which are related, as described. But there is no weakly equipartite cubic bipartite graph of order 12 in Lemma 11. By the same argument, G is not equipartite.

We may assume that G contains a triangle, say $v_1v_2v_3$. Let $\{v_1, v_2, v_3\} \subset A_1$ be a subset of six vertices of $V(G)$ and let $B_1 = V(G) \setminus A_1$. Since the subgraphs $G[A_1]$ and $G[B_1]$ are isomorphic, $G[B_1]$ contains a triangle, say $v_4v_5v_6$. Let $A_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $B_2 = V(G) \setminus A_2$. Since $G[A_2]$ and $G[B_2]$ are isomorphic, $G[B_2]$ contains two vertex-disjoint triangles, say $v_7v_8v_9$ and $v_{10}v_{11}v_{12}$. Consider now the following two sets A and B (see Figure 4):

$$A = \{v_1, v_2, v_3, v_4, v_7, v_{10}\}$$
$$B = \{v_5, v_6, v_8, v_9, v_{11}, v_{12}\}.$$

Again, since $G[A]$ contains a triangle, $G[B]$ contains a triangle, too. Observe now that the maximum degree of $G[B]$ is two because each vertex of B has at least one neighbor among the vertices v_4, v_7 and v_{10} (which are contained in A). Thus, a triangle contained in $G[B]$ actually forms a component of $G[B]$. In addition, the graph $G[B]$ contains a perfect matching (consider the edges v_5v_6, v_8v_9 and $v_{11}v_{12}$). But this is impossible because one of the components of $G[B]$ is a triangle, a contradiction. ∎

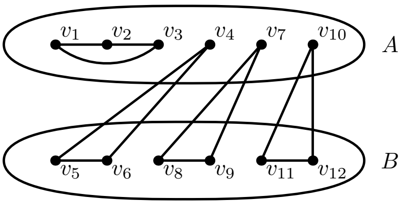

Figure 4. The notation used in the proof of Lemma 12.

We are ready now to characterize weakly equipartite graphs. Let us recall that all weakly equipartite graphs of order six and eight are depicted in Figures 1 and 2.

Theorem 13: *A graph G is weakly equipartite if and only if it is one of the following graphs:*

$$2nK_1,\ nK_2,\ 2C_4,\ K_{n,\,n} \setminus nK_2 \text{ and } 2K_n$$

or one of their complements:

$$K_{2n},\ K_{2n} \setminus 2K_n,\ K_8 \setminus 2C_4,\ 2K_n + nK_2 \text{ and } K_{n,\,n}.$$

Proof. It is straightforward to verify that all the graphs listed in the statement of the theorem are weakly equipartite. We prove that no other graph is weakly equipartite. Fix a weakly equipartite graph G of order $2n$. By Lemma 3, G is d-regular for some d. By Lemma 2, we can assume that $d \leq n-1$ (otherwise, we consider the complement of G), and, by Lemma 4, we get $d \in \{0, 1, n-3, n-2, n-1\}$.

By Theorem 8, if G is disconnected, then it is one of the graphs $2nK_1$, nK_2, $2C_4$ and $2K_n$. Let us assume in the rest that G is connected and $d \geq 2$ (if $d = 1$, then $G = nK_2$). If $d = 2$, then G is a cycle and its length is either four or six by Lemma 4. Note that C_4 is $K_{2,2}$ and C_6 is $K_{3,3} \setminus 3K_2$. In the rest, we assume that $d \geq 3$.

This means that $d \geq n-3$, $n \geq 6$ and the graph G is bipartite by Lemma 10.

The case that remains to be considered is that $d = n-3$ and $n \leq 6$. Recall that $d \geq 3$. Therefore, $d = 3$ and $n = 6$. However, there is no weakly equipartite cubic graph of order 12 by Lemma 12. ∎

An immediate corollary of Theorem 13 is the following:

Corollary 14: *A graph G of order $2n$ is equipartite if and only if it is weakly equipartite, and every weakly equipartite graph is vertex-transitive.*

5. Concluding remarks

A relaxation of the notion of equipartite graphs also seems to be of some interest:

Problem 1: A graph G of order $2n$ is degree-equipartite if for every n-element set $A \subseteq V(G)$, the degree sequences of the graphs $G[A]$ and $G[V(G) \setminus A]$ are the same. Which graphs G are degree-equipartite? In particular, is there a degree-equipartite graph which is not equipartite?

We also note that if G and H are edge disjoint equipartite graphs on the same set of vertices then so is the graph $G+H$. The three graphs: nK_2, $K_{n,n} \setminus nK_2$, $2K_n$ generate all equpartite graphs for $n \geq 5$. When $n = 4$, $2C_4$ does not have a "parallel" for $n > 4$.

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